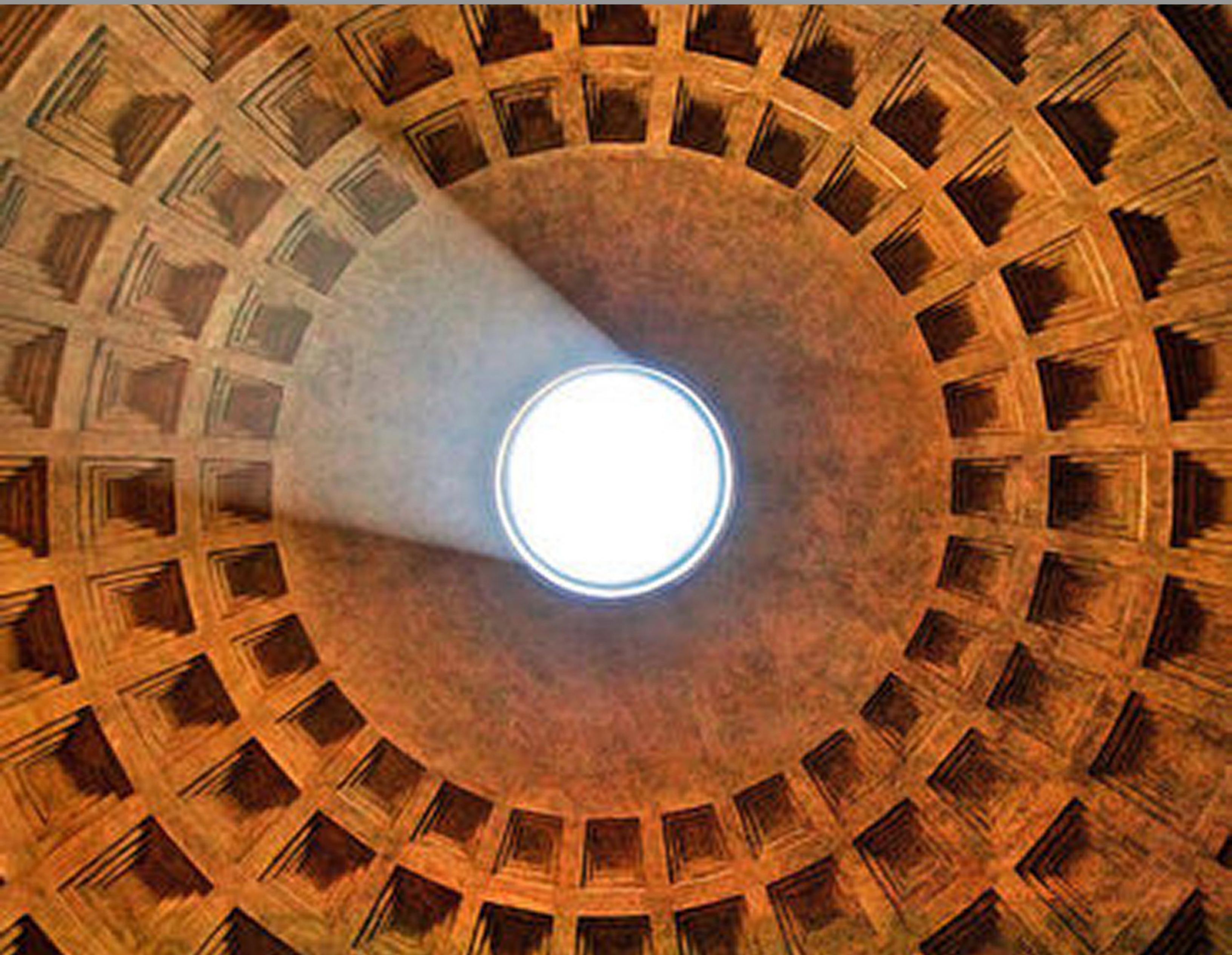
Mr D's Geometry Flexbook



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Contents

CHAPTER 1 Basics of Geometry

Chapter Outline

In this chapter, students will learn about the building blocks of geometry. We will start with the basics: point, line and plane and build upon those terms. From here, students will learn about segments, midpoints, angles, bisectors, angle relationships, and how to classify polygons.

1.1 Points, Lines, and Planes

Learning Objectives

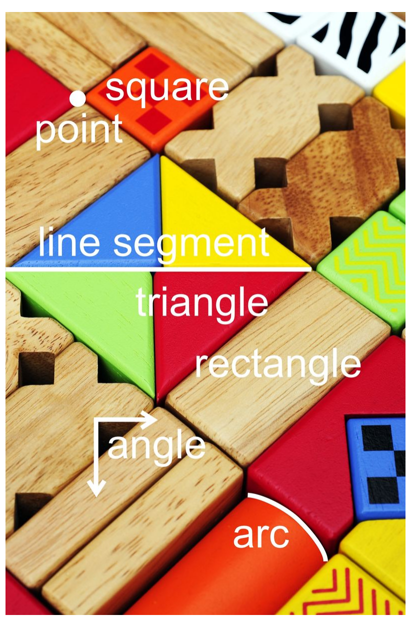
- Understand and build upon the terms:*point, line,* and *plane*.
- Apply and use basic postulates.
- Draw and label terms in a diagram.

Review Queue

This subsection will provide a review of Algebra and previous lessons.

a. List and draw pictures of five geometric figures you are familiar with.
b. A plane is any flat, two-dimensional surface. List three examples of planes in real life.
c. Solve the algebraic equations.
    a. $4x-7=29$
    b. $2(-3x+5)-8=-x+17$
    c. $x^2-2x-15=0$
    d. $x^2=121$

Know What? Geometry is everywhere. Remember these wooden blocks that you played with as a kid? If you played with these blocks, then you have been "studying" geometry since you were a child. For example, if you were to move the four triangles at the center of the picture, so that the vertex in the middle was on the outside, what shape would you make? (Flip each triangle outward)



Geometry: The study of shapes and their spatial properties.

Building Blocks

Point: An exact location in space.

A point describes a location, but has no size. Dots are used to represent points in pictures and diagrams.

These points are said "Point A," "Point L", and "Point F." Points are labeled with a CAPITAL letter.

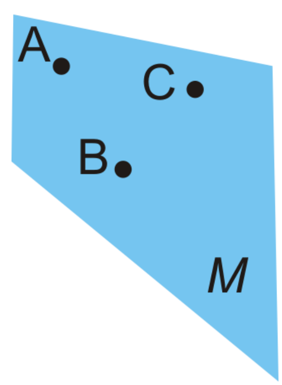
Line: Infinitely many points that extend forever in both directions.

A line, like a point, does not take up space. It has direction, location and is always straight. Lines are one-dimensional because they only have length (no width). A line can by named or identified using any two points on that line or with a lower-case, italicized letter.

This line can be labeled $\overleftrightarrow{PQ}$, $\overleftrightarrow{QP}$ or just g. You would say "line PQ," "line QP," or "line g," respectively. Notice that the line over the $\overleftrightarrow{PQ}$ and $\overleftrightarrow{QP}$ has arrows over both the P and Q. The order of P and Q does not matter.

Plane: Infinitely many intersecting lines that extend forever in all directions.

Think of a plane as a huge sheet of paper that goes on forever. Planes are considered to be two-dimensional because they have a length and a width. A plane can be classified by any three points in the plane.



This plane would be labeled Plane ABC or Plane $\mathcal{M}$. Again, the order of the letters does not matter. Sometimes, planes can also be labeled by a capital cursive letter. Typically, the cursive letter written in a corner of the plane.

Example 1: Which term best describes how San Diego, California, would be represented on a globe?

A. point

B. line

C. plane

Solution: A city is usually labeled with a dot, or point, on a globe. A.

Example 2: Which geometric object best models the surface of a movie screen?

A. point

B. line

C. plane

Solution: The surface of a movie screen extends in two dimensions: up and down and left to right. This description most closely resembles a plane, C.

Beyond the Basics

Now we can use **point, line**, and **plane** to define new terms.

Space: The set of all points expanding in ***three*** dimensions.

Think back to the plane. It extended along two different lines: up and down, and side to side. If we add a third direction, we have something that looks like three-dimensional space, or the real-world.

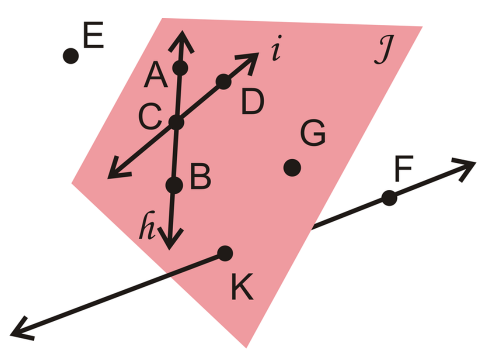
Collinear: Points that lie on the same line.

Example 3: Which points are collinear?

Solution: P, Q, R, S, and T are collinear because they are all on line w. If a point U was above or below line w, it would be **non-collinear**.

Coplanar: Points and/or lines within the same plane.

Example 4:



a) List two other ways to label Plane $\mathcal{J}$.

b) List one other way to label line h.

c) Are K and F collinear? Are they coplanar?

d) Are E, B and F coplanar?

e) List four points that are non-collinear.

Solution:

a) Plane BDG, Plane KAG, among several others. Any combination of three coplanar points that are not collinear would be correct.

b) $\overleftrightarrow{AB}$ or any combination of two of the letters A, C or B in any order.

c) Yes, they lie on the same line.Yes, you need three points to create a plane, so any two or three points are coplanar.

d) Yes, any three points are coplanar.

e) A, C, D and E would be non-collinear. Since three points define a plane, any three of the points would line in the same plane but the fourth must not.

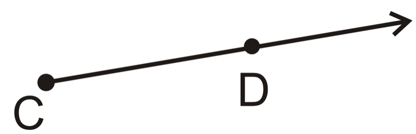
Endpoint: A point at the end of a line.

Line Segment: Part of a line with two endpoints. Or a line that stops at both ends.

Line segments are labeled by their endpoints, $\overline{AB}$ or $\overline{BA}$. Notice that the bar over the endpoints has NO arrows. Order does not matter.

Ray: Part of a line with one endpoint and extends forever in the other direction.

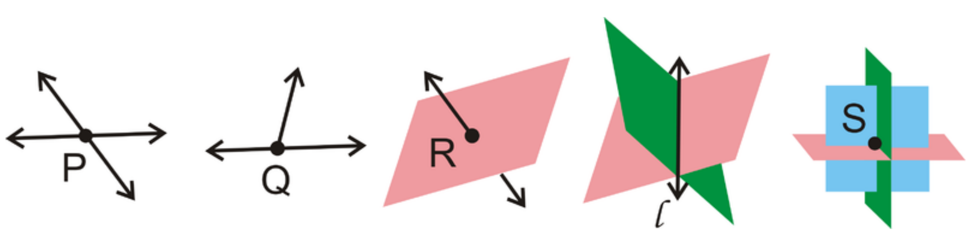
A ray is labeled by its endpoint and one other point on the line.



Of lines, line segments and rays, rays are the only one where order matters. When labeling, always write the endpoint under the side WITHOUT the arrow, $\overrightarrow{CD}$ or $\overleftarrow{DC}$.

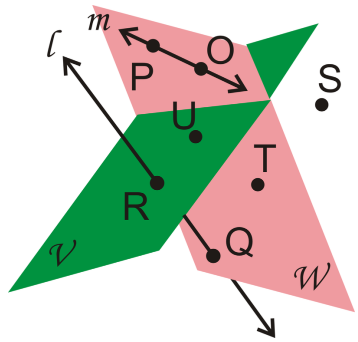
Intersection: A point or set of points where lines, planes, segments or rays cross each other.

Example 5: How do the figures below intersect?



Solution: The first three figures intersect at a point, P, Q and R, respectively. The fourth figure, two planes, intersect in a line, l. And the last figure, three planes, intersect at one point, S.

Example 6: Answer the following questions about the picture to the right.



a) How do the two planes intersect?

b) Is line l coplanar with Plane $\mathcal{V}$ or $\mathcal{W}$?

c) Are R and Q collinear?

d) What point is non-coplanar with either plane?

e) List three coplanar points in Plane $\mathcal{W}$.

Solution:

a) In a line.

b) No.

c) Yes.

d) S

e) Any combination of P, O, T and Q would be correct.

Further Beyond

With these new definitions, we can make statements and generalizations about these geometric figures. This section introduces a few basic postulates. Throughout this book we will be introducing Postulates and Theorems so it is important that you understand what they are and how they differ.

Postulates: Basic rules of geometry. We can assume that all postulates are true, much like a definition.

Theorem: A statement that can be proven true using postulates, definitions, and other theorems that have already proven.

The only difference between a theorem and postulate is that a postulate is *assumed* true because it cannot be shown to be false, a theorem must be *proven* true. We will prove theorems later in this text.

Postulate 1-1: There is exactly one (straight) line through any two points.

Postulate 1-2: There is exactly one plane that contains any three non-collinear points.

Postulate 1-3: A line with points in a plane also lies within that plane.

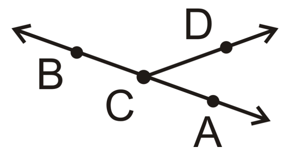
Postulate 1-4: The intersection of two distinct lines will be one point.

Postulate 1-5: The intersection of two planes is a line.

When making geometric drawings, you need to be sure to be clear and label. For example, if you draw a line, be sure to include arrows at both ends. Make sure you label your points, lines, and planes clearly, and refer to them by name when writing explanations.

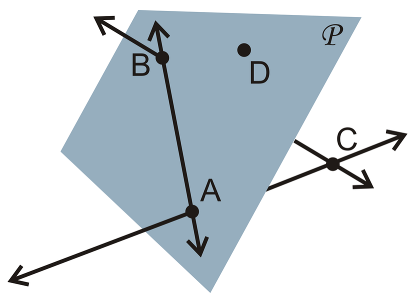
Example 7: Draw and label the intersection of line $\overleftrightarrow{AB}$ and ray $\overrightarrow{CD}$ at point C.

Solution: It does not matter the placement of A or B along the line nor the direction that $\overrightarrow{CD}$ points.

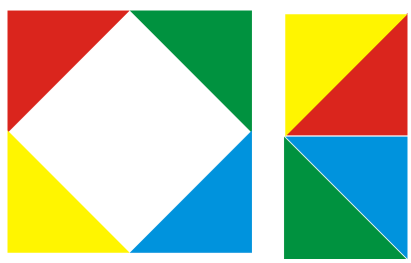


Example 8: Describe the picture below using all the geometric terms you have learned.

Solution: $\overleftrightarrow{AB}$ and D are coplanar in Plane $\mathcal{P}$, while $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ intersect at point C which is non-coplanar.



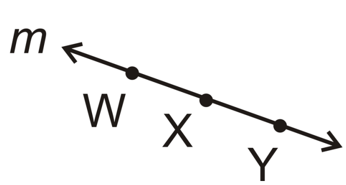
Know What? Revisited If you take the triangles and move them so that the point that met at the center of the square was on the outside, you would get the figure at the right. However, you could also argue that this is not a shape because it has a square hole in the center of it. Another shape that can be made from the four triangles is a rectangle. Part of geometry is justifying and explaining reasoning. You could reason that both of these answers are acceptable.



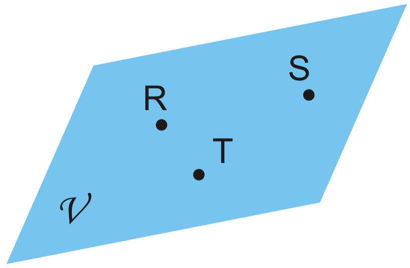
Review Questions

For questions 1-5, draw and label an image to fit the descriptions.

1. $\overrightarrow{CD}$ intersecting $\overline{AB}$ and Plane P containing $\overline{AB}$ but not $\overrightarrow{CD}$.
2. Three collinear points A, B, and C such that B is also collinear with points D and E.
3. $\overrightarrow{XY}$, $\overrightarrow{XZ}$, and $\overrightarrow{XW}$ such that $\overrightarrow{XY}$ and $\overrightarrow{XZ}$ are coplanar, but $\overrightarrow{XW}$ is not.
4. Two intersecting planes, $\mathcal{P}$ and Q, with $\overline{GH}$ where G is in plane $\mathcal{P}$ and H is in plane Q.
5. Four non-collinear points, I, J, K, and L, with line segments connecting all points to each other.
6. Name this line in five ways.

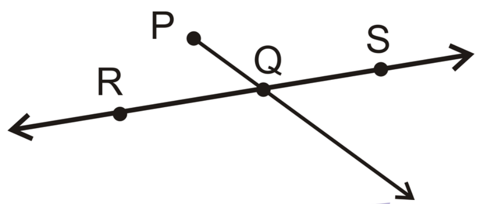


7. Name the geometric figure below in two different ways.



8. Draw three ways three different planes can (or cannot) intersect.
9. What type of geometric object is made by the intersection of a sphere (a ball) and a plane? Draw your answer.

For 10-13, use geometric notation to explain each picture in as much detail as possible.



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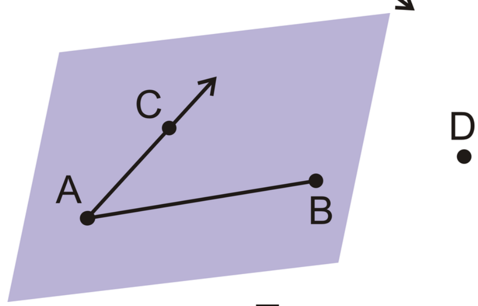

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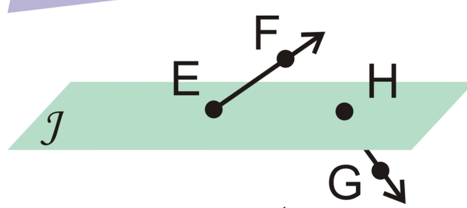

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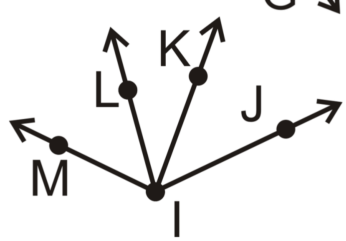


13.

For 14-23, determine if the following statements are ALWAYS true, SOMETIMES true, or NEVER true.

14. Any two distinct points are collinear.
15. Any three points determine a plane.
16. A line is composed to two rays with a common endpoint.
17. A line segment is infinitely many points between two endpoints.
18. A point takes up space.
19. A line is one-dimensional.
20. Any four distinct points are coplanar.

21. $\overrightarrow{AB}$ could be read "ray AB" or "ray BA."
22. $\overleftrightarrow{AB}$ could be read "line AB" or "line BA."
23. Theorems are proven true with postulates.
24. Two of the above statements are "never." Explain why.
25. Four of the above statements are "sometimes." Explain why.

For 26-28, describe the following real world objects in geometric terms.

26. The walls of your classroom and the intersections of these walls with each other and the floor or ceiling. What about where two walls and the floor intersect?
27. The spokes of a bicycle wheel. What about their intersection?
28. Cities on a map. What geometric figure would you draw to measure the distance between them?

In Algebra you plotted points on the coordinate plane and graphed lines. For 29-35, use graph paper and follow the steps to make the diagram on the same graph.

29. Plot the point (2, -3) and label it A.
30. Plot the point (-4, 3) and label it B.
31. Draw the segment $\overline{AB}$.
32. Locate point C, the intersection of this line with the $x-$axis.
33. Draw the ray $\overrightarrow{CD}$ with point $D(1,4)$.

Review Queue Answers

a. Examples could be triangles, squares, rectangles, lines, circles, points, pentagons, stop signs (octagons), boxes (prisms, or dice (cubes).
b. Examples of a plane would be: a desktop, the chalkboard/whiteboard, a piece of paper, a TV screen, window, wall or a door.

   a. $$\begin{aligned} 4x-7&=29\\ 4x&=36\\ x&=9 \end{aligned}$$
   b. $$\begin{aligned} 2(-3x+5)-8&=-x+17\\ -6x+10-8&=-x+17\\ -6x+2&=-x+17\\ -5x&=15\\ x&=3 \end{aligned}$$
   c. Factor, $x=5,\ -3$
   d. $x=\pm 11$

1.2 Segments and Distance

Learning Objectives

- Understand the ruler postulate.
- Understand the segment addition postulate.
- Place line segments on a coordinate grid.

Review Queue

Answer the following questions.

a. How would you label the following geometric figure? List 3 different ways

b. Draw three collinear points and a fourth that is coplanar with these points.
c. Plot the following points on the $x-y$ plane.

  a. (3, -3)
  b. (-4, 2)
  c. (0, -7)
  d. (6, 0)

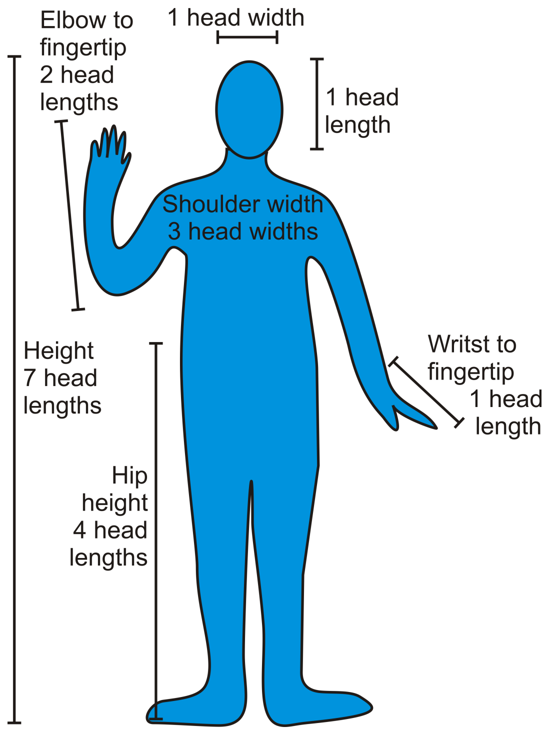
d. Find the equation of the line containing the points (-4, 3) and (6, -2).

Know What? The average adult human body can be measured in "heads." For example, the average human is 7-8 heads tall. When doing this, keep in mind that each person uses their own head to measure their own body. Other interesting measurements are in the picture to the right.

After analyzing the picture, we can determine a few other measurements that aren't listed.

- The length from the wrist to the elbow
- The length from the top of the neck to the hip
- The width of each shoulder

What are these measurements?



Measuring Distances

Distance: The length between two points.

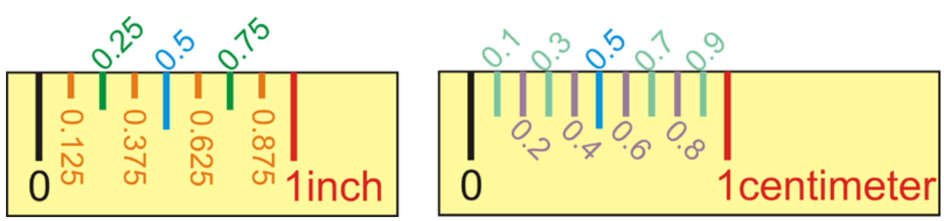
Measure: To determine how far apart two geometric objects are.

The most common way to measure distance is with a ruler. In this class we will use both inches and centimeters.

Example 1: Determine how long the line segment is, in inches. Round to the nearest quarter-inch.

Solution: To measure this line segment with a ruler, it is very important to line up the "0" with the one of the endpoints. DO NOT USE THE EDGE OF THE RULER. This segment is about 3.5 inches (in) long.

As a reminder, inch-rulers are usually divided up by $\frac{1}{8}$-in. (or 0.125 in) segments. Centimeter rulers are divided up by $\frac{1}{10}$-centimenter (or 0.1 cm) segments.



The two rulers above are NOT DRAWN TO SCALE. Anytime you see this statement, it means that the measured length is not actually the distance apart that it is labeled. Different problems and examples will be labeled this way because it can be difficult to draw problems in this text to full scale. You should never assume that objects are drawn to scale. Always rely on the measurements or markings given in a diagram.

Example 2: Determine the measurement between the two points to the nearest tenth of a centimeter.

Solution: Even though there is no line segment between the two points, we can still measure the distance using a ruler. It looks like the two points are 4.5 centimeters (cm) apart.

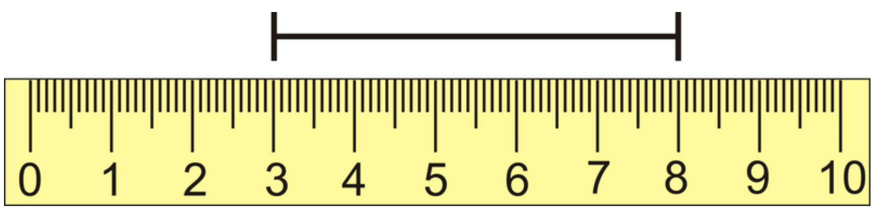
NOTE: We label a line segment, $\overline{AB}$. The ***distance*** between A and B is labeled as AB or $m\overline{AB}$, where m means measure. AB and $m\overline{AB}$ can be used interchangeably. In this text we will primarily use the first.

Ruler Postulate

Ruler Postulate: The distance between two points will be the absolute value of the difference between the numbers shown on the ruler.

The ruler postulate implies that you do not need to start measuring at "0", as long as you subtract the first number from the second. "Absolute value" is used because ***distance is always positive.***

Example 3: What is the distance marked on the ruler below? The ruler is in centimeters.



Solution: Find the absolute value of difference between the numbers shown. The line segment spans from 3 cm to 8 cm.

$$|8-3| = |5| = 5$$

The line segment is 5 cm long. Notice that you also could have done $|3-8| = |-5| = 5$.

Example 4: Draw $\overline{CD}$, such that $CD = 3.825\ in$.

Solution: To draw a line segment, start at "0" and draw a segment to 3.825 in. Put points at each end and label.

Segment Addition Postulate

Before we introduce this postulate, we need to address what the word "between" means in geometry.

B is between A and C in this picture. As long as B is ***anywhere on the segment***, it can be considered to be ***between*** the endpoints.

Segment Addition Postulate: If A, B, and C are collinear and B is between A and C, then $AB + BC = AC$.

The picture above illustrates the Segment Addition Postulate. If $AB = 5\ cm$ and $BC = 12\ cm$, then AC must equal $5 + 12$ or 17 cm. You may also think of this as the "sum of the partial lengths, will be equal to the whole length."

Example 5: Make a sketch of $\overline{OP}$, where Q is between O and P.

Solution: Draw $\overline{OP}$ first, then place Q somewhere along the segment.

Example 6: In the picture from Example 5, if $OP = 17$ and $QP = 6$, what is OQ?

Solution: Use the Segment Additional Postulate. $OQ + QP = OP$, so $OQ + 6 = 17$, or $OQ = 17 - 6 = 9$. So, $OQ = 9$.

Example 7: Make a sketch that matches the description: S is between T and V. R is between S and T. $TR = 6\ cm$, $RV = 23\ cm$, and $TR = SV$. Then, find SV, TS, RS and TV.

Solution: Interpret the first sentence first: S is between T and V.

Then add in what we know about R: It is between S and T.

$\underline{\text{To find } SV}$, we know it is equal to TR, so $SV = 6\ cm$.

$$\begin{aligned} \underline{\text{For } RS:}\ RV &= RS + SV \\ 23 &= RS + 6 \\ RS &= 17\ cm \end{aligned} \qquad \begin{aligned} \underline{\text{For } TS:}\ TS &= TR + RS \\ TS &= 6 + 17 \\ TS &= 23\ cm \end{aligned} \qquad \begin{aligned} \underline{\text{For } TV:}\ TV &= TR + RS + SV \\ TV &= 6 + 17 + 6 \\ TV &= 29\ cm \end{aligned}$$

Example 8: ***Algebra Connection*** For $\overline{HK}$, suppose that J is between H and K. If $HJ = 2x + 4$, $JK = 3x + 3$, and $KH = 22$, find the lengths of HJ and JK.

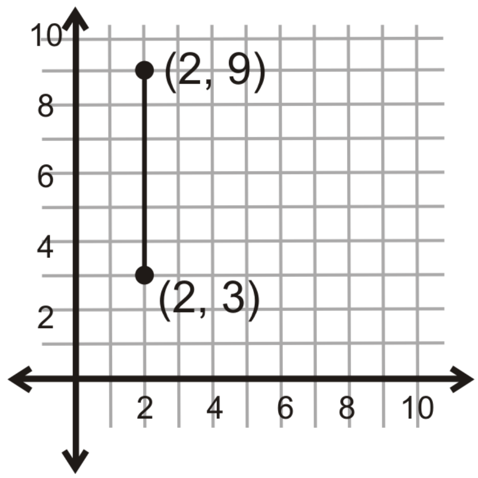
Solution: Use the Segment Addition Postulate and then substitute what we know.

$$\begin{aligned} HJ \quad + \quad JK &= KH \\ (2x+4) + (3x+3) &= 22 \\ 5x + 7 &= 22 \qquad \text{So, if } x = 3, \text{then } HJ = 10 \text{ and } JK = 12. \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

Distances on a Grid

In Algebra, you worked with graphing lines and plotting points in the $x-y$ plane. At this point, you can find the distances between points plotted in the $x-y$ plane if the lines are horizontal or vertical. ***If the line is vertical, find the change in the $y-$ coordinates. If the line is horizontal, find the change in the $x-$ coordinates.***

Example 8: What is the distance between the two points shown below?

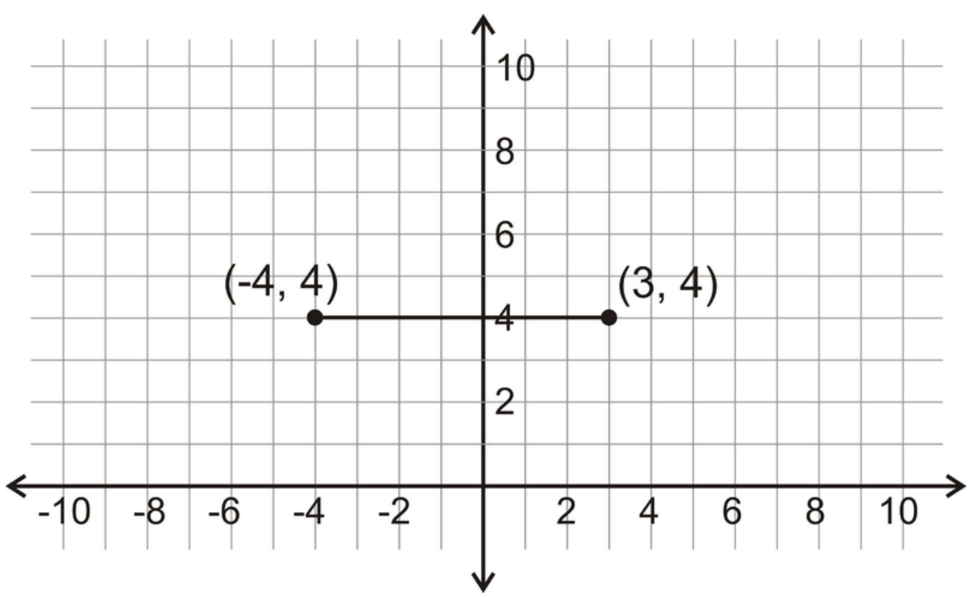


Solution: Because this line is vertical, look at the change in the $y-$coordinates.

$$|9-3| = |6| = 6$$

The distance between the two points is 6 units.

Example 9: What is the distance between the two points shown below?



Solution: Because this line is horizontal, look at the change in the $x-$coordinates.

$$|(-4)-3| = |-7| = 7$$

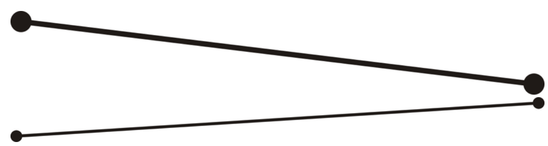
The distance between the two points is 7 units.

Know What? Revisited The length from the wrist to the elbow is one head, the length from the top of the neck to the hip is two heads, and the width of each shoulder one head width. There are several other interesting body proportion measurements. For example, your foot is the same length as your forearm (wrist to elbow, on the interior of the arm).There are also facial proportions. All of these proportions are what artists use to draw the human body and what da Vinci used to draw his Vitruvian Man, http://en.wikipedia.org/wiki/Vitruvian_Man .

Review Questions

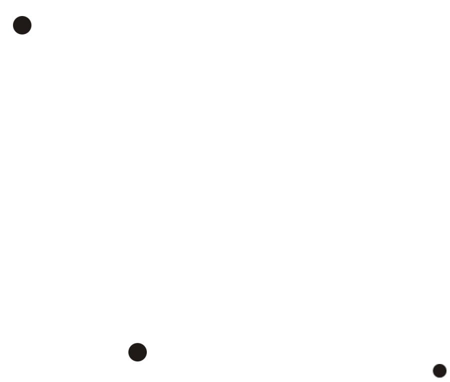
Find the length of each line segment in inches. Round to the nearest $\frac{1}{8}$ of an inch.

1.
2.



For ind the distance between each pair of points in centimeters. Round to the nearest tenth.

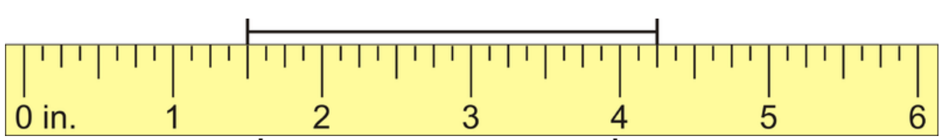
3.



4.

For 5-8, use the ruler in each picture to determine the length of the line segment.

5.



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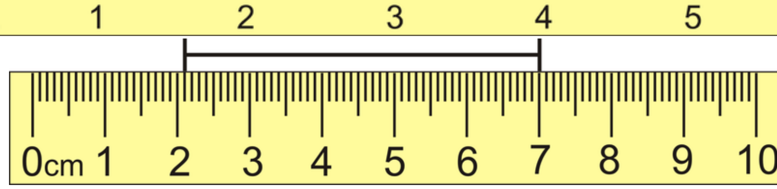
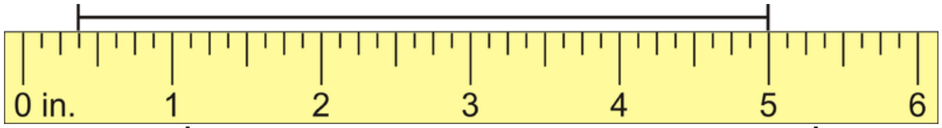
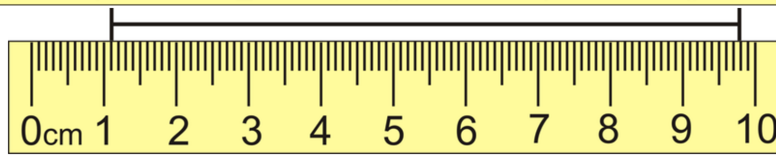


7. 

8. 

9. Make a sketch of $\overline{BT}$, with A between B and T.
10. If O is in the middle of $\overline{LT}$, where exactly is it located? If $LT = 16\ cm$, what is LO and OT?
11. For three collinear points, A between T and Q.

    a. Draw a sketch.
    b. Write the Segment Addition Postulate.
    c. If $AT = 10\ in$ and $AQ = 5\ in$, what is TQ?

12. For three collinear points, M between H and A.

    a. Draw a sketch.
    b. Write the Segment Addition Postulate.
    c. If $HM = 18\ cm$ and $HA = 29\ cm$, what is AM?

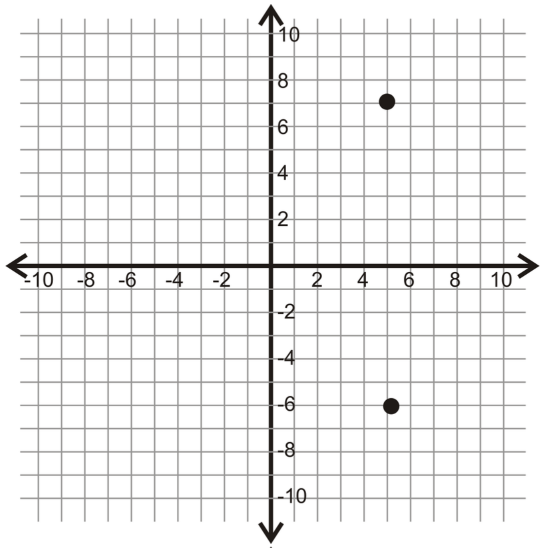
13. Make a sketch that matches the description: B is between A and D. C is between B and D. $AB = 7\ cm$, $AC = 15\ cm$, and $AD = 32\ cm$. Find BC, BD, and CD.
14. Make a sketch that matches the description: E is between F and G. H is between F and E. $FH = 4\ in$, $EG = 9\ in$, and $FH = HE$. Find FE, HG, and FG.
15. Make a sketch that matches the description: S is between T and V. R is between S and T. T is between R and Q. $QV = 18$, $QT = 6$, and $TR = RS = SV$.

    a. Find RS.
    b. Find QS.
    c. Find TS.
    d. Find TV.

For 16-20, Suppose J is between H and K. Use the Segment Addition Postulate to solve for x. Then find the length of each segment.

16. $HJ = 4x + 9,\ JK = 3x + 3,\ KH = 33$
17. $HJ = 5x - 3,\ JK = 8x - 9,\ KH = 131$
18. $HJ = 2x + \frac{1}{3},\ JK = 5x + \frac{2}{3},\ KH = 12x - 4$
19. $HJ = x + 10,\ JK = 9x,\ KH = 14x - 58$
20. $HJ = \frac{3}{4}x - 5,\ JK = x - 1,\ KH = 22$
21. Draw four points, A, B, C, and D such that $AB = BC = AC = AD = BD$ (HINT: A, B, C and D should NOT be collinear)

For 22-25, determine the vertical or horizontal distance between the two points.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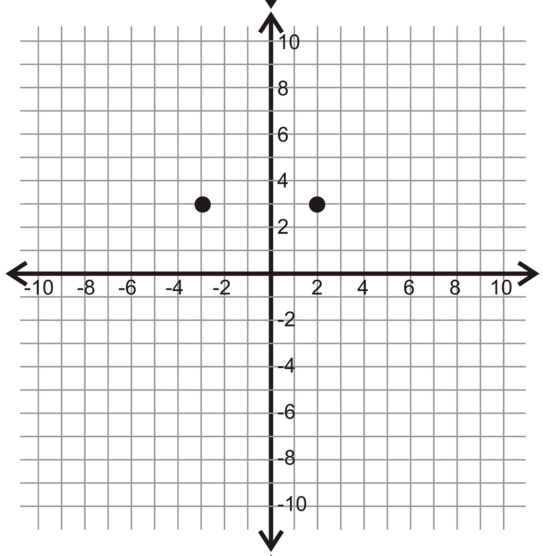
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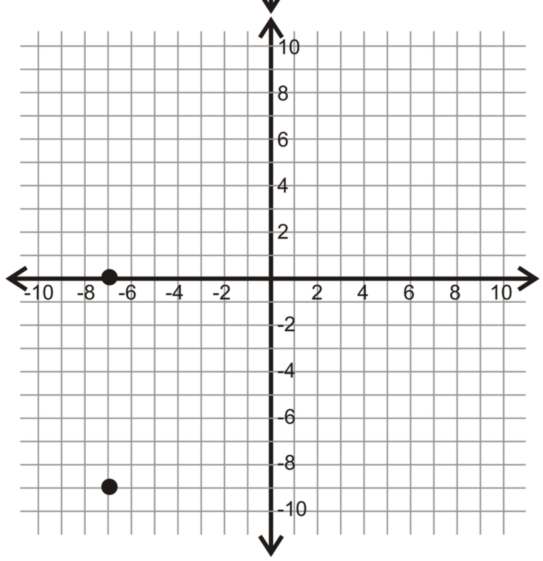
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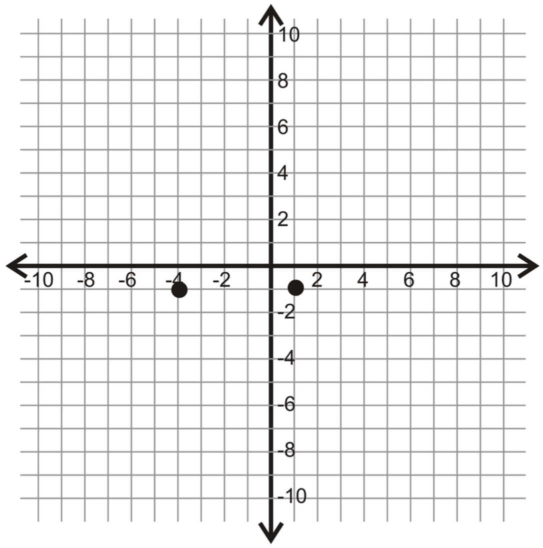
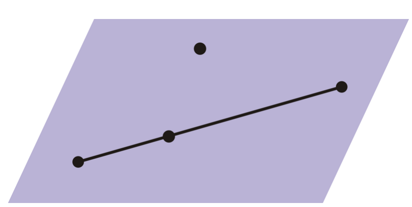
25. 

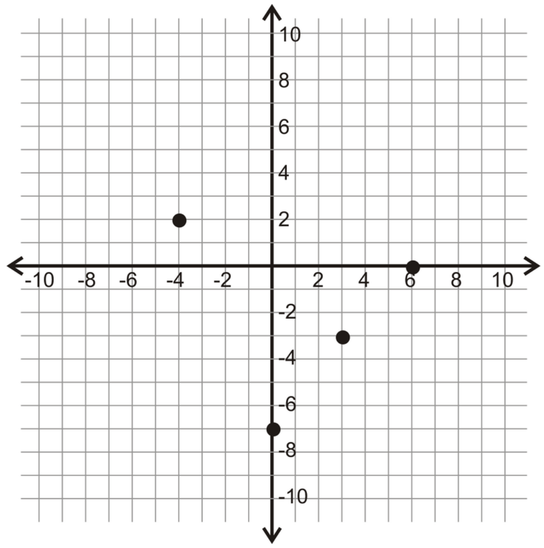
Each of the following problems presents an opportunity for students to extend their knowledge of measurement to the real world. Each of these concepts could be further developed into a mini-project.

26. Measure the length of your head and create a "ruler" of this length out of cardstock or cardboard. Use your ruler to measure your height. Share your height in terms of your head length with your class and compare results.
27. Describe the advantages of using the metric system to measure length over the English system. Use the examples of the two rulers (one in inches and one in centimeters) to aid in your description.
28. A speedometer in a car measures distance traveled by tracking the number of rotations on the wheels on the car. A pedometer is a device that a person can wear that tracks the number of steps a person takes and calculates the distance traveled based on the person's stride length. Which would produce a more accurate measure of distance? Why? What could you do to make the less accurate measure more precise?
29. Research the origins of ancient measurement units such as the cubit. Research the origins of the units of measure we use today such as: foot, inch, mile, meter. Why are standard units important?
30. Research the facial proportions that da Vinci used to create his Vitruvian man. Write a summary of your findings.

Review Queue Answers

a. line $l, \overline{MN}, \overline{NM}$

b. 

c. 

d. $m = \frac{3-(-2)}{-4-6} = \frac{5}{-10} = -\frac{1}{2}$
$y = -\frac{1}{2}x + b$
$-2 = -\frac{1}{2}(6) + b$
$-2 = -3 + b$
$1 = b$

So, the equation is $y = -\frac{1}{2}x + 1$

1.3 Angles and Measurement

Learning Objectives

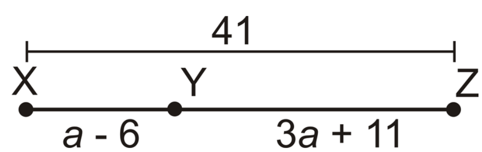
- Define and classify angles.
- Apply the Protractor Postulate and the Angle Addition Postulate.

Review Queue

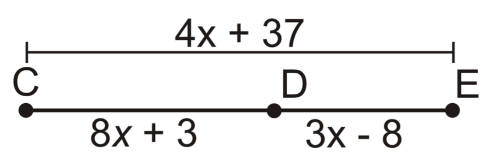
Answer the following questions.

a. Label the following geometric figure. What is it called?

b. Find a, XY and YZ.



c. Find x, CD and DE.

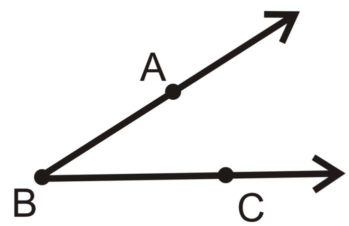


d. B is between A and C on $\overline{AC}$. If $AB = 4$ and $BC = 9$, what is AC? What postulate do you use to solve this problem?

Know What? Back to the building blocks. Every block has its own dimensions, angles and measurements. Using a protractor, find the measure of the three outlined angles in the "castle" to the right. Also, determine which other angles are equal to these measurements. Use appropriate angle markings. Do not measure any arcs.

Two Rays = One Angle

In #1 above, the figure was a ray. It is labeled $\overrightarrow{AB}$, with the arrow over the point that is NOT the endpoint. When two rays have the same endpoint, an angle is created.



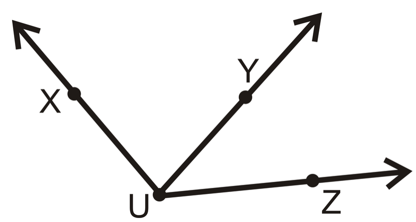
Here, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ meet to form an angle. An angle is labeled with an "$\angle$" symbol in front of the three letters used to label it. This angle can be labeled $\angle ABC$ or $\angle CBA$. **Always put the vertex in the middle of the three points.** It doesn't matter which side point is written first.

Angle: When two rays have the same endpoint.

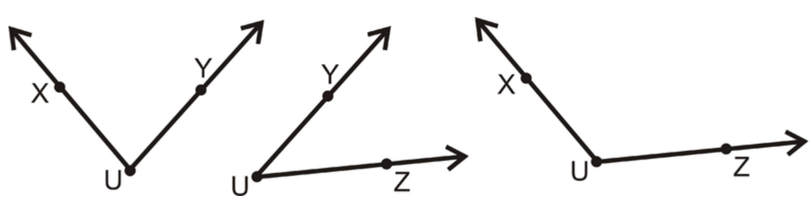
Vertex: The common endpoint of the two rays that form an angle.

Sides: The two rays that form an angle.

Example 1: How many angles are in the picture below? Label each one two different ways.



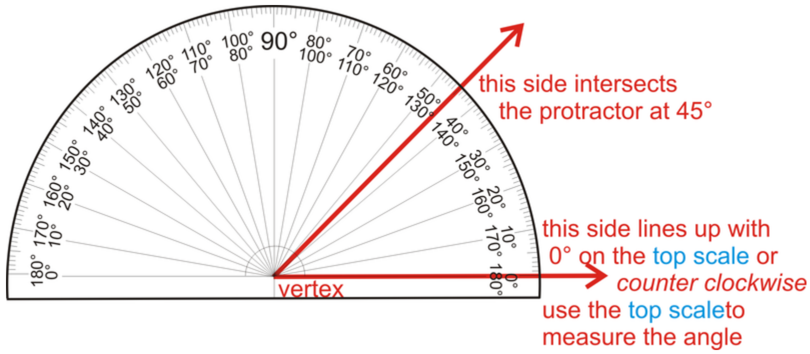
Solution: There are three angles with vertex U. It might be easier to see them all if we separate them out.



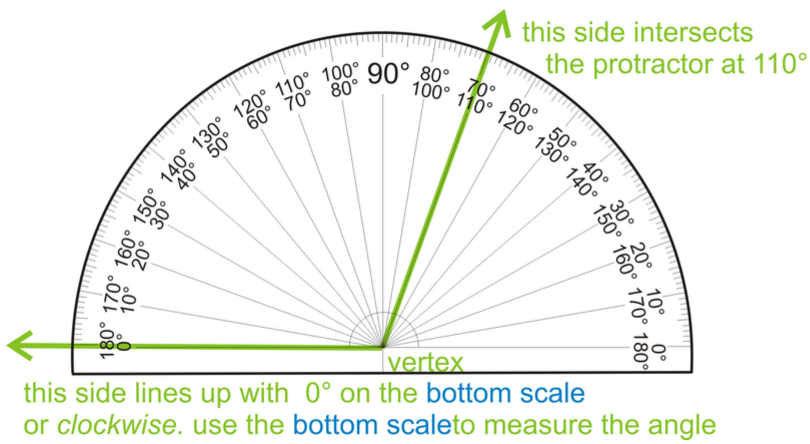
So, the three angles can be labeled, $\angle XUY$ or $\angle YUX$, $\angle YUZ$ or $\angle ZUY$, and $\angle XUZ$ or $\angle ZUX$.

Protractor Postulate

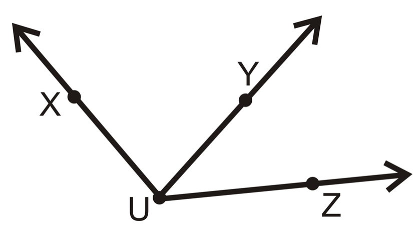
We measure a line segment's *length* with a ruler. Angles are measured with something called a ***protractor***. A protractor is a measuring device that measures how "open" an angle is. Angles are measured in degrees, and labeled with a $^\circ$ symbol.



Notice that there are two sets of measurements, one opening clockwise and one opening counter-clockwise, from 0° to 180°. When measuring angles, always line up one side with 0°, and see where the other side hits the protractor. The vertex lines up in the middle of the bottom line, where all the degree lines meet.



Example 2: Measure the three angles from Example 1, using a protractor.



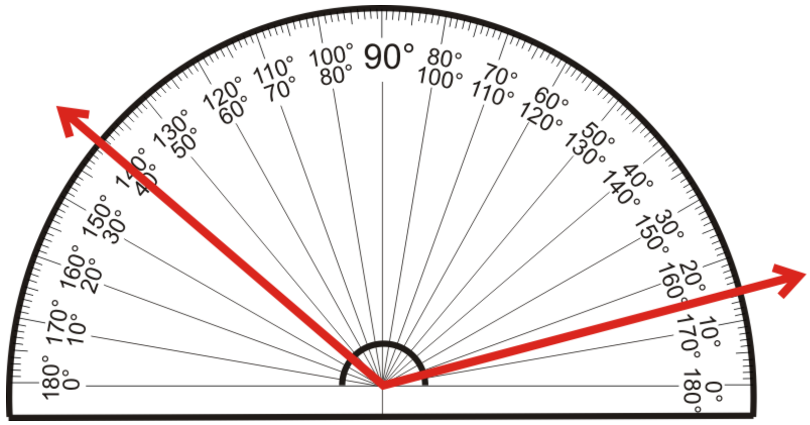
Solution: Just like in Example 1, it might be easier to measure these three angles if you separate them. With measurement, we put an m in front of the $\angle$ sign to indicate measure. So, $m\angle XUY = 84^\circ$, $m\angle YUZ = 42^\circ$ and $m\angle XUZ = 126^\circ$.

In the last lesson, we introduced the Ruler Postulate. Here we introduce the Protractor Postulate.

Protractor Postulate: For every angle there is a number between 0° and 180° that is the measure of the angle in degrees. The angle's measure is then the absolute value of the difference of the numbers shown on the protractor where the sides of the angle intersect the protractor.

In other words, you do not have to start measuring an angle at 0°, as long as you subtract one measurement from the other.

Example 3: What is the measure of the angle shown below?

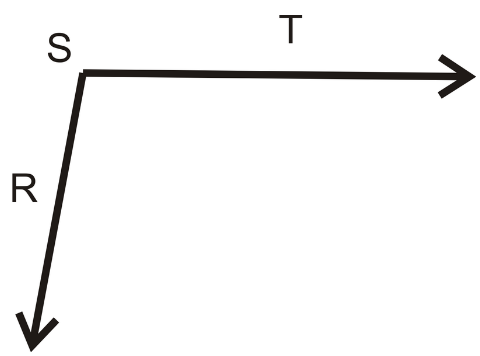


Solution: This angle is not lined up with 0°, so use subtraction to find its measure. It does not matter which scale you use.

Using the inner scale, $|140-25| = 125^\circ$

Using the outer scale, $|165-40| = 125^\circ$

Example 4: Use a protractor to measure $\angle RST$ below.

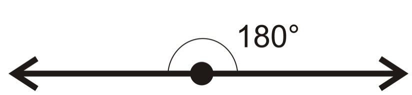


Solution: The easiest way to measure any angle is to line one side up with 0°. This angle measures 100°.

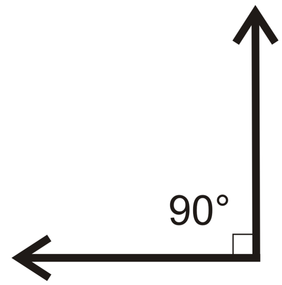
Classifying Angles

By looking at the protractor we measure angles from 0° to 180°. Angles can be classified, or grouped, into four different categories.

Straight Angle: When an angle measures 180°. The angle measure of a straight line. The rays that form this angle are called opposite rays.

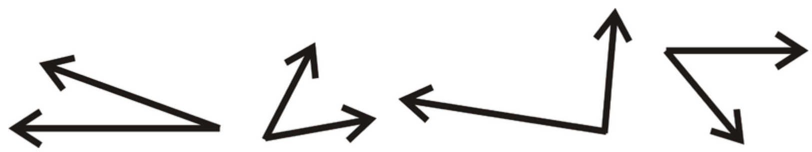


Right Angle: When an angle measures 90°.

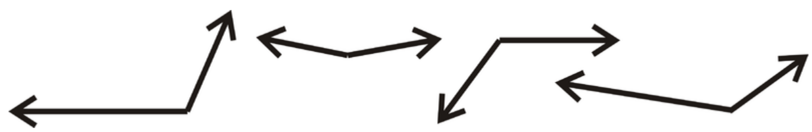


Notice the half-square, marking the angle. This marking is always used to mark right, or 90°, angles.

Acute Angles: Angles that measure between 0° and 90°.



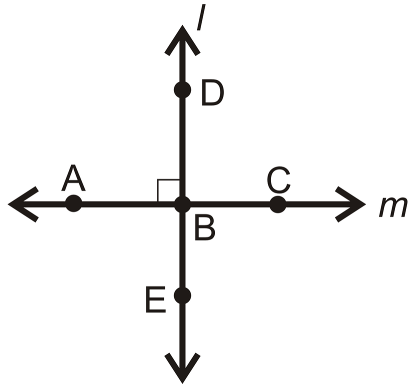
Obtuse Angles: Angles that measure between 90° and 180°.



It is important to note that 90° is NOT an acute angle and 180° is NOT an obtuse angle.

Additionally, any two lines or line segments can intersect to form four angles. If the two lines intersect to form right angles, we say the lines are perpendicular.

Perpendicular: When two lines intersect to form four right angles.

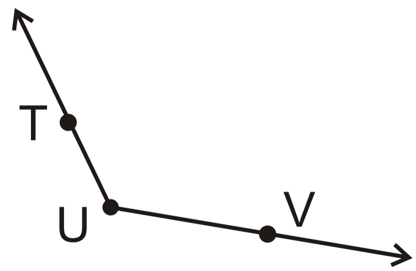


Even though all four angles are 90°, only one needs to be marked. It can be assumed thatall four are 90°.

The symbol for perpendicular is $\perp$, so these two lines would be labeled $l \perp m$ or $\overleftrightarrow{AC} \perp \overleftrightarrow{DE}$.

There are several other ways to label these two intersecting lines. This picture shows **two perpendicular lines, four right angles, four** 90° **angles**, and even **two straight angles,** $\angle ABC$ and $\angle DBE$.

Example 5: Name the angle and determine what type of angle it is.



Solution: The vertex is U. So, the angle can be $\angle TUV$ or $\angle VUT$. To determine what type of angle it is, compare it to a right angle. Because it opens wider than a right angle, and less than a straight angle it is **obtuse**.

Example 6: What type of angle is 84°? What about 165°?

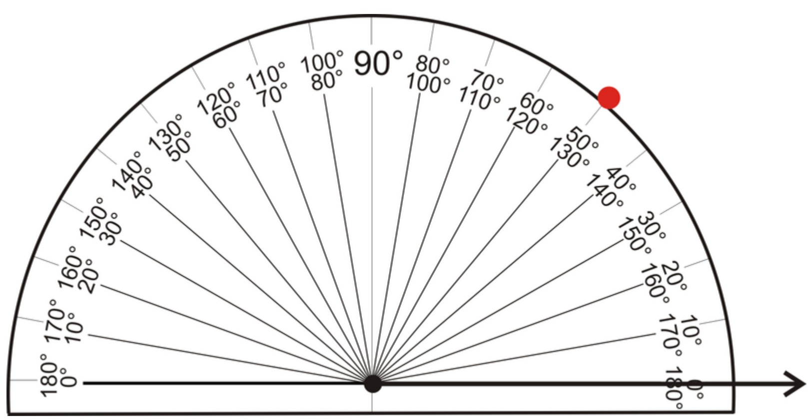
Solution: 84° is less than 90°, so it is **acute**. 165° is greater than 90°, but less than 180°, so it is **obtuse**.

Drawing an Angle

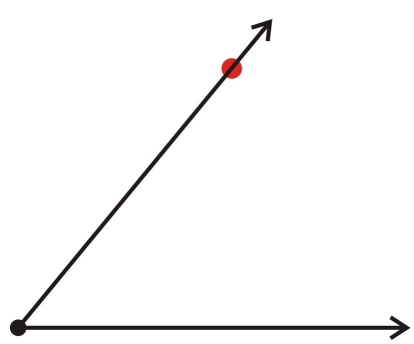
Investigation 1-1: Drawing a 50° Angle with a Protractor

a. Start by drawing a horizontal line across the page, about 2 in long.
b. Place an endpoint at the left side of your line.

c. Place the protractor on this point. Make sure to put the center point on the bottom line of the protractor on the vertex. Mark 50° on the appropriate scale.



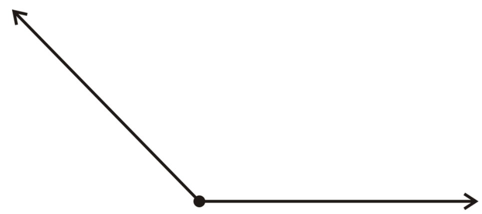
d. Remove the protractor and connect the vertex and the 50° mark.



This process can be used to draw any angle between 0° and 180°. See http://www.mathsisfun.com/geometry/protractor-using.html for an **animation** of this investigation.

Example 7: Draw a 135° angle.

Solution: Following the steps from above, your angle should look like this:

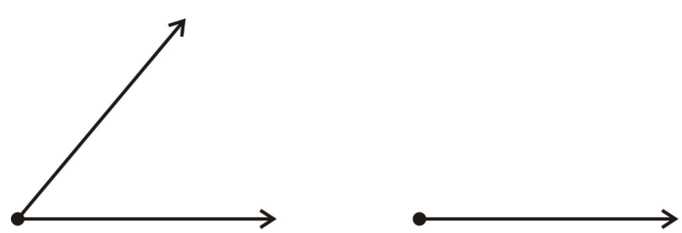


Now that we know how to draw an angle, we can also copy that angle with a compass and a straightedge, usually a ruler. Anytime we use a compass and ruler to draw different geometric figures, it called a **construction**.

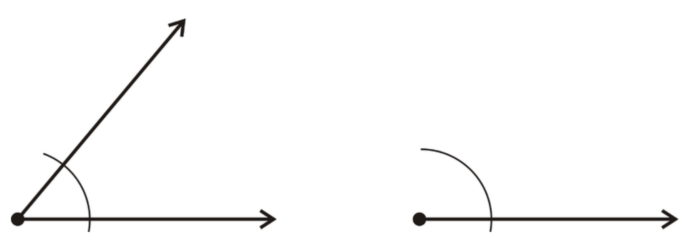
Compass: A tool used to draw circles and arcs.

Investigation 1-2: Copying an Angle with a Compass and Straightedge

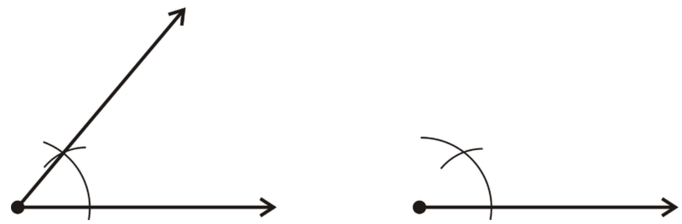
a. We are going to copy the angle created in the previous investigation, a 50° angle. First, draw a straight line, about 2 inches long, and place an endpoint at one end.



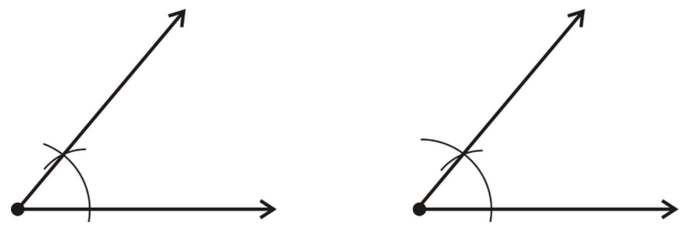
b. With the point (non-pencil side) of the compass on the vertex, draw an arc that passes through both sides of the angle. Repeat this arc with the line we drew in #1.



c. Move the point of the compass to the horizontal side of the angle we are copying. Place the point where the arc intersects this side. Open (or close) the "mouth" of the compass so you can draw an arc that intersects the other side of the arc drawn in #2. Repeat this on the line we drew in #1.



d. Draw a line from the new vertex to the arc intersections.

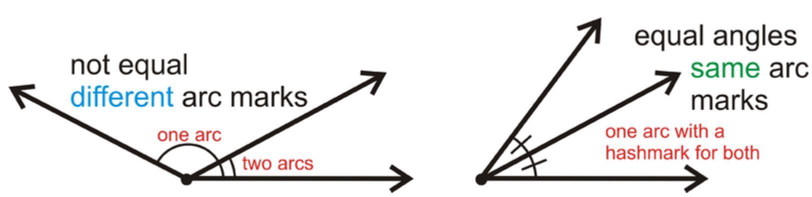


To watch an **animation** of this construction, see http://www.mathsisfun.com/geometry/construct-anglesame.html

Marking Angles and Segments in a Diagram

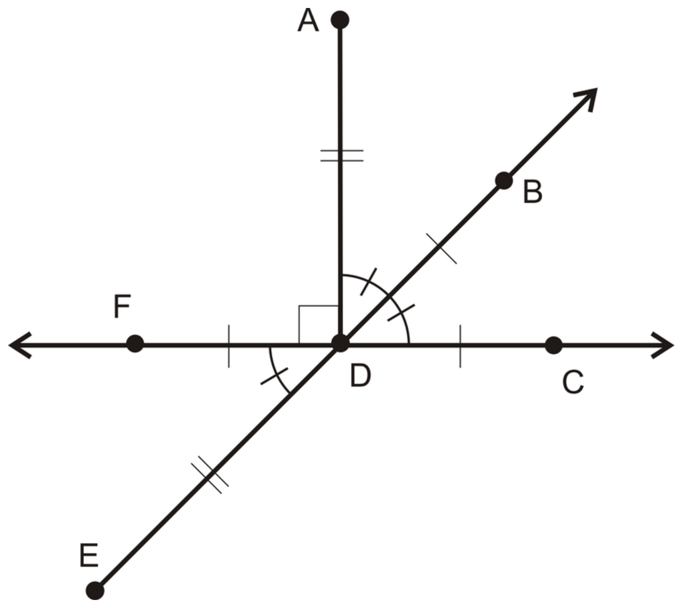
With all these segments and angles, we need to have different ways to label equal angles and segments.

Angle Markings



Segment Markings

Example 8: Interpret the picture below. Write all equal angle and segment statements.



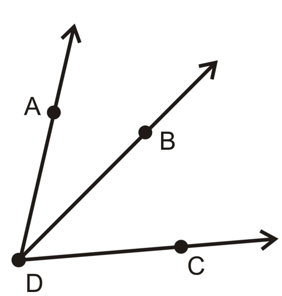
Solution:

$$\overline{AD} \perp \overleftrightarrow{FC}$$
$$m\angle ADB = m\angle BDC = m\angle FDE = 45^\circ$$
$$AD = DE$$
$$FD = DB = DC$$
$$m\angle ADF = m\angle ADC = 90^\circ$$

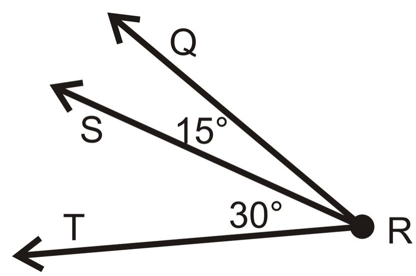
Angle Addition Postulate

Much like the Segment Addition Postulate, there is an Angle Addition Postulate.

Angle Addition Postulate: If B is on the interior of $\angle ADC$, then $m\angle ADC = m\angle ADB + m\angle BDC$. See the picture below.

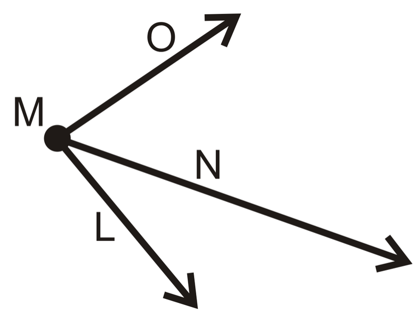


Example 9: What is $m\angle QRT$ in the diagram below?



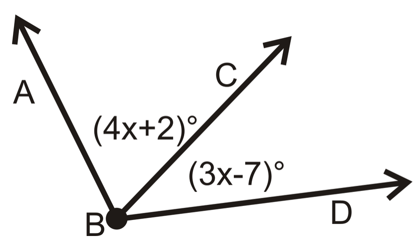
Solution: Using the Angle Addition Postulate, $m\angle QRT = 15^\circ + 30^\circ = 45^\circ$.

Example 10: What is $m\angle LMN$ if $m\angle LMO = 85^\circ$ and $m\angle NMO = 53^\circ$?



Solution: From the Angle Addition Postulate, $m\angle LMO = m\angle NMO + m\angle LMN$. Substituting in what we know, $85^\circ = 53^\circ + m\angle LMN$, so $85^\circ - 53^\circ = m\angle LMN$ or $m\angle LMN = 32^\circ$.

Example 11: ***Algebra Connection*** If $m\angle ABD = 100^\circ$, find x and $m\angle ABC$ and $m\angle CBD$?



Solution: From the Angle Addition Postulate, $m\angle ABD = m\angle ABC + m\angle CBD$. Substitute in what you know and solve the equation.

$$\begin{aligned}
100^\circ &= (4x+2)^\circ + (3x-7)^\circ \\
100^\circ &= 7x - 5^\circ \\
105^\circ &= 7x \\
15^\circ &= x
\end{aligned}$$

So, $m\angle ABC = 4(15^\circ) + 2^\circ = 62^\circ$ and $m\angle CBD = 3(15^\circ) - 7^\circ = 38^\circ$.

Know What? Revisited Using a protractor, the measurement marked in the red triangle is 90°, the measurement in the blue triangle is 45° and the measurement in the orange square is 90°.

All of the equal angles are marked in the picture to the right. All of the acute angles in the triangles are equal and all the other angles are right, or 90°.

Review Questions

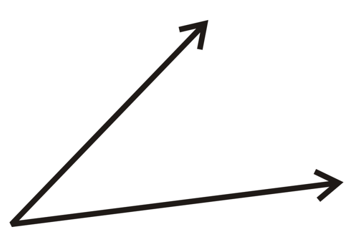
For questions 1-10, determine if the statement is true or false. If you answered FALSE for any question, state why.

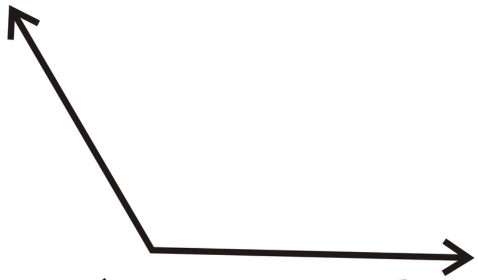
1. Two angles always add up to be greater than 90°.
2. 180° is an obtuse angle.
3. 180° is a straight angle.
4. Two perpendicular lines intersect to form four right angles.
5. A construction uses a protractor and a ruler.
6. For an angle $\angle ABC, C$ is the vertex.
7. For an angle $\angle ABC, \overline{AB}$ and $\overline{BC}$ are the sides.
8. The m in front of $m\angle ABC$ means measure.
9. Angles are always measured in degrees.
10. The Angle Addition Postulate says that an angle is equal to the sum of the smaller angles around it.

For 11-16, draw the angle with the given degree, using a protractor and a ruler. Also, state what type of angle it is.

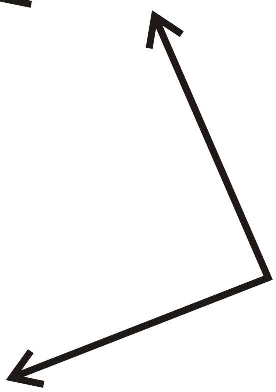
11. 55°
12. 92°
13. 178°
14. 5°
15. 120°
16. 73°
17. ***Construction*** Copy the angle you made from #12, using a compass and a straightedge.
18. ***Construction*** Copy the angle you made from #16, using a compass and a straightedge.

For 19-22, use a protractor to determine the measure of each angle.

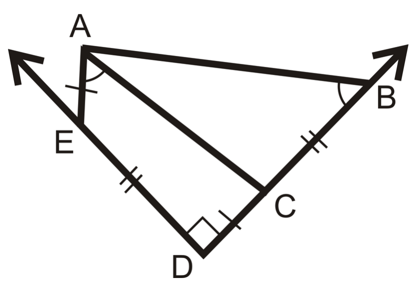
19. 

20. 

21. 

22. 

23. Interpret the picture to the right. Write down all equal angles, segments and if any lines are perpendicular.

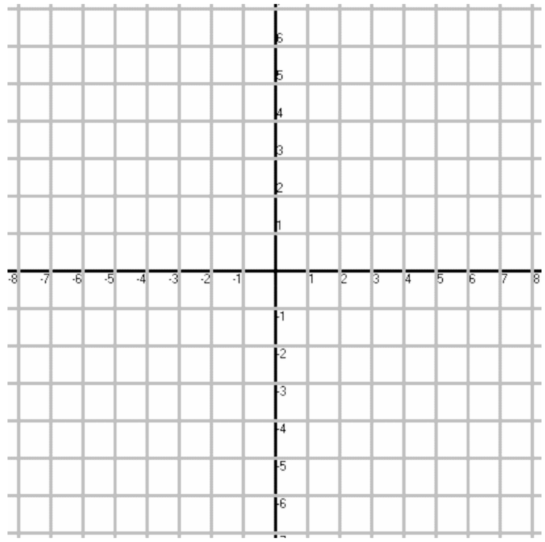


24. Draw a picture with the following requirements.

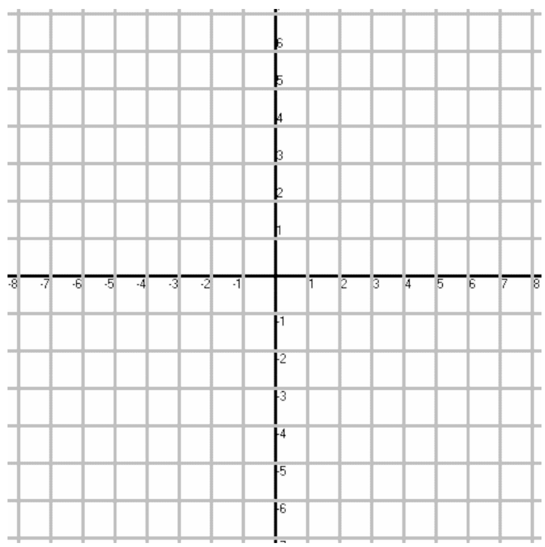
$amp; AB = BC = BD$	$m\angle ABD = 90^\circ$
$amp; m\angle ABC = m\angle CBD$	A, B, C and D are coplanar

In 25 and 26, plot and sketch $\angle ABC$. Classify the angle. Write the coordinates of a point that lies in the interior of the angle.

25. $A(5, -3)$
$B(-3, -1)$
$C(2, 2)$



26. $A(-3,0)$
    $B(1,3)$
    $C(5,0)$

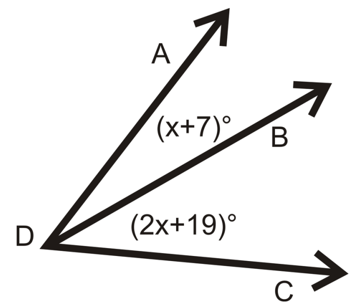


In Exercises 27-31, use the following information: Q is in the interior of $\angle ROS$. S is in the interior of $\angle QOP$. P is in the interior of $\angle SOT$. S is in the interior of $\angle ROT$ and $m\angle ROT = 160^\circ$, $m\angle SOT = 100^\circ$, and $m\angle ROQ = m\angle QOS = m\angle POT$.

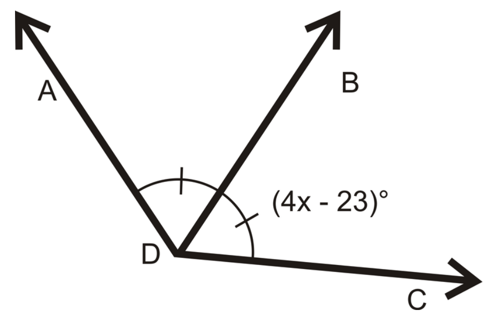
27. Make a sketch.
28. Find $m\angle QOP$
29. Find $m\angle QOT$
30. Find $m\angle ROQ$
31. Find $m\angle SOP$

Algebra Connection Solve for x.

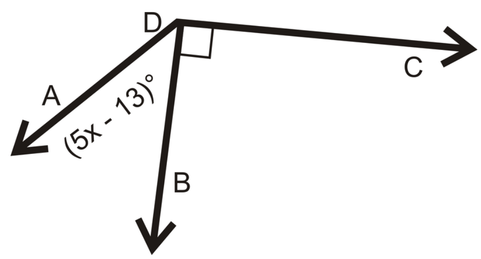
32. $m\angle ADC = 56^\circ$



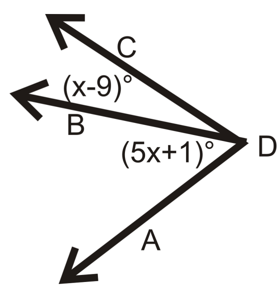
33. $m\angle ADC = 130^\circ$



34. $m\angle ADC = (16x - 55)^\circ$



35. $m\angle ADC = (9x - 80)^\circ$



36. ***Writing*** Write a paragraph about why the degree measure of a straight line is 180, the degree measure of a right angle is 90, etc. In other words, answer the question, "Why is the straight line divided into exactly 180 degrees and not some other number of degrees?"

Review Queue Answers

1. $\overrightarrow{AB}$, a ray

2. $XY = 3,\ YZ = 38$

$$\begin{aligned} a - 6 + 3a + 11 &= 41 \\ 4a + 5 &= 41 \end{aligned}$$

$$\begin{aligned} 4a &= 36 \\ a &= 9 \end{aligned}$$

3. $CD = 51,\ DE = 10$

$$\begin{aligned} 8x+3+3x-8 &= 4x+37 \\ 11x-5 &= 4x+37 \\ 7x &= 42 \\ x &= 6 \end{aligned}$$

4. Use the Segment Addition Postulate, $AC = 13$.

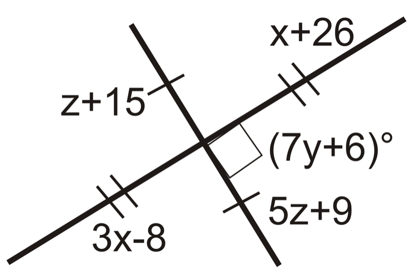
1.4 Angle Pairs

Learning Objectives

- Recognize complementary angles, supplementary angles, linear pairs and vertical angles.
- Apply the Linear Pair Postulate and the Vertical Angles Theorem.

Review Queue

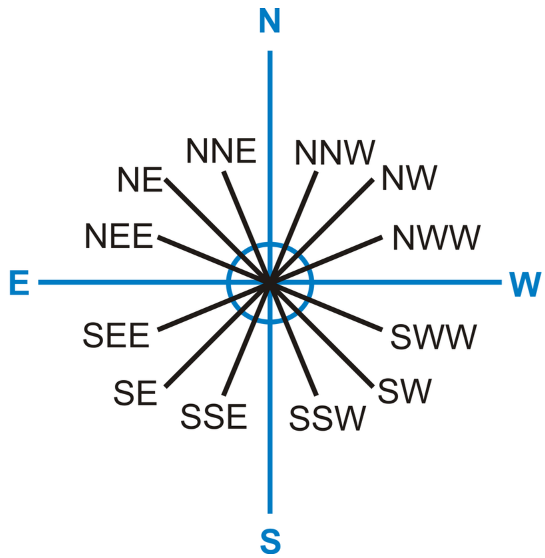
Use the picture below to answer questions 1-3.



a. Find x.
b. Find y.
c. Find z.

Know What? A compass (as seen to the right) is used to determine the direction a person is traveling in. The angles between each direction are very important because they enable someone to be more specific and precise with their direction. In boating, captains use headings to determine which direction they are headed. A heading is the angle at which these compass lines intersect. So, a heading of $45^\circ NW$, would be straight out along that northwest line.

What headings have the same angle measure? What is the angle measure between each compass line?

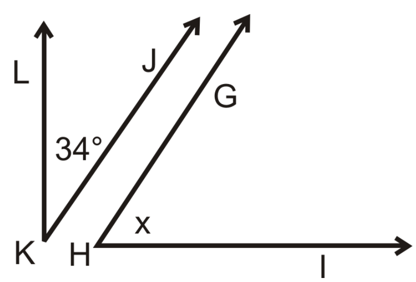


Complementary Angles

Complementary: When two angles add up to 90°.

Complementary angles do not have to be congruent to each other, nor do they have to be next to each other.

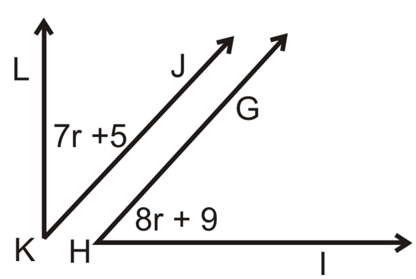
Example 1: The two angles below are complementary. $m\angle GHI = x$. What is x?



Solution: Because the two angles are complementary, they add up to 90°. Make an equation.

$$\begin{aligned} x + 34^\circ &= 90^\circ \\ x &= 56^\circ \end{aligned}$$

Example 2: The two angles below are complementary. Find the measure of each angle.



Solution: Again, the two angles add up to 90°. Make an equation.

$$\begin{aligned} 8r + 9^\circ + 7r + 5^\circ &= 90^\circ \\ 15r + 14^\circ &= 90^\circ \\ 15r &= 74^\circ \\ r &= 4.93^\circ \end{aligned}$$

However, this is not what the question asks for. You need to plug r back into each expression to find each angle.

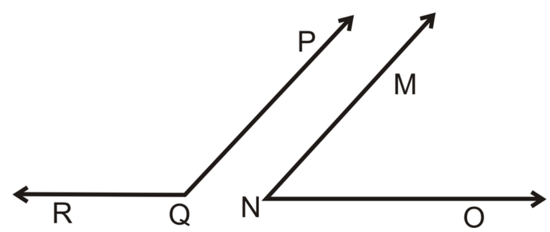
$$\begin{aligned} m\angle GHI &= 8(5^\circ) + 9^\circ = 49^\circ \\ m\angle JKL &= 7(5^\circ) + 6^\circ = 41^\circ \end{aligned}$$

Supplementary Angles

Supplementary: When two angles add up to 180°.

Just like complementary angles, supplementary angles do not have to be congruent or touching.

Example 3: The two angles below are supplementary. If $m\angle MNO = 78^\circ$ what is $m\angle PQR$?



Solution: Just like Examples 1 and 2, set up an equation. However, instead of equaling 90°, now it is 180°.

$$\begin{aligned} 78^\circ + m\angle PQR &= 180^\circ \\ m\angle PQR &= 102^\circ \end{aligned}$$

Example 4: What is the measure of two congruent, supplementary angles?

Solution: Supplementary angles add up to 180°. Congruent angles have the same measure. Divide 180° by 2, to find the measure of each angle.

$$180^\circ \div 2 = 90^\circ$$

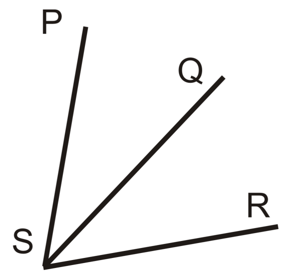
So, two congruent, supplementary angles are right angles, or 90°.

Linear Pairs

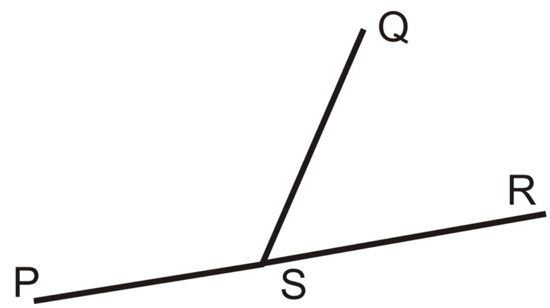
Adjacent Angles: Two angles that have the same vertex, share a side, and do not overlap.

$\angle PSQ$ and $\angle QSR$ are adjacent.

$\angle PQR$ and $\angle PQS$ are NOT adjacent because they overlap.



Linear Pair: Two angles that are adjacent and whose non-common sides form a straight line.

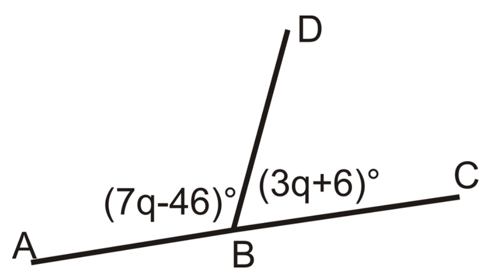


$\angle PSQ$ and $\angle QSR$ are a linear pair.

$$\begin{aligned} m\angle PSR &= 180^\circ \\ m\angle PSQ + m\angle QSR &= m\angle PSR \\ m\angle PSQ + m\angle QSR &= 180^\circ \end{aligned}$$

Linear Pair Postulate: If two angles are a linear pair, then they are supplementary.

Example 5: ***Algebra Connection*** What is the value of each angle?



Solution: These two angles are a linear pair, so they are supplementary, or add up to 180°. Write an equation.

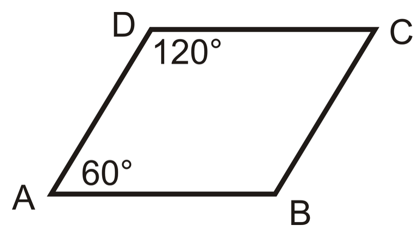
$$\begin{aligned} (7q-46)^\circ + (3q+6)^\circ &= 180^\circ \\ 10q - 40^\circ &= 180^\circ \\ 10q &= 220^\circ \\ q &= 22^\circ \end{aligned}$$

So, plug in q to get the measure of each angle.

$$m\angle ABD = 7(22^\circ) - 46^\circ = 108^\circ \quad m\angle DBC = 180^\circ - 108^\circ = 72^\circ$$

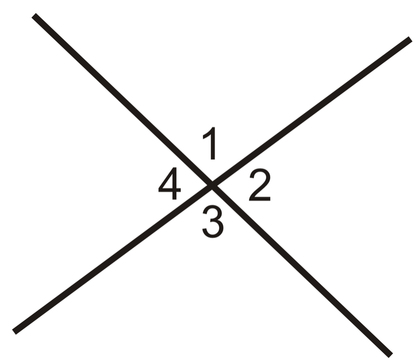
Example 6: Are $\angle CDA$ and $\angle DAB$ a linear pair? Are they supplementary?

Solution: The two angles are not a linear pair because they do not have the same vertex. However, they are supplementary, $120^\circ + 60^\circ = 180^\circ$.



Vertical Angles

Vertical Angles: Two non-adjacent angles formed by intersecting lines.



$\angle 1$ and $\angle 3$ are vertical angles

$\angle 2$ and $\angle 4$ are vertical angles

Notice that these angles are labeled with numbers. You can tell that these are labels because they do not have a degree symbol.

Investigation 1-5: Vertical Angle Relationships

a. Draw two intersecting lines on your paper. Label the four angles created $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$. See the picture above.
b. Take your protractor and find $m\angle 1$.
c. What is the angle relationship between $\angle 1$ and $\angle 2$? Find $m\angle 2$.
d. What is the angle relationship between $\angle 1$ and $\angle 4$? Find $m\angle 4$.
e. What is the angle relationship between $\angle 2$ and $\angle 3$? Find $m\angle 3$.
f. Are any angles congruent? If so, write down the congruence statement.

From this investigation, hopefully you found out that $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$. This is our first theorem. That means it must be proven true in order to use it.

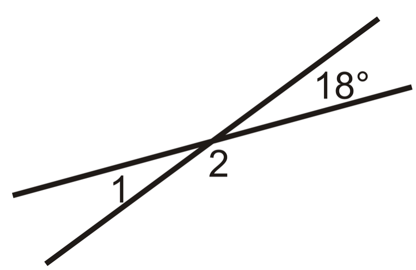
Vertical Angles Theorem: If two angles are vertical angles, then they are congruent.

We can prove the Vertical Angles Theorem using the same process we used above. However, let's not use any specific values for the angles.

From the picture above:	
$\angle 1$ and $\angle 2$ are a linear pair	$m\angle 1 + m\angle 2 = 180^\circ$
$\angle 2$ and $\angle 3$ are a linear pair	$m\angle 2 + m\angle 3 = 180^\circ$
$\angle 3$ and $\angle 4$ are a linear pair	$m\angle 3 + m\angle 4 = 180^\circ$
All of the equations $= 180^\circ$, so set the first and second equation equal to each other and the second and third.	$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$ *AND* $m\angle 2 + m\angle 3 = m\angle 3 + m\angle 4$
Cancel out the like terms	$m\angle 1 = m\angle 3,\ m\angle 2 = m\angle 4$

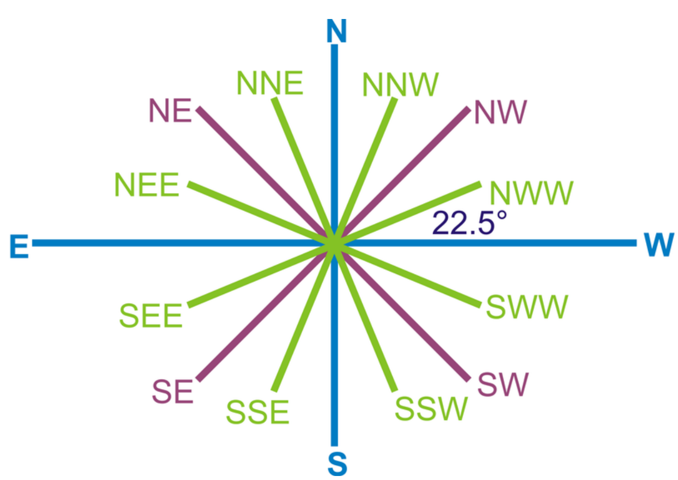
Recall that anytime the measures of two angles are equal, the angles are also congruent.

Example 7: Find $m\angle 1$ and $m\angle 2$.



Solution: $\angle 1$ is vertical angles with 18°, so $m\angle 1 = 18^\circ$. $\angle 2$ is a linear pair with $\angle 1$ or 18°, so $18^\circ + m\angle 2 = 180^\circ$. $m\angle 2 = 180^\circ - 18^\circ = 162^\circ$.

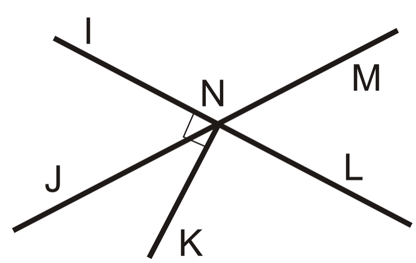
Know What? Revisited The compass has several vertical angles and all of the smaller angles are $22.5^\circ, 180^\circ \div 8$. Directions that are opposite each other, have the same angle measure, but of course, a different direction. All of the green directions have the same angle measure, 22.5°, and the purple have the same angle measure, 45°. N, S, E and W all have different measures, even though they are all 90° apart.



Review Questions

1. Find the measure of an angle that is complementary to $\angle ABC$ if $m\angle ABC$ is
   a. 45°
   b. 82°
   c. 19°
   d. z°
2. Find the measure of an angle that is supplementary to $\angle ABC$ if $m\angle ABC$ is
   a. 45°
   b. 118°
   c. 32°
   d. x°

Use the diagram below for exercises 3-7. Note that $\overline{NK} \perp \overleftrightarrow{IL}$.

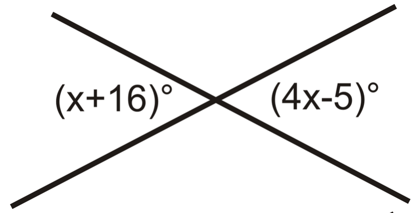


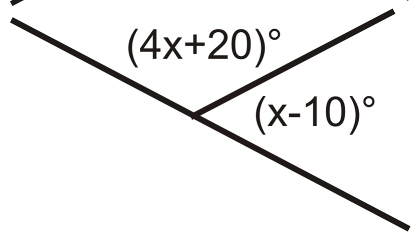
3. Name one pair of vertical angles.
4. Name one linear pair of angles.
5. Name two complementary angles.
6. Name two supplementary angles.
7. Given that $m\angle IJN = 63^\circ$, find:
   a. $m\angle JNL$
   b. $m\angle KNL$
   c. $m\angle MNL$
   d. $m\angle MNI$

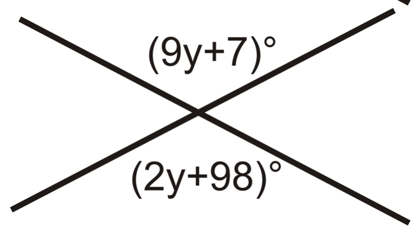
For 8-15, determine if the statement is ALWAYS true, SOMETIMES true or NEVER true.

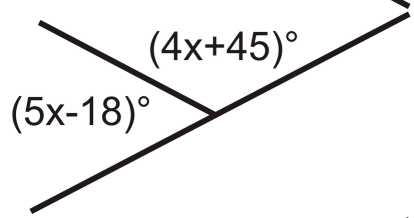
8. Vertical angles are congruent.
9. Linear pairs are congruent.
10. Complementary angles add up to 180°.
11. Supplementary angles add up to 180°
12. Adjacent angles share a vertex.
13. Adjacent angles overlap.
14. Complementary angles are 45°.
15. The complement of x° is $(90-x)^\circ$.

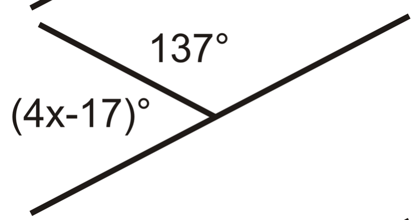
For 16-25, find the value of x or y.

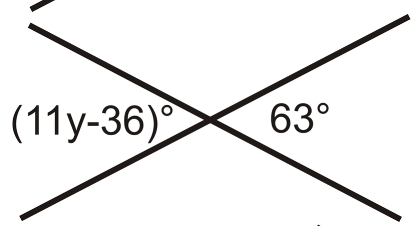
16. 

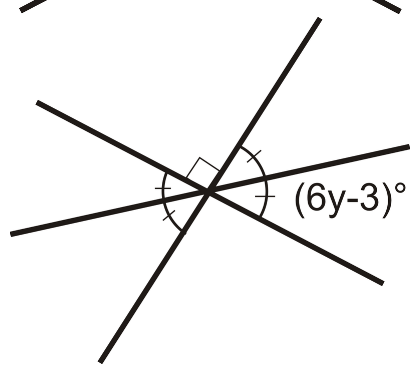
17. 

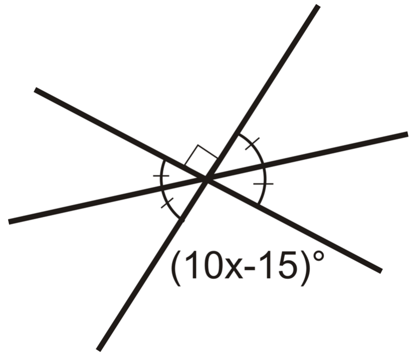
18. 

19. 

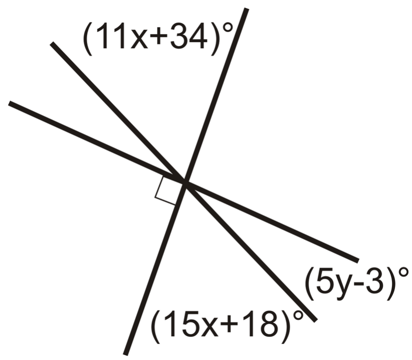
20. 

21. 

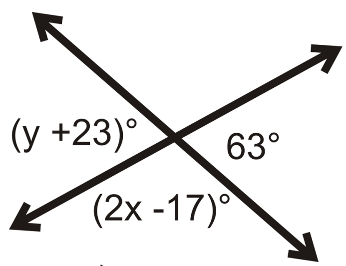
22. 

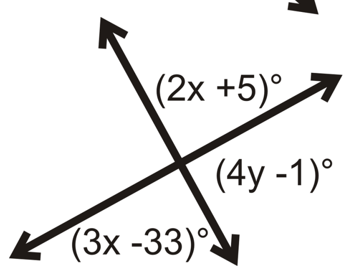
23. 

24. Find *x*.
25. Find *y*.

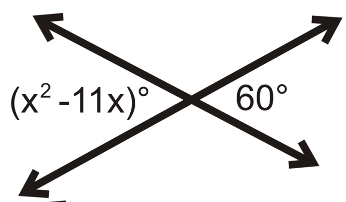


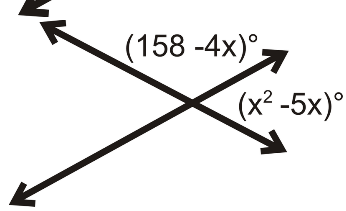
Find *x* and *y* in the following diagrams.

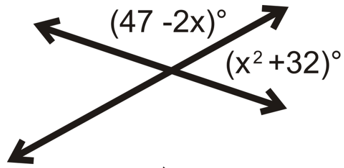
26. 

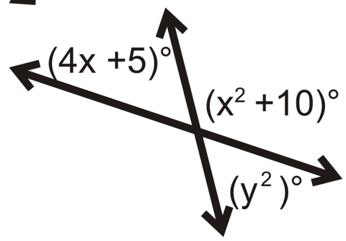
27. 

Algebra Connection. Use factoring or the quadratic formula to solve for the variables.

28. 

29. 

30. 

31. 

Review Queue Answers

a. $x+26=3x-8$
$34=2x$
$17=x$

b. $(7y+6)^\circ=90^\circ$
$7y=84^\circ$
$y=12^\circ$

c. $z+15=5z+9$
$6=4z$
$1.5=z$

1.5 Classifying Polygons

Learning Objectives

- Define triangle and polygon.
- Classify triangles by their sides and angles.
- Understand the difference between convex and concave polygons.
- Classify polygons by number of sides.

Review Queue

a. Draw a triangle.
b. Where have you seen 4, 5, 6 or 8-sided polygons in real life? List 3 examples.
c. Fill in the blank.
   a. Vertical angles are always ________________.
   b. Linear pairs are ___________________.
   c. The parts of an angle are called ______________ and a ______________.

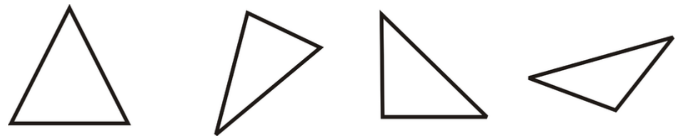
Know What? The pentagon in Washington DC is a pentagon with congruent sides and angles. There is a smaller pentagon inside of the building that houses an outdoor courtyard. Looking at the picture, the building is divided up into 10 smaller sections. What are the shapes of these sections? Are any of these division lines diagonals? How do you know?

Triangles

The first enclosed shape to examine is the triangle.

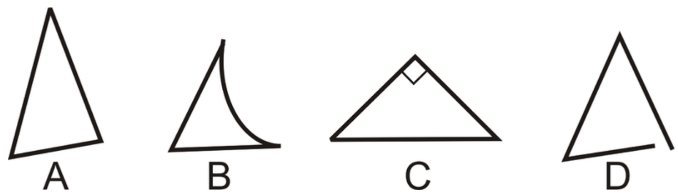
Triangle: Any closed figure made by three line segments intersecting at their endpoints.

Every triangle has three **vertices** (the points where the segments meet), three **sides** (the segments), and three **interior angles** (formed at each vertex). All of the following shapes are triangles.



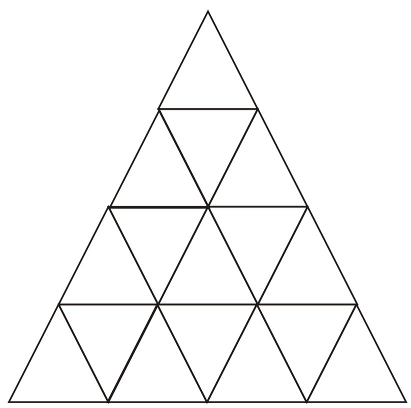
You might have also learned that the sum of the interior angles in a triangle is 180°. Later we will prove this, but for now you can use this fact to find missing angles.

Example 1: Which of the figures below are not triangles?

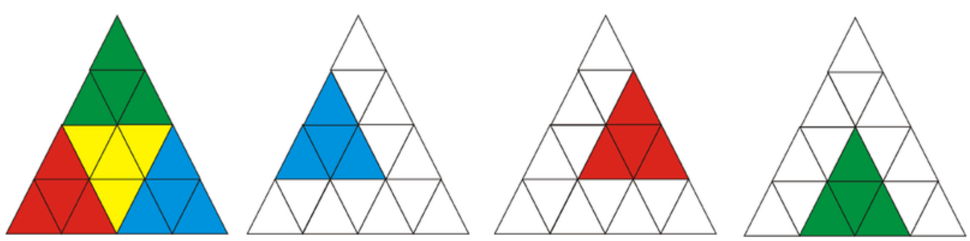


Solution: *B* is not a triangle because it hasone curved side. *D* is not a closed shape, so it is not a triangle either.

Example 2: How many triangles are in the diagram below?



Solution: Start by counting the smallest triangles, 16. Now count the triangles that are formed by four of the smaller triangles.



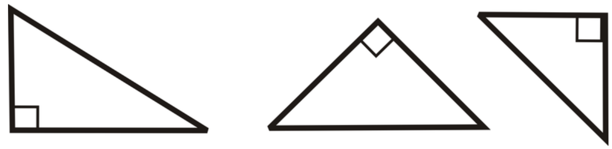
There are a total of seven triangles of this size, including the inverted one in the center of the diagram.Next, count the triangles that are formed by nine of the smaller triangles. There are three of this size. And finally, there is one triangle formed by the 16 smaller triangles. Adding these numbers together, we get $16+7+3+1=27$.

Classifying by Angles

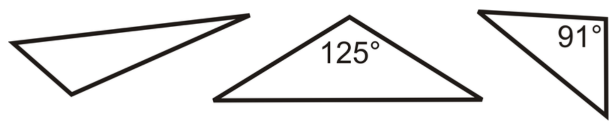
Angles can be classified by their angles;acute, obtuse or right. In any triangle, two of the angles will always be acute. The third angle can be acute, obtuse, or right.

We classify each triangle by this angle.

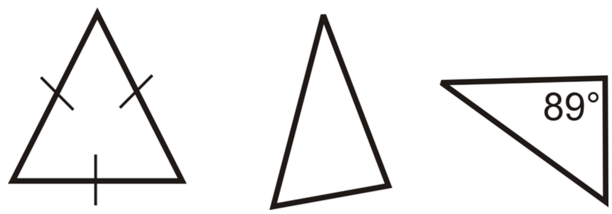
Right Triangle: When a triangle has one right angle.



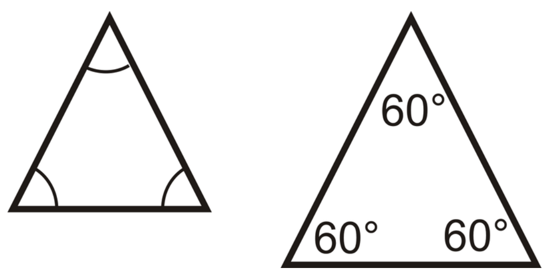
Obtuse Triangle: When a triangle has one obtuse angle.



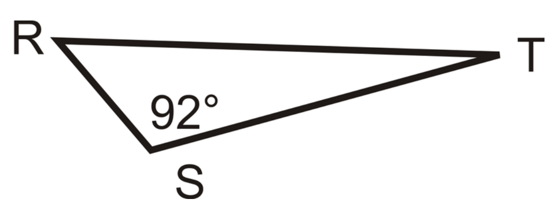
Acute Triangle: When all three angles in the triangle are acute.



Equiangular Triangle: When all the angles in a triangle are congruent.



Example 3: Which term best describes $\triangle RST$ below?

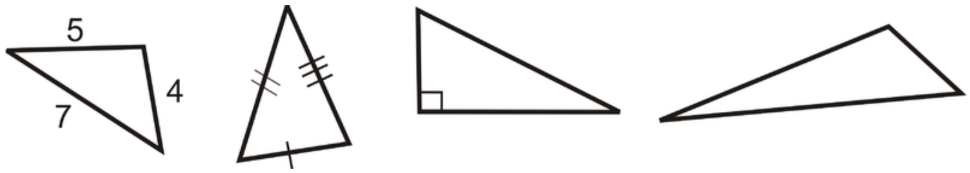


Solution: This triangle has one labeled obtuse angle of 92°. Triangles can only have one obtuse angle, so it is an obtuse triangle.

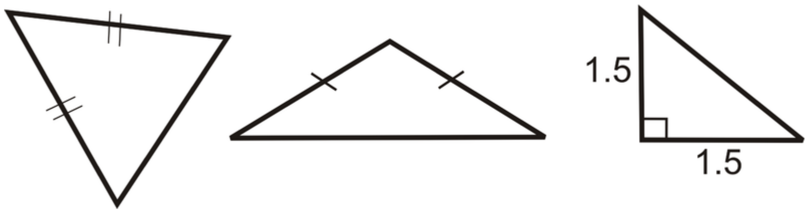
Classifying by Sides

These classifications have to do with the sides of the triangle and their relationships to each other.

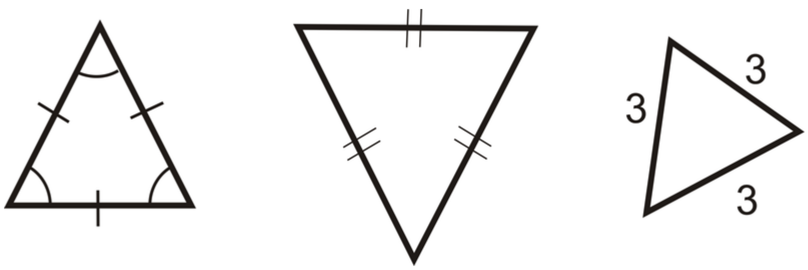
Scalene Triangle: When a triangles sides are all different lengths.



Isosceles Triangle: When *at least* two sides of a triangle are congruent.

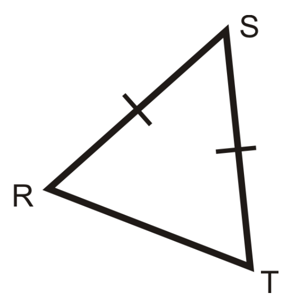


Equilateral Triangle: When all the sides of a triangle are congruent.



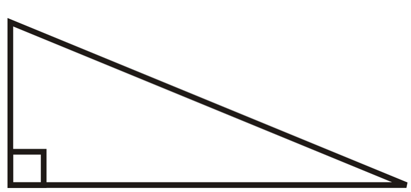
Note that by the definitions, an equilateral triangle is also an isosceles triangle.

Example 4: Classify the triangle by its sides and angles.



Solution: We are told there are two congruent sides, so it is an isosceles triangle. By its angles, they all look acute, so it is an acute triangle. Typically, we say this is an acute isosceles triangle.

Example 5: Classify the triangle by its sides and angles.



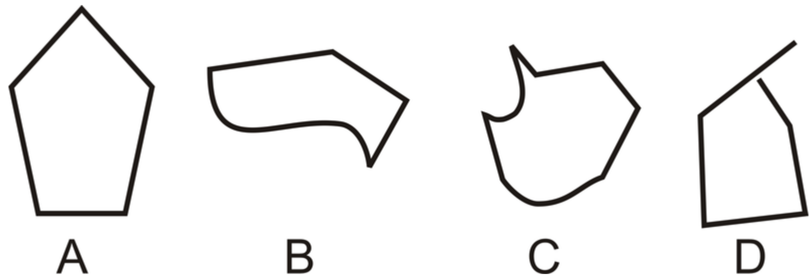
Solution: This triangle has a right angle and no sides are marked congruent. So, it is a right scalene triangle.

Polygons

Polygon: Any closed planar figure that is made entirely of line segments that intersect at their endpoints.

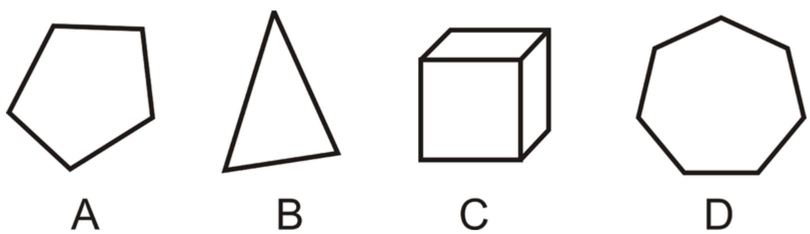
Polygons can have any number of sides and angles, but the sides can never be curved. The segments are called the **sides** of the polygons, and the points where the segments intersect are called **vertices**. The easiest way to identify a polygon is to look for a closed figure with no curved sides.

Example 6: Which of the figures below is a polygon?



Solution: The easiest way to identify the polygon is to identify which shapes are not polygons. B and C each have at least one curved side, so they cannot be polygons. D has all straight sides, but one of the vertices is not at the endpoint of the adjacent side, so it is not a polygon either. A is the only polygon.

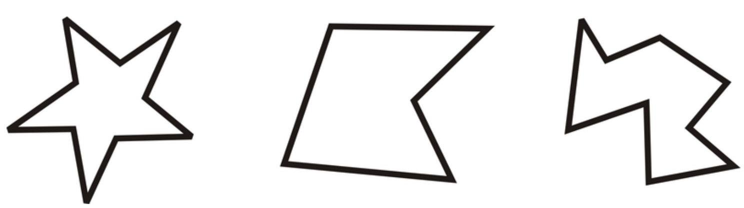
Example 7: Which of the figures below is *not* a polygon?



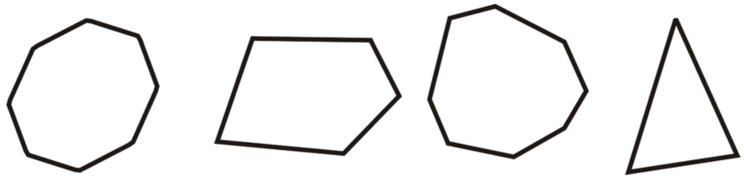
Solution: C is a three-dimensional shape, so it does not lie within one plane, so it is not a polygon.

Convex and Concave Polygons

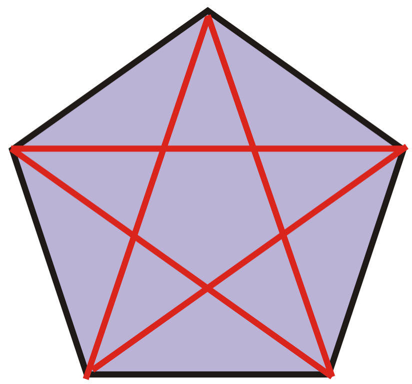
Polygons can be either **convex** or **concave**. Think of the term concave as referring to a cave, or "caving in". A concave polygon has a section that "points inward" toward the middle of the shape. All stars are concave polygons.



A convex polygon does not share this property.



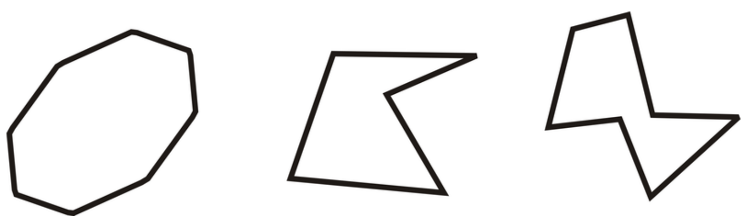
Diagonals: Line segments that connects the vertices of a convex polygon that are not sides.



The red lines are all diagonals.

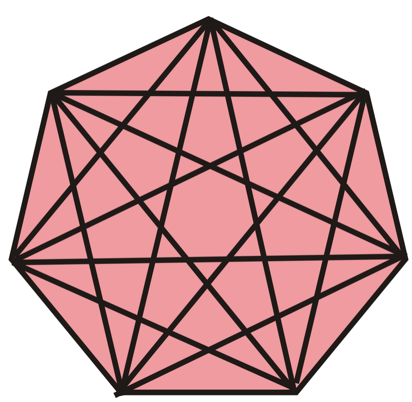
This pentagon has 5 diagonals.

Example 8: Determine if the shapes below are convex or concave.



Solution: To see if a polygon is concave, look at the polygons and see if any angle "caves in" to the interior of the polygon. The first polygon does not do this, so it is convex. The other two do, so they are concave. You could add here that concave polygons have at least one diagonal outside the figure.

Example 9: How many diagonals does a 7-sided polygon have?



Solution: Draw a 7-sided polygon, also called a heptagon. Drawing in all the diagonals and counting them, we see there are 14.

Classifying Polygons

Whether a polygon is convex or concave, it can always be named by the number of sides. See the chart below.

TABLE 1.1:

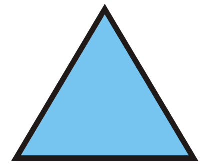
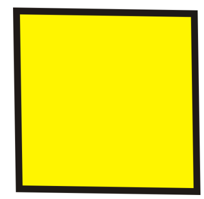
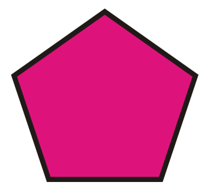
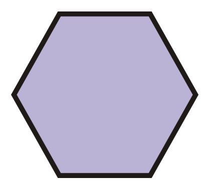
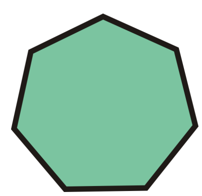
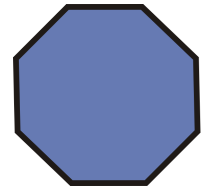
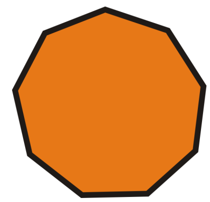
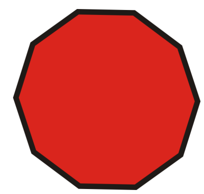
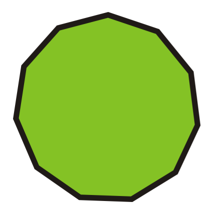
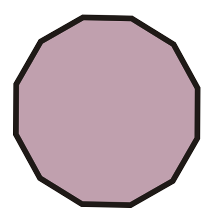
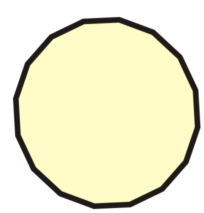
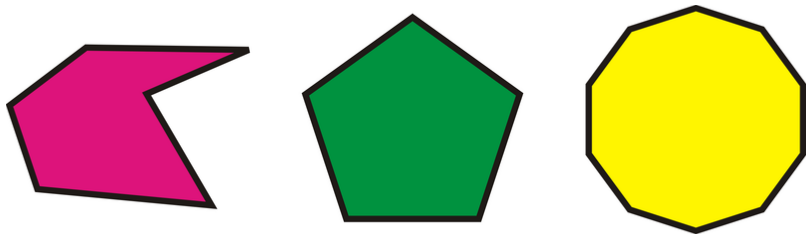
Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Triangle	3	0	
Quadrilateral	4	2	
Pentagon	5	5	
Hexagon	6	9	
Heptagon	7	14	

TABLE 1.1: (continued)

Polygon Name	Number of Sides	Number of Diagonals	Convex Example
Octagon	**8**	**?**	
Nonagon	**9**	**?**	
Decagon	**10**	**?**	
Undecagon or hendecagon	**11**	**?**	
Dodecagon	**12**	**?**	
$n-$**gon**	n (**where** $n > 12$)	**?**	

Example 10: Name the three polygons below by their number of sides and if it is convex or concave.



Solution:

A. This shape has six sides and concave, so it is a concave hexagon.

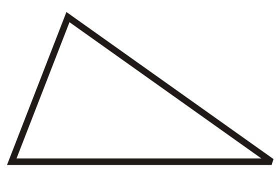
B. This shape has five sides and is convex, so it is a convex pentagon.

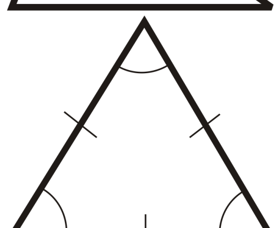
C. This shape has ten sides and is convex, so it is a convex decagon.

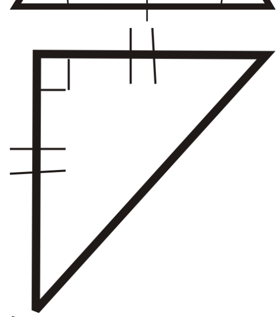
Know What? Revisited The pentagon is divided up into 10 sections, all quadrilaterals. More specifically, there are 5 rectangles and 5 kites. None of these dividing lines are diagonals because they are not drawn from vertices.

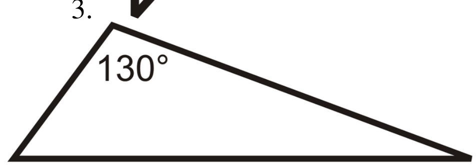
Review Questions

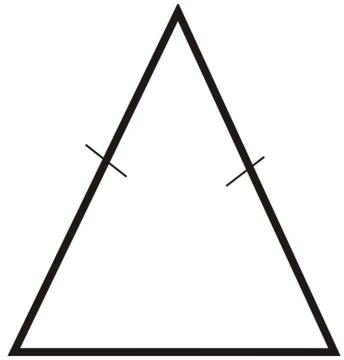
For questions 1-6, classify each triangle by its sides and by its angles.

1. 

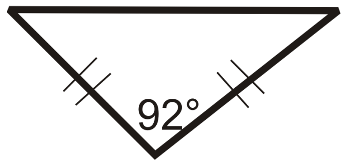
2. 

3. 

4. 



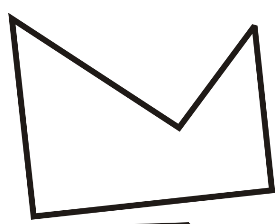
5.



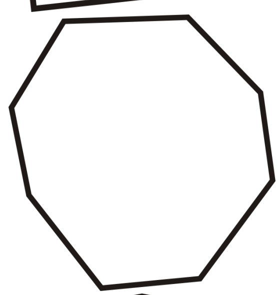
6.

7. Can you draw a triangle with a right angle and an obtuse angle? Why or why not?
8. In an isosceles triangle, can the angles opposite the congruent sides be obtuse?
9. ***Construction*** Construct an equilateral triangle with sides of 3 cm. Start by drawing a horizontal segment of 3 cm and measure this side with your compass from both endpoints.
10. What must be true about the angles of your equilateral triangle from #9?

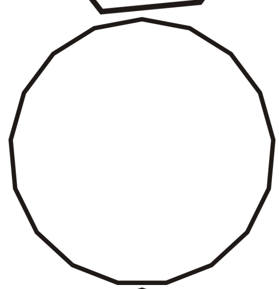
In problems 11-16, name each polygon in as much detail as possible.



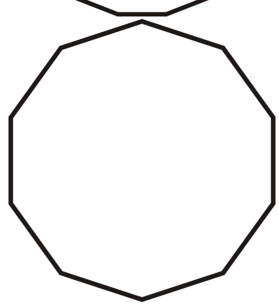
11.



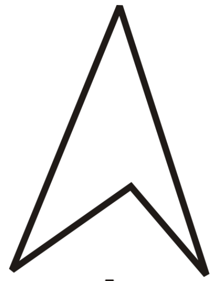
12.

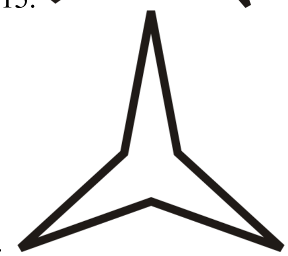


13.

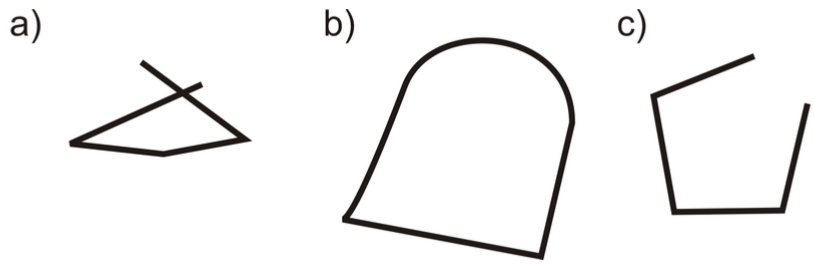


14.

15. 

16. 

17. Explain why the following figures are NOT polygons:



18. How many diagonals can you draw from **one vertex** of a pentagon? Draw a sketch of your answer.
19. How many diagonals can you draw from **one vertex** of an octagon? Draw a sketch of your answer.
20. How many diagonals can you draw from **one vertex** of a dodecagon?
21. Use your answers from 17-19 to figure out how many diagonals you can draw from **one vertex** of an $n-$gon?
22. Determine the number of total diagonals for an octagon, nonagon, decagon, undecagon, and dodecagon. Do you see a pattern? BONUS: Find the equation of the total number of equations for an $n-$gon.

For 23-30, determine if the statement is ALWAYS true, SOMETIMES true, or NEVER true.

23. Obtuse triangles are isosceles.
24. A polygon must be enclosed.
25. A star is a concave polygon.
26. A right triangle is acute.
27. An equilateral triangle is equiangular.
28. A quadrilateral is a square.
29. You can draw $(n-1)$ triangles from one vertex of a polygon.
30. A decagon is a 5-point star.

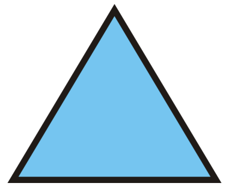
In geometry it is important to know the difference between a sketch, a drawing and a construction. A sketch is usually drawn free-hand and marked with the appropriate congruence markings or labeled with measurement. It may or may not be drawn to scale. A drawing is made using a ruler, protractor or compass and should be made to scale. A construction is made using only a compass and ruler and should be made to scale.

For 31-36, draw, sketch or construct the indicated figures.

31. Sketch a convex heptagon with two sides congruent and three angles congruent.
32. Sketch a non-polygon figure.
33. Draw a concave pentagon with exactly two right angles and at least two congruent sides.

34. Draw an equilateral quadrilateral that is NOT a square.
35. Construct a right triangle with side lengths 3 cm, 4 cm and 5 cm.
36. ***Construction Challenge*** Construct a 60° angle. (*Hint: Think about an equilateral triangle.*)

Review Queue Answers

a. 

b. Examples include: stop sign (8), table top (4), the Pentagon (5), snow crystals (6), bee hive combs (6), soccer ball pieces (5 and 6)

   a. congruent or equal
   b. supplementary
   c. sides, vertex

1.6 Chapter 1 Review

Symbol Toolbox

$\overleftrightarrow{AB}$, $\overrightarrow{AB}$, $\overline{AB}$ Line, ray, line segment

$\angle ABC$ Angle with vertex B

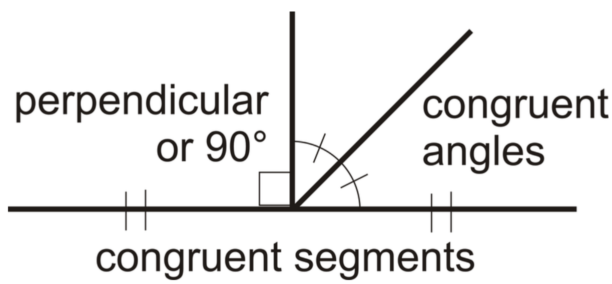
$m\overline{AB}$ or AB Distance between A and B

$m\angle ABC$ Measure of $\angle ABC$

$\perp$ Perpendicular

$=$ Equal

$\cong$ Congruent



Keywords

Geometry
Geometry is founded upon some very important basic concepts. These include points, angles, lines, and line segments.

Point
An exact location in space.

Line
Infinitely many points that extend forever in both directions.

Plane
Infinitely many intersecting lines that extend forever in all directions.

Space
The set of all points expanding in ***three*** dimensions.

Collinear
Points that lie on the same line.

Coplanar
Points and/or lines within the same plane.

Endpoint
A point at the end of a line.

Line Segment
Part of a line with two endpoints. Or a line that stops at both ends.

Ray
Part of a line with one endpoint and extends forever in the other direction.

Intersection
A point or set of points where lines, planes, segments or rays cross each other

Postulates
Basic rules of geometry.

Theorem
A statement that can be proven true using postulates, definitions, and other theorems that have already proven.

Distance
How far apart two geometric objects are.

Measure
Angles are classified by their measure.

Ruler Postulate
The distance between two points will be the absolute value of the difference between the numbers shown on the ruler.

Segment Addition Postulate
The measure of any line segment can be found by adding the measures of the smaller segments that make it up

If A, B, and C are collinear and B is between A and C, then $AB+BC=AC$.

Angle
When two rays have the same endpoint.

Vertex
The common endpoint of the two rays that form an angle.

Sides
The two rays that form an angle.

Protractor Postulate
For every angle there is a number between 0° and 180° that is the measure of the angle in degrees. The angle's measure is then the absolute value of the difference of the numbers shown on the protractor where the sides of the angle intersect the protractor.

Straight Angle
When an angle measures 180°. The angle measure of a straight line.

Right Angle
When an angle measures 90°.

Acute Angles
Angles that measure between 0° and 90°.

Obtuse Angles
Angles that measure between 90° and 180°.

Perpendicular
When two lines intersect to form four right angles.

Construction
Anytime we use a compass and ruler to draw different geometric figures, it called a **construction**.

Compass
A tool used to draw circles and arcs.

Angle Addition Postulate
The measure of any angle can be found by adding the measures of the smaller angles that comprise it.
If B is on the interior of $\angle ADC$, then $m\angle ADC = m\angle ADB + m\angle BDC$.

Congruent
When two geometric figures have the same shape and size.

Midpoint
A point on a line segment that divides it into two congruent segments

Midpoint Postulate
Any line segment will have exactly one midpoint.

Segment Bisector
A line, segment, or ray that passes through a midpoint of another segment.

Perpendicular Bisector
A line, ray or segment that passes through the midpoint of another segment and intersects the segment at a right angle.

Perpendicular Bisector Postulate
For every line segment, there is one perpendicular bisector that passes through the midpoint.

Angle Bisector
A ray that divides an angle into two congruent angles, each having a measure exactly half of the original angle.

Angle Bisector Postulate
Every angle has exactly one angle bisector.

Complementary
When two angles add up to 90°.

Supplementary
When two angles add up to 180°.

Adjacent Angles
Two angles that have the same vertex, share a side, and do not overlap.

Linear Pair
Two angles that are adjacent and whose non-common sides form a straight line.

Linear Pair Postulate
If two angles are a linear pair, then they are supplementary.

Vertical Angles
Two non-adjacent angles formed by intersecting lines.

Vertical Angles Theorem
If two angles are vertical angles, then they are congruent.

Triangle
Any closed figure made by three line segments intersecting at their endpoints.

Right Triangle
When a triangle has one right angle.

Obtuse Triangle
When a triangle has one obtuse angle.

Acute Triangle
When all three angles in the triangle are acute.

Equiangular Triangle
When all the angles in a triangle are congruent.

Scalene Triangle
When a triangles sides are all different lengths.

Isosceles Triangle
A triangle with at least two sides of equal length.

Equilateral Triangle
A triangle with three sides of equal length.

Polygon
Any closed planar figure that is made entirely of line segments that intersect at their endpoints.

Diagonals
Line segments that connects the vertices of a convex polygon that are not sides.

Review

Match the definition or description with the correct word.

1. When three points lie on the same line. — A. Measure
2. All vertical angles are ________. — B. Congruent
3. Linear pairs add up to _______. — C. Angle Bisector
4. The m in from of $m\angle ABC$. — D. Ray
5. What you use to measure an angle. — E. Collinear
6. When two sides of a triangle are congruent. — F. Perpendicular
7. $\perp$ — G. Line
8. A line that passes through the midpoint of another line. — H. Protractor
9. An angle that is greater than 90°. — I. Segment Addition Postulate
10. The intersection of two planes is a ___________. — J. Obtuse
11. $AB+BC=AC$ — K. Point
12. An exact location in space. — L. 180°
13. A sunbeam, for example. — M. Isosceles
14. Every angle has exactly one. — N. Pentagon
15. A closed figure with 5 sides. — O. Hexagon — P. Bisector

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9686 .

Chapter 2 Reasoning and Proof

Chapter Outline

This chapter explains how to reason and how to use reasoning to prove theorems about angle pairs and segments. This chapter also introduces the properties of congruence, which will also be used in proofs. Subsequent chapters will combine what you have learned in Chapters 1 and 2 and build upon them.

2.1 Conditional Statements

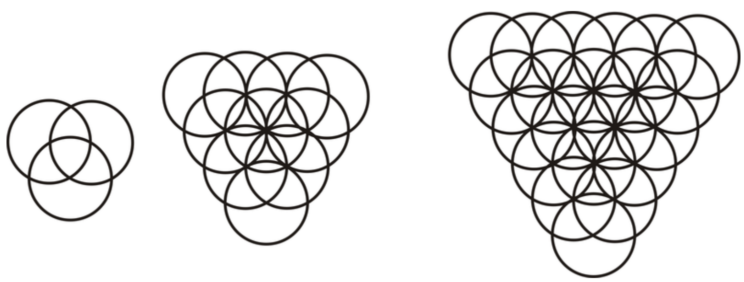
Learning Objectives

- Identify the hypothesis and conclusion of an if-then or conditional statement.
- Write the converse, inverse, and contrapositive of an if-then statement.
- Recognize a biconditional statement.

Review Queue

Find the next figure or term in the pattern.

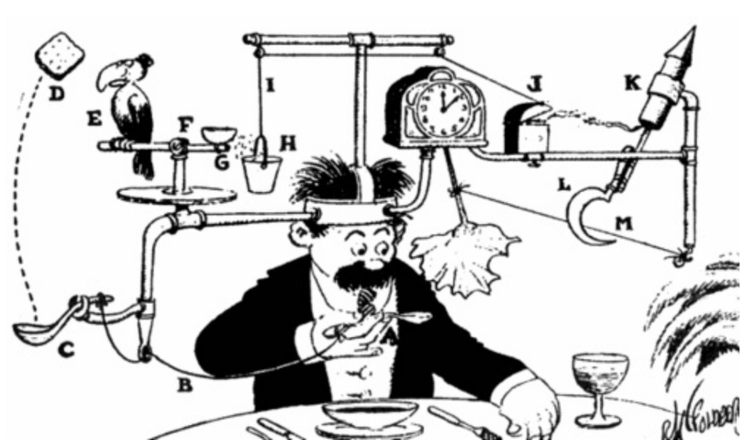
a. 5, 8, 12, 17, 23,...
b. $\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{9}, \frac{6}{10}, \ldots$



c.

d. Find a counterexample for the following conjectures.
   a. If it is April, then it is Spring Break.
   b. If it is June, then I am graduating.

Know What? Rube Goldman was a cartoonist in the 1940s who drew crazy inventions to do very simple things. The invention to the right has a series of smaller tasks that leads to the machine wiping the man's face with a napkin.



Write a series of if-then statements to that would caption this cartoon, from A to M.

If-Then Statements

Conditional Statement (also called an **If-Then Statement**): A statement with a hypothesis followed by a conclusion.

Another way to define a conditional statement is to say, "If this happens, then that will happen."

Hypothesis: The first, or "if," part of a conditional statement. An educated guess.

Conclusion: The second, or "then," part of a conditional statement. The conclusion is the result of a hypothesis.

Keep in mind that conditional statements might not always be written in the "if-then" form. Here are a few examples.

Statement 1: If you work overtime, then you'll be paid time-and-a-half.

Statement 2: I'll wash the car if the weather is nice.

Statement 3: If 2 divides evenly into x, then x is an even number.

Statement 4: I'll be a millionaire when I win monopoly.

Statement 5: All equiangular triangles are equilateral.

Statements 1 and 3 are written in the "if-then" form. The hypothesis of Statement 1 is "you work overtime." The conclusion is "you'll be paid time-and-a-half."

So, if Sarah works overtime, then what will happen? From Statement 1, we can conclude that she will be paid time-and-a-half.

If 2 goes evenly into 16, what can you conclude? From Statement 3, we know that 16 must be an even number.

Statement 2 has the hypothesis after the conclusion. Even though the word "then" is not there, the statement can be rewritten as: If the weather is nice, then I'll wash the car. If the word "if" is in the middle of a conditional statement, the hypothesis is always after it.

Statement 4 uses the word "when" instead of "if." It should be treated like Statement 2, so it can be written as: If I win monopoly, then I will be a millionaire.

Statement 5 "if" and "then" are not there, but can be rewritten as: If a triangle is equiangular, then it is equilateral.

Converse, Inverse, and Contrapositive of a Conditional Statement

Look at **Statement 2** again: *If the weather is nice, then I'll wash the car.*

This can be rewritten using letters to represent the hypothesis and conclusion.

$$\text{If } p, \text{then } q. \qquad p = \text{the weather is nice}$$
$$q = \text{I'll wash the car}$$

$\text{Or}, p \rightarrow q$

In addition to these positives, we can also write the negations, or "not"s of p and q. The symbolic version of not p, is $\sim p$.

$\sim p = \text{the weather is not nice}$
$\sim q = \text{I won't wash the car}$

Using these negations and switching the order of p and q, we can create three more conditional statements.

Converse	$q \to p$	$\underbrace{\text{If I wash the car}}_{q}, \underbrace{\text{then the weather is nice}}_{p}.$
Inverse	$\sim p \to \sim q$	$\underbrace{\text{If the weather is not nice}}_{\sim p}, \underbrace{\text{then I won't wash the car}}_{\sim q}.$
Contrapositive	$\sim q \to \sim p$	$\underbrace{\text{If I don't wash the car}}_{\sim q}, \underbrace{\text{then the weather is not nice}}_{\sim p}.$

If we accept "If the weather is nice, then I'll wash the car" as true, then the converse and inverse are not necessarily true. However, if we take original statement to be true, then the contrapositive is also true. We say that the contrapositive is ***logically equivalent*** to the original if-then statement.

Example 1: Use the statement: If $n > 2$, then $n^2 > 4$.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

$\underline{\text{Converse}}$:	If $n^2 > 4$, then $n > 2$.	*False*. n could be -3, making $n^2 = 9$.
$\underline{\text{Inverse}}$:	If $n < 2$, then $n^2 < 4$.	*False*. Again, if $n = -3$, then $n^2 = 9$.
$\underline{\text{Contrapositive}}$:	If $n^2 < 4$, then $n < 2$.	*True*, the only square number less than 4 is 1, which has square roots of 1 or -1, both less than 2.

Example 2: Use the statement: If I am at Disneyland, then I am in California.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: The original statement is true.

$\underline{\text{Converse}}$:	If I am in California, then I am at Disneyland.
	False. I could be in San Francisco.
$\underline{\text{Inverse}}$:	If I am not at Disneyland, then I am not in California.
	False. Again, I could be in San Francisco.
$\underline{\text{Contrapositive}}$:	If I am not in California, then I am not at Disneyland.
	True. If I am not in the state, I couldn't be at Disneyland.

Notice for the inverse and converse ***we can use the same counterexample.*** This is because the inverse and converse are also ***logically equivalent.***

Example 3: Use the statement: Any two points are collinear.

a) Find the converse, inverse, and contrapositive.

b) Determine if the statements from part a are true or false. If they are false, find a counterexample.

Solution: First, change the statement into an "if-then" statement: If two points are on the same line, then they are collinear.

Converse : If two points are collinear, then they are on the same line. *True*.
Inverse : If two points are not on the same line, then they are not collinear. *True*.
Contrapositive : If two points are not collinear, then they do not lie on the same line. *True*.

Biconditional Statements

Example 3 is an example of a biconditional statement.

Biconditional Statement: When the original statement and converse are both true.

So, $p \to q$ is true and $q \to p$ is true. It is written $p \leftrightarrow q$, with a double arrow to indicate that it does not matter if p or q is first. It is said, "p if and only if q"

Example 4: Rewrite Example 3 as a biconditional statement.

Solution: *If two points are on the same line, then they are collinear* can be rewritten as: *Two points are on the same line if and only if they are collinear.*

Replace the "if-then" with "if and only if" in the middle of the statement. "If and only if" can be abbreviated "iff."

Example 5: The following is a true statement:

$m\angle ABC > 90^\circ$ if and only if $\angle ABC$ is an obtuse angle.

Determine the two true statements within this biconditional.

Solution:

Statement 1: If $m\angle ABC > 90^\circ$, then $\angle ABC$ is an obtuse angle

Statement 2: If $\angle ABC$ is an obtuse angle, then $m\angle ABC > 90^\circ$.

You should recognize this as the definition of an obtuse angle. All geometric definitions are biconditional statements.

Example 6: $p : x < 10 \qquad q : 2x < 50$

a) Is $p \to q$ true? If not, find a counterexample.

b) Is $q \to p$ true? If not, find a counterexample.

c) Is $\sim p \to \sim q$ true? If not, find a counterexample.

d) Is $\sim q \to \sim p$ true? If not, find a counterexample.

Solution:

$$\begin{aligned}
amp; p \to q : &\quad \text{If } x < 10, \text{ then } 2x < 50. \quad True. \\
amp; q \to p : &\quad \text{If } 2x < 50, \text{ then } x < 10. \quad False, \ x = 15 \text{ would be a counterexample.} \\
\sim p \to \sim q : &\quad \text{If } x > 10, \text{ then } 2x > 50. \quad False, \ x = 15 \text{ would also work here.} \\
\sim q \to \sim p : &\quad \text{If } 2x > 50, \text{ then } x > 10. \quad True.
\end{aligned}$$

Know What? Revisited The conditional statements are as follows:

$A \to B$: If the man raises his spoon, then it pulls a string.

$B \to C$: If the string is pulled, then it tugs back a spoon.

$C \to D$: If the spoon is tugged back, then it throws a cracker into the air.

$D \rightarrow E$: If the cracker is tossed into the air, the bird will eat it.

$E \rightarrow F$: If the bird eats the cracker, then it turns the pedestal.

$F \rightarrow G$: If the bird turns the pedestal, then the water tips over.

$G \rightarrow H$: If the water tips over, it goes into the bucket.

$H \rightarrow I$: If the water goes into the bucket, then it pulls down the string.

$I \rightarrow J$: If the bucket pulls down the string, then the string opens the box.

$J \rightarrow K$: If the box is opened, then a fire lights the rocket.

$K \rightarrow L$: If the rocket is lit, then the hook pulls a string.

$L \rightarrow M$: If the hook pulls the string, then the man's faces is wiped with the napkin.

This is a very complicated contraption used to wipe a man's face. Purdue University liked these cartoons so much, that they started the Rube Goldberg Contest in 1949. This past year, the task was to pump hand sanitizer into someone's hand in no less than 20 steps. http://www.purdue.edu/newsroom/rubegoldberg/index.html

Review Questions

For questions 1-6, determine the hypothesis and the conclusion.

1. If 5 divides evenly into x, then x ends in 0 or 5.
2. If a triangle has three congruent sides, it is an equilateral triangle.
3. Three points are coplanar if they all lie in the same plane.
4. If $x = 3$, then $x^2 = 9$.
5. If you take yoga, then you are relaxed.
6. All baseball players wear hats.
7. Write the converse, inverse, and contrapositive of #1. Determine if they are true or false. If they are false, find a counterexample.
8. Write the converse, inverse, and contrapositive of #5. Determine if they are true or false. If they are false, find a counterexample.
9. Write the converse, inverse, and contrapositive of #6. Determine if they are true or false. If they are false, find a counterexample.
10. Find the converse of #2. If it is true, write the biconditional of the statement.
11. Find the converse of #3. If it is true, write the biconditional of the statement.
12. Find the converse of #4. If it is true, write the biconditional of the statement.

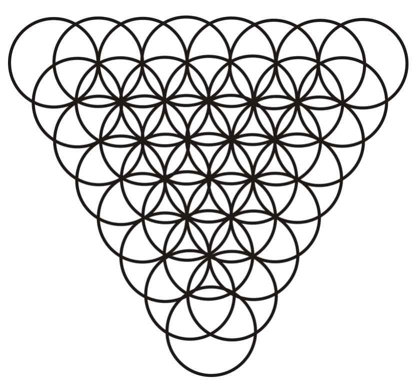
For questions 13-16, use the statement: If $AB = 5$ and $BC = 5$, then B is the midpoint of $\overline{AC}$.

13. If this is the converse, what is the original statement? Is it true?
14. If this is the original statement, what is the inverse? Is it true?
15. Find a counterexample of the statement.
16. Find the contrapositive of the original statement from #13.
17. What is the inverse of the inverse of $p \rightarrow q$? HINT: Two wrongs make a right in math!
18. What is the one-word name for the converse of the inverse of an if-then statement?
19. What is the one-word name for the inverse of the converse of an if-then statement?
20. What is the contrapositive of the contrapositive of an if-then statement?

For questions 21-24, determine the two true conditional statements from the given biconditional statements.

21. A U.S. citizen can vote if and only if he or she is 18 or more years old.
22. A whole number is prime if and only if it has exactly two distinct factors.
23. Points are collinear if and only if there is a line that contains the points.
24. $2x = 18$ if and only if $x = 9$.
25. $p : x = 4 \quad q : x^2 = 16$
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
26. $p : x = -2 \quad q : -x + 3 = 5$
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
27. p : the measure of $\angle ABC = 90^\circ$ q : $\angle ABC$ is a right angle
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
28. p : the measure of $\angle ABC = 45^\circ$ q : $\angle ABC$ is an acute angle
    a. Is $p \to q$ true? If not, find a counterexample.
    b. Is $q \to p$ true? If not, find a counterexample.
    c. Is $\sim p \to \sim q$ true? If not, find a counterexample.
    d. Is $\sim q \to \sim p$ true? If not, find a counterexample.
29. Write a conditional statement. Write the converse, inverse and contrapositive of your statement. Are they true or false? If they are false, write a counterexample.
30. Write a true biconditional statement. Separate it into the two true conditional statements.

Review Queue Answers

a. 30
b. $\frac{7}{11}$
c. 
    a. It could be another day that isn't during Spring Break. Spring Break doesn't last the entire month.
    b. You could be a freshman, sophomore or junior. There are several counterexamples.

2.2 Inductive Reasoning

Learning Objectives

- Recognize visual and number patterns.
- Extend and generalize patterns.
- Write a counterexample.

Review Queue

a. Look at the patterns of numbers below. Determine the next three numbers in the list. Describe the pattern.
   a. 1, 2, 3, 4, 5, 6, _____, _____, _____
   b. 3, 6, 9, 12, 15, _____, _____, _____
   c. 1, 4, 9, 16, 25, _____, _____, _____
b. Are the statements below true or false? If they are false, state why.
   a. Perpendicular lines form four right angles.
   b. Angles that are congruent are also equal.
   c. Linear pairs are always congruent.
c. For the line, $y = 3x + 1$, make an $x - y$ table for $x = 1, 2, 3, 4$, and 5. What do you notice? How does it relate to $1b$?

Know What? This is the "famous" locker problem:

A new high school has just been completed. There are 1000 lockers in the school and they have been numbered from 1 through 1000. During recess, the students decide to try an experiment. When recess is over each student walks into the school one at a time. The first student will open all of the locker doors. The second student will close all of the locker doors with even numbers. The third student will change all of the locker doors that are multiples of 3 (*change means closing lockers that are open, and opening lockers that are closed*). The fourth student will change the position of all locker doors numbered with multiples of four and so on.

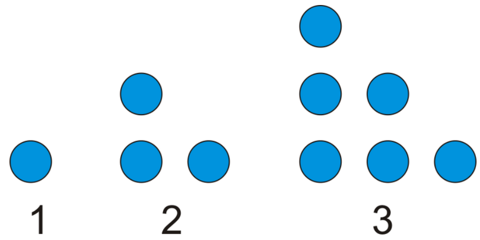
Imagine that this continues until the 1000 students have followed the pattern with the 1000 lockers. At the end, which lockers will be open and which will be closed? Which lockers were touched the most often? Which lockers were touched exactly 5 times?

Visual Patterns

Inductive Reasoning: Making conclusions based upon observations and patterns.

Let's look at some visual patterns to get a feel for what inductive reasoning is.

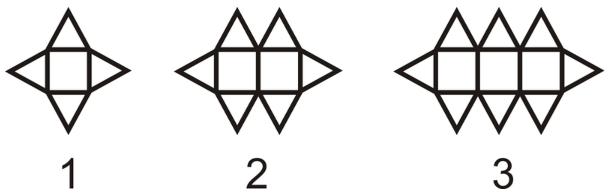
Example 1: A dot pattern is shown below. How many dots would there be in the bottom row of the 4^{th} figure? What would the *total number* of dots be in the 6^{th} figure?



Solution: There will be 4 dots in the bottom row of the 4^{th} figure. There is one more dot in the bottom row of each figure than in the previous figure.

There would be a total of 21 dots in the 6^{th} figure, $6+5+4+3+2+1$.

Example 2: How many *triangles* would be in the 10^{th} figure?



Solution: There are 10 squares, with a triangle above and below each square. There is also a triangle on each end of the figure. That makes $10+10+2=22$ triangles in all.

Example 2b: If one of these figures contains 34 triangles, how many *squares* would be in that figure?

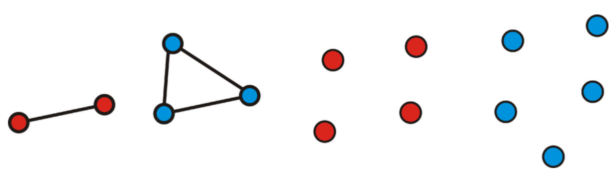
Solution: First, the pattern has a triangle on each end. Subtracting 2, we have 32 triangles. Now, divide 32 by 2 because there is a row of triangles above and below each square. $32 \div 2 = 16$ squares.

Example 2c: How can we find the number of triangles if we know the figure number?

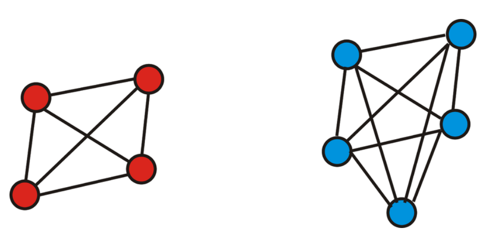
Solution: Let n be the figure number. This is also the number of squares. $2n$ is the number of triangles above and below the squares. Add 2 for the triangles on the ends.

If the figure number is n, then there are $2n+2$ triangles in all.

Example 3: For two points, there is one line segment between them. For three non-collinear points, there are three line segments with those points as endpoints. For four points, no three points being collinear, how many line segments are between them? If you add a fifth point, how many line segments are between the five points?



Solution: Draw a picture of each and count the segments.



For 4 points there are 6 line segments and for 5 points there are 10 line segments.

Number Patterns

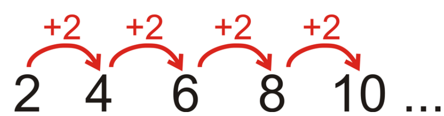
Let's look at a few examples.

Example 4: Look at the pattern 2, 4, 6, 8, 10,...

a) What is the 19^{th} term in the pattern?

b) Describe the pattern and try and find an equation that works for every term in the pattern.

Solution: For part a, each term is 2 more than the previous term.



You could count out the pattern until the 19^{th} term, but that could take a while. The easier way is to recognize the pattern. Notice that the 1^{st} term is $2 \cdot 1$, the 2^{nd} term is $2 \cdot 2$, the 3^{rd} term is $2 \cdot 3$, and so on. So, the 19^{th} term would be $2 \cdot 19$ or 38.

For part b, we can use this pattern to generate a formula. Typically with number patterns we use n to represent the term number. So, this pattern is 2 times the term number, or $2n$.

Example 5: Look at the pattern 1, 3, 5, 7, 9, 11,...

a) What is the 34^{th} term in the pattern?

b) What is the n^{th} term?

Solution: The pattern increases by 2 and is odd. From the previous example, we know that if a pattern increases by 2, you would multiply n by 2. However, this pattern is odd, so we need to add or subtract a number. Let's put what we know into a table:

TABLE 2.1:

n	$2n$	-1	**Pattern**
1	2	-1	1
2	4	-1	3
3	6	-1	5
4	8	-1	7
5	10	-1	9
6	12	-1	11

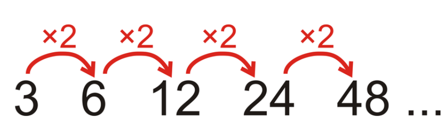
From this we can reason that the 34^{th} term would be $34 \cdot 2$ minus 1, which is 67. Therefore, the n^{th} term would be $2n-1$.

Example 6: Look at the pattern: 3, 6, 12, 24, 48,...

a) What is the next term in the pattern? The 10^{th} term?

b) Make a rule for the n^{th} term.

Solution: This pattern is different than the previous two examples. Here, each term is multiplied by 2 to get the next term.



Therefore, the next term will be $48 \cdot 2$ or 96. To find the 10^{th} term, we need to work on the pattern, let's break apart each term into the factors to see if we can find the rule.

TABLE 2.2:

n	**Pattern**	**Factors**	**Simplify**
1	3	3	$3 \cdot 2^0$
2	6	$3 \cdot 2$	$3 \cdot 2^1$
3	12	$3 \cdot 2 \cdot 2$	$3 \cdot 2^2$
4	48	$3 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^3$
5	48	$3 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$3 \cdot 2^4$

Using this equation, the 10^{th} term will be $3 \cdot 2^9$, or 1536. Notice that the exponent is one less than the term number. So, for the n^{th} term, the equation would be $3 \cdot 2^{n-1}$.

Example 7: Find the 8^{th} term in the list of numbers as well as the rule.

$$2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \dots$$

Solution: First, change 2 into a fraction, or $\frac{2}{1}$. So, the pattern is now $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25} \dots$ Separate the top and the bottom of the fractions into two different patterns. The top is 2, 3, 4, 5, 6. It increases by 1 each time, so the 8^{th} term's numerator is 9. The denominators are the square numbers, so the 8^{th} term's denominator is 10^2 or 100. Therefore, the 8^{th} term is $\frac{9}{100}$. The rule for this pattern is $\frac{n+1}{n^2}$.

To summarize:

- If the same number is **added** from one term to the next, then you multiply n by it.
- If the same number is **multiplied** from one term to the next, then you would multiply the first term by increasing powers of this number. n or $n-1$ is in the exponent of the rule.
- If the pattern has **fractions**, separate the numerator and denominator into two different patterns. Find the rule for each separately.

Conjectures and Counterexamples

Conjecture: An "educated guess" that is based on examples in a pattern.

Numerous examples may make you believe a conjecture. However, no number of examples can actually *prove* a conjecture. It is always possible that the next example would show that the conjecture is false.

Example 8: Here's an algebraic equation and a table of values for n and with the result for t.

$$t = (n-1)(n-2)(n-3)$$

TABLE 2.3:

n	$(n-1)(n-2)(n-3)$	t
1	$(0)(-1)(-2)$	0
2	$(1)(0)(-1)$	0
3	$(2)(1)(0)$	0

After looking at the table, Pablo makes this conjecture:

The value of $(n-1)(n-2)(n-3)$ *is 0 for any whole number value of n.*

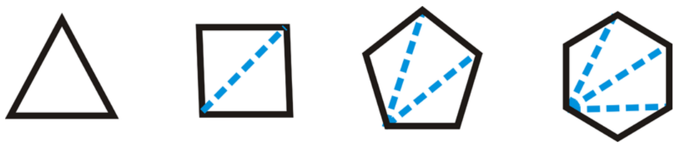
Is this a valid, or true, conjecture?

Solution: No, this is not a valid conjecture. If Pablo were to continue the table to $n=4$, he would have see that $(n-1)(n-2)(n-3)=(4-1)(4-2)(4-3)=(3)(2)(1)=6$.

In this example $n=4$ is called a counterexample.

Counterexample: An example that disproves a conjecture.

Example 9: Arthur is making figures for a graphic art project. He drew polygons and some of their diagonals.



Based on these examples, Arthur made this conjecture:

If a convex polygon has n sides, then there are $n-3$ triangles drawn from any given vertex of the polygon.

Is Arthur's conjecture correct? Can you find a counterexample to the conjecture?

Solution: The conjecture appears to be correct. If Arthur draws other polygons, in every case he will be able to draw $n-3$ triangles if the polygon has n sides.

Notice that we have ***not proved*** Arthur's conjecture, but only found several examples that hold true. This type of conjecture would need to be proven by induction.

Know What? Revisited Start by looking at the pattern. Red numbers are OPEN lockers.

Student 1 changes every locker:

1, 2, 3, 4, 5, 6, 7, 8,... 1000

Student 2 changes every 2^{nd} locker:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... 1000

Student 3 changes every 3^{rd} locker:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... 1000

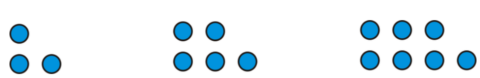
Student 4 changes every 4^{th} locker:

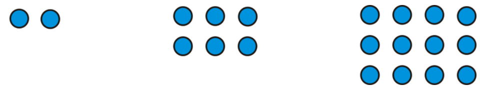
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,... 1000

If you continue on in this way, the only lockers that will be left open are the numbers with an odd number of factors, or the square numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, and 961. The lockers that were touched the most are the numbers with the most factors. The one locker that was touched the most was 840, which has 32 factors and thus, touched 32 times. There are three lockers that were touched exactly five times: 16, 81, and 625.

Review Questions

For questions 1 and 2, determine how many dots there would be in the 4^{th} and the 10^{th} pattern of each figure below.

1. 

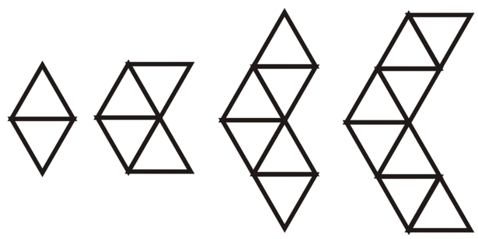


2.

3. Use the pattern below to answer the questions.

   a. Draw the next figure in the pattern.
   b. How does the number of points in each star relate to the figure number?
   c. Use part b to determine a formula for the n^{th} figure.

4. Use the pattern below to answer the questions. All the triangles are equilateral triangles.



   a. Draw the next figure in the pattern. How many triangles does it have?
   b. Determine how many triangles are in the 24^{th} figure.
   c. How many triangles are in the n^{th} figure?

For questions 5-12, determine: 1) the next two terms in the pattern, 2) the 35^{th} figure and 3) the formula for the n^{th} figure.

5. 5, 8, 11, 14, 17,...
6. 6, 1, -4, -9, -14,...
7. 2, 4, 8, 16, 32,...
8. 67, 56, 45, 34, 23,...
9. 9, -4, 6, -8, 3,...
10. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$
11. $\frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \frac{8}{15}, \frac{10}{19}, \ldots$
12. 3, -5, 7, -9, 11,...
13. -1, 5, -9, 13, -17,...
14. $\frac{-1}{2}, \frac{1}{4}, \frac{-1}{6}, \frac{1}{8}, \frac{-1}{10}, \ldots$
15. 5, 12, 7, 10, 9,...
16. 1, 4, 9, 16, 25,...

For questions 13-16, determine the next two terms and describe the pattern.

17. 3, 6, 11, 18, 27,...
18. 3, 8, 15, 24, 35,...
19. 1, 8, 27, 64, 125,...
20. 1, 1, 2, 3, 5,...

We all use inductive reasoning in our daily lives. The process consists of making observations, recognizing a pattern and making a generalization or conjecture. Read the following examples of reasoning in the real world and determine if they are examples of Inductive reasoning. Do you think the conjectures are true or can you give a counterexample?

21. For the last three days Tommy has gone for a walk in the woods near his house at the same ime of day. Each time he has seen at least one deer. Tommy reasons that if he goes for a walk tomorrow at the same time, he will see deer again.
22. Maddie likes to bake. She especially likes to take recipes and make substitutions to try to make them healthier. She might substitute applesauce for sugar or oat flour for white flour. She has noticed that she needs to add more baking powder or baking soda than the recipe indicates in these situations in order for the baked goods to rise appropriately.
23. One evening Juan saw a chipmunk in his backyard. He decided to leave a slice of bread with peanut butter on it for the creature to eat. The next morning the bread was gone. Juan concluded that chipmunks like to eat bread with peanut butter.
24. Describe an instance in your life when either you or someone you know used inductive reasoning to correctly make a conclusion.
25. Describe an instance when you observed someone using invalid reasoning skills.

Challenge For the following patterns find a) the next two terms, b) the 40^{th} term and c) the n^{th} term rule. You will need to think about each of these in a different way. *Hint: Double all the values and look for a pattern in their factors. Once you come up with the rule remember to divide it by two to undo the doubling.*

26. 2, 5, 9, 14,...
27. 3, 6, 10, 15,...
28. 3, 12, 30, 60,...

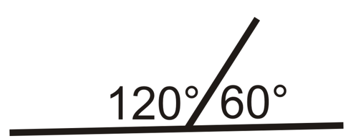
Connections to Algebra

29. Plot the values of the terms in the sequence 3, 8, 13,... against the term numbers in the coordinate plane. In other words, plot the points (1, 3), (2, 8), and (3, 13). What do you notice? Could you use algebra to figure out the "rule" or equation which maps each term number (x) to the correct term value (y)? Try it.
30. Which sequences in problems 5-16 follow a similar pattern to the one you discovered in #29? Can you use inductive reasoning to make a conclusion about which sequences follow the same type of rule?

Review Queue Answers

a. 7, 8, 9
b. 18, 21, 24
c. 36, 49, 64

a. true
b. true
c. false,



2.3 Deductive Reasoning

Learning Objectives

- Apply some basic rules of logic.
- Compare inductive reasoning and deductive reasoning.
- Use truth tables to analyze patterns of logic.

Review Queue

1. Write the converse, inverse, and contrapositive of the following statement:

Football players wear shoulder pads.

2. Is the converse, inverse or contrapositive of #1 true? If not, find a counterexample.

3. If flowers are in bloom, then it is spring.

If it is spring, then the weather is nice.

So, if flowers are blooming, what can we conclude?

Know What? In a fictitious far-away land, a poor peasant is awaiting his fate from the king. He is standing in a stadium, filled with spectators pointing and wondering what is going to happen. Finally, the king directs everyone's attention to two doors, at the floor level with the peasant. Both doors have signs on them, which are below:

TABLE 2.4:

Door A	Door B
IN THIS ROOM THERE IS A LADY, AND IN THE OTHER ROOM THERE IS A TIGER.	IN ONE OF THESE ROOMS THERE IS A LADY, AND IN ONE OF THE OTHER ROOMS THERE IS A TIGER.

The king states, "Only one of these statements is true. If you pick correctly, you will find the lady. If not, the tiger will be waiting for you." Which door should the peasant pick?

Deductive Reasoning

Logic: The study of reasoning.

In the first section, you learned about inductive reasoning, which is to make conclusions based upon patterns and observations. Now, we will learn about deductive reasoning. Deductive reasoning draws conclusions from facts.

Deductive Reasoning: When a conclusion is drawn from facts. Typically, conclusions are drawn from general statements about something more specific.

Example 1: Suppose Bea makes the following statements, which are known to be true.

If Central High School wins today, they will go to the regional tournament.

Central High School won today.

What is the logical conclusion?

Solution: This is an example of deductive reasoning. There is one logical conclusion if these two statements are true: *Central High School will go to the regional tournament.*

Example 2: Here are two true statements.

Every odd number is the sum of an even and an odd number.

5 is an odd number.

What can you conclude?

Solution: Based on only these two true statements, there is one conclusion: *5 is the sum of an even and an odd number.* (This is true, $5 = 3+2$ or $4+1$).

Law of Detachment

Let's look at Example 2 and change it into symbolic form.

$p:$ A number is odd $q:$ It is the sum of an even and odd number

So, the first statement is $p \rightarrow q$.

- The second statement in Example 2, "5 is an odd number," is a specific example of p. "A number" is 5.
- The conclusion is q. Again it is a specific example, such as $4+1$ or $2+3$.

The symbolic form of Example 2 is:

$$\begin{aligned} & p \rightarrow q \\ & p \\ & \therefore q \qquad \qquad \therefore \text{ symbol for "therefore"} \end{aligned}$$

All deductive arguments that follow this pattern have a special name, the Law of Detachment.

Law of Detachment: Suppose that $p \rightarrow q$ is a true statement and given p. Then, you can conclude q.

Another way to say the Law of Detachment is: "If $p \rightarrow q$ is true, and p is true, then q is true."

Example 3: Here are two true statements.

If $\angle A$ and $\angle B$ are a linear pair, then $m\angle A + m\angle B = 180^\circ$.

$\angle ABC$ and $\angle CBD$ are a linear pair.

What conclusion can you draw from this?

Solution: This is an example of the Law of Detachment, therefore:

$$m\angle ABC + m\angle CBD = 180^\circ$$

Example 4: Here are two true statements. *Be careful!*

If $\angle A$ and $\angle B$ are a linear pair, then $m\angle A + m\angle B = 180^\circ$.

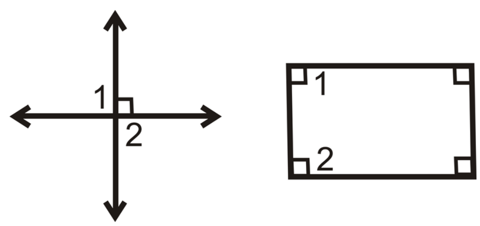
$m\angle 1 = 90^\circ$ and $m\angle 2 = 90^\circ$.

What conclusion can you draw from these two statements?

Solution: Here there is NO conclusion. These statements are in the form:

$$\begin{aligned} &p \rightarrow q \\ &q \end{aligned}$$

We ***cannot*** conclude that $\angle 1$ and $\angle 2$ are a linear pair. We are told that $m\angle 1 = 90^\circ$ and $m\angle 2 = 90^\circ$ and while $90^\circ + 90^\circ = 180^\circ$, this does not mean they are a linear pair. Here are two counterexamples.



In both of these counterexamples, $\angle 1$ and $\angle 2$ are right angles. In the first, they are vertical angles and in the second, they are two angles in a rectangle.

This is called the *Converse Error* because the second statement is the conclusion of the first, like the converse of a statement.

Law of Contrapositive

Example 5: The following two statements are true.

If a student is in Geometry, then he or she has passed Algebra I.

Daniel has not passed Algebra I.

What can you conclude from these two statements?

Solution: These statements are in the form:

$$\begin{aligned} &p \rightarrow q \\ &\sim q \end{aligned}$$

Not q is the beginning of the contrapositive $(\sim q \rightarrow \sim p)$, therefore the logical conclusion is *not p: Daniel is not in Geometry.*

This example is called the Law of Contrapositive.

Law of Contrapositive: Suppose that $p \rightarrow q$ is a true statement and given $\sim q$. Then, you can conclude $\sim p$.

Recall that the logical equivalent to a conditional statement is its contrapositive. Therefore, the Law of Contrapositive is a logical argument.

Example 6: Determine the conclusion from the true statements below.

Babies wear diapers.

My little brother does not wear diapers.

Solution: The second statement is the equivalent of $\sim q$. Therefore, the conclusion is $\sim p$, or: *My little brother is not a baby.*

Example 7a: Determine the conclusion from the true statements below.

If you are not in Chicago, then you can't be on the L.

Bill is in Chicago.

Solution: If we were to rewrite this symbolically, it would look like:

$$\begin{aligned} &\sim p \rightarrow \sim q \\ &p \end{aligned}$$

This is not in the form of the Law of Contrapositive or the Law of Detachment, so there is no logical conclusion. You cannot conclude that Bill is on the *L* because he could be anywhere in Chicago. This is an example of the *Inverse Error* because the second statement is the negation of the hypothesis, like the beginning of the inverse of a statement.

Example 7b: Determine the conclusion from the true statements below.

If you are not in Chicago, then you can't be on the L.

Sally is on the L.

Solution: If we were to rewrite this symbolically, it would look like:

$$\begin{aligned} &\sim p \rightarrow \sim q \\ &q \end{aligned}$$

Even though it looks a little different, this is an example of the Law of Contrapositive. Therefore, the logical conclusion is: *Sally is in Chicago.*

Law of Syllogism

Example 8: Determine the conclusion from the following true statements.

If Pete is late, Mark will be late.

If Mark is late, Karl will be late.

So, if Pete is late, what will happen?

Solution: If Pete is late, this starts a domino effect of lateness. Mark will be late and Karl will be late too. So, if Pete is late, then *Karl will be late*, is the logical conclusion.

Each "then" becomes the next "if" in a chain of statements. The chain can consist of any number of connected statements. This is called the Law of Syllogism

Law of Syllogism: If $p \rightarrow q$ and $q \rightarrow r$ are true, then $p \rightarrow r$ is the logical conclusion.

Typically, when there are more than two linked statements, we continue to use the next letter(s) in the alphabet to represent the next statement(s); $r \rightarrow s, s \rightarrow t$, and so on.

Example 9: Look back at the **Know What? Revisited** from the previous section. There were 12 linked if-then statements, making one LARGE Law of Syllogism. Write the conclusion from these statements.

Solution: Symbolically, the statements look like this:

$$\begin{array}{llllll} A \to B & B \to C & C \to D & D \to E & E \to F & F \to G \\ G \to H & H \to I & I \to J & J \to K & K \to L & L \to M \\ \therefore A \to M & & & & & \end{array}$$

So, *If the man raises his spoon, then his face is wiped with the napkin.*

Inductive vs. Deductive Reasoning

You have now worked with both inductive and deductive reasoning. They are different but not opposites. Inductive reasoning means reasoning from examples or patterns. Enough examples might make you suspect that a relationship is always true. But, until you go beyond the inductive stage, you can't be absolutely sure that it is always true. That is, you cannot ***prove*** something is true with inductive reasoning.

That's where deductive reasoning takes over. Let's say we have a conjecture that was arrived at inductively, but is not proven. We can use the Law of Detachment, Law of Contrapositive, Law of Syllogism, and other logic rules to prove this conjecture.

Example 10: Determine if the following statements are examples of inductive or deductive reasoning.

a) Solving an equation for x.

b) 1, 10, 100, 1000,...

c) Doing an experiment and writing a hypothesis.

Solution: Inductive Reasoning = Patterns, Deductive Reasoning = Logic from Facts

a) Deductive Reasoning: Each step follows from the next.

b) Inductive Reasoning: This is a pattern.

c) Inductive Reasoning: You make a hypothesis or conjecture comes from the patterns that you found in the experiment (not facts). If you were to *prove* your hypothesis, then you would have to use deductive reasoning.

Truth Tables

So far we know these symbols for logic:

$\sim$ not (negation)

$\to$ if-then

$\therefore$ therefore

Two more symbols are:

$\wedge$ and

$\vee$ or

We would write "p and q" as $p \wedge q$ and "p or q" as $p \vee q$.

Truth tables use these symbols and are another way to analyze logic.

First, let's relate p and $\sim p$. To make it easier, set p as: *An even number.*

Therefore, $\sim p$ is *An odd number*. Make a truth table to find out if they are both true. Begin with all the "truths" of p, true (T) or false (F).

TABLE 2.5:

p
T
F

Next we write the corresponding truth values for $\sim p$. $\sim p$ has the opposite truth values of p. So, if p is true, then $\sim p$ is false and vise versa.

TABLE 2.6:

p	$\sim p$
T	F
F	T

Example 11: Draw a truth table for p, q and $p \wedge q$.

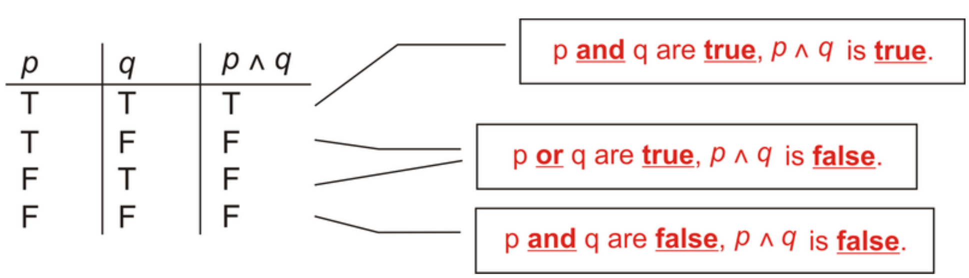
Solution: First, make columns for p and q. Fill the columns with all the possible true and false combinations for the two.

TABLE 2.7:

p	q
T	T
T	F
F	T
F	F

Notice all the combinations of p and q. **Anytime we have truth tables with two variables, this is always how we fill out the first two columns.**

Next, we need to figure out when $p \wedge q$ is true, based upon the first two columns. $p \wedge q$ **can only be true if BOTH** p **and** q **are true.** So, the completed table looks like this:



This is how a truth table with two variables and their "and" column is always filled out.

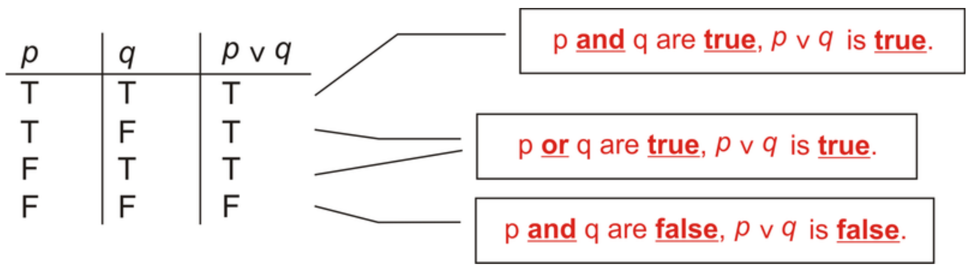
Example 12: Draw a truth table for p, q and $p \vee q$.

Solution: First, make columns for p and q, just like Example 11.

TABLE 2.8:

p	q
T	T
T	F
F	T
F	F

Next, we need to figure out when $p \vee q$ is true, based upon the first two columns. $p \vee q$ **is true if** p **OR** q **are true, or both are true.** So, the completed table looks like this:



The difference between $p \wedge q$ and $p \vee q$ is the second and third rows. For "and" both p and q have to be true, but for "or" only one has to be true.

Example 13: Determine the truths for $p \wedge (\sim q \vee r)$.

Solution: First, there are three variables, so we are going to need all the combinations of their truths. **For three variables, there are always 8 possible combinations.**

TABLE 2.9:

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Next, address the $\sim q$. It will just be the opposites of the q column.

TABLE 2.10:

p	q	r	$\sim q$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

Now, let's do what's in the parenthesis, $\sim q \vee r$. Remember, for "or" only $\sim q$ OR r has to be true. Only use the $\sim q$ and r columns to determine the values in this column.

TABLE 2.11:

p	q	r	$\sim q$	$\sim q \vee r$
T	T	T	F	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

Finally, we can address the entire problem, $p \wedge (\sim q \vee r)$. Use the p and $\sim q \vee r$ to determine the values. Remember, for "and" both p and $\sim q \vee r$ must be true.

TABLE 2.12:

p	q	r	$\sim q$	$\sim q \vee r$	$p \wedge (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	F	F	F
F	F	T	T	T	F
F	F	F	T	T	F

To Recap:

- Start truth tables with all the possible combinations of truths. For 2 variables there are 4 combinations for 3 variables there are 8. **You always start a truth table this way.**
- Do any negations on the any of the variables.
- Do any combinations in parenthesis.
- Finish with completing what the problem was asking for.

Know What? Revisited Analyze the two statements on the doors.

Door A: IN THIS ROOM THERE IS A LADY, AND IN THE OTHER ROOM THERE IS A TIGER.

Door B: IN ONE OF THESE ROOMS THERE IS A LADY, AND IN ONE OF THE OTHER ROOMS THERE IS A TIGER.

We know that one door is true, so the other one must be false. Let's assume that Door A is true. That means the lady is behind Door A and the tiger is behind Door B. However, if we read Door B carefully, it says "in one of these rooms," which means the lady could be behind either door, which is actually the true statement. So, because Door B is the true statement, Door A is false and the tiger is actually behind it. Therefore, the peasant should pick Door B.

Review Questions

Determine the logical conclusion and state which law you used (Law of Detachment, Law of Contrapositive, or Law of Syllogism). If no conclusion can be drawn, write "no conclusion."

1. People who vote for Jane Wannabe are smart people. I voted for Jane Wannabe.
2. If Rae is the driver today then Maria is the driver tomorrow. Ann is the driver today.
3. If a shape is a circle, then it never ends. If it never ends, then it never starts. If it never starts, then it doesn't exist. If it doesn't exist, then we don't need to study it.
4. If you text while driving, then you are unsafe. You are a safe driver.
5. If you wear sunglasses, then it is sunny outside. You are wearing sunglasses.
6. If you wear sunglasses, then it is sunny outside. It is cloudy.
7. I will clean my room if my mom asks me to. I am not cleaning my room.
8. If I go to the park, I bring my dog. If I bring my dog, we play fetch with a stick. If we play fetch, my dog gets dirty. If my dog gets dirty, I give him a bath.
9. Write the symbolic representation of #3. Include your conclusion. Is this a sound argument? Does it make sense?
10. Write the symbolic representation of #1. Include your conclusion.
11. Write the symbolic representation of #7. Include your conclusion.

For questions 12 and 13, rearrange the order of the statements (you may need to use the Law of Contrapositive too) to discover the logical conclusion.

12. If I shop, then I will buy shoes. If I don't shop, then I didn't go to the mall. If I need a new watch battery, then I go to the mall.
13. If Anna's parents don't buy her ice cream, then she didn't get an A on her test. If Anna's teacher gives notes, Anna writes them down. If Anna didn't get an A on her test, then she couldn't do the homework. If Anna writes down the notes, she can do the homework.

Determine if the problems below represent inductive or deductive reasoning. Briefly explain your answer.

14. John is watching the weather. As the day goes on it gets more and more cloudy and cold. He concludes that it is going to rain.
15. Beth's 2-year-old sister only eats hot dogs, blueberries and yogurt. Beth decides to give her sister some yogurt because she is hungry.
16. Nolan Ryan has the most strikeouts of any pitcher in Major League Baseball. Jeff debates that he is the best pitcher of all-time for this reason.
17. Ocean currents and waves are dictated by the weather and the phase of the moon. Surfers use this information to determine when it is a good time to hit the water.
18. As Rich is driving along the 405, he notices that as he gets closer to LAX the traffic slows down. As he passes it, it speeds back up. He concludes that anytime he drives past an airport, the traffic will slow down.
19. Amani notices that the milk was left out on the counter. Amani remembers that she put it away after breakfast so it couldn't be her who left it out. She also remembers hearing her mother tell her brother on several occasions to put the milk back in the refrigerator. She concludes that he must have left it out.
20. At a crime scene, the DNA of four suspects is found to be present. However, three of them have an alibi for the time of the crime. The detectives conclude that the fourth possible suspect must have committed the crime.

Write a truth table for the following variables.

21. $p \wedge \sim p$

22. $\sim p \vee \sim q$
23. $p \wedge (q \vee \sim q)$
24. $(p \wedge q) \vee \sim r$
25. $p \vee (\sim q \vee r)$
26. $p \wedge (q \vee \sim r)$
27. The only difference between 19 and 21 is the placement of the parenthesis. How does the truth table differ?
28. When is $p \vee q \vee r$ true?

Is the following a valid argument? If so, what law is being used? HINT: Statements could be out of order.

29. $p \rightarrow q$
    $r \rightarrow p$
    $\therefore r \rightarrow q$
30. $p \rightarrow q$
    $r \rightarrow q$
    $\therefore p \rightarrow r$
31. $p \rightarrow \sim r$
    r
    $\therefore \sim p$
32. $\sim q \rightarrow r$
    q
    $\therefore \sim r$
33. $p \rightarrow (r \rightarrow s)$
    p
    $\therefore r \rightarrow s$
34. $r \rightarrow q$
    $r \rightarrow s$
    $\therefore q \rightarrow s$

Review Queue Answers

1. Converse: If you wear shoulder pads, then you are a football player.

Inverse: If you are not a football player, then you do not wear shoulder pads.

Contrapositive: If you do not wear shoulder pads, then you are not a football player.

2. The converse and inverse are both false. A counterexample for both could be a woman from the 80's. They definitely wore shoulder pads!

3. You could conclude that the weather is nice.

2.4 Algebraic and Congruence Properties

Learning Objectives

- Understand basic properties of equality and congruence.
- Solve equations and justify each step in the solution.
- Use a 2-column format to prove theorems.

Review Queue

Solve the following problems.

1. *Explain* how you would solve $2x-3=9$.

2. *If two angles are a linear pair, they are supplementary.*

If two angles are supplementary, their sum is 180°.

What can you conclude? By which law?

3. Draw a picture with the following:

$\angle LMN$ is bisected by $\overline{MO}$	$\overline{LM} \cong \overline{MP}$
$\angle OMP$ is bisected by $\overline{MN}$	N is the midpoint of $\overline{MQ}$

Know What? Three identical triplets are sitting next to each other. The oldest is Sara and she always tells the truth. The next oldest is Sue and she always lies. Sally is the youngest of the three. She sometimes lies and sometimes tells the truth.

Scott came over one day and didn't know who was who, so he asked each of them one question. Scott asked the sister that was sitting on the left, "Which sister is in the middle?" and the answer he received was, "That's Sara." Scott then asked the sister in the middle, "What is your name?" The response given was, "I'm Sally." Scott turned to the sister on the right and asked, "Who is in the middle?" The sister then replied, "She is Sue." Who was who?

Properties of Equality

Recall from Chapter 1 that the = sign and the word "equality" are used with numbers.

The basic properties of equality were introduced to you in Algebra I. Here they are again:

<u>For all real numbers a, b, and c:</u>

TABLE 2.13:

		Examples
Reflexive Property of Equality	$a=a$	$25=25$
Symmetric Property of Equality	$a=b$ and $b=a$	$m\angle P=90^\circ$ or $90^\circ=m\angle P$

TABLE 2.13: (continued)

		Examples
Transitive Property of Equality	$a = b$ and $b = c$, then $a = c$	$a+4 = 10$ and $10 = 6+4$, then $a+4 = 6+4$
Substitution Property of Equality	If $a = b$, then b can be used in place of a and vise versa.	If $a = 9$ and $a-c = 5$, then $9-c = 5$
Addition Property of Equality	If $a = b$, then $a+c = b+c$.	If $2x = 6$, then $2x+5 = 6+11$
Subtraction Property of Equality	If $a = b$, then $a-c = b-c$.	If $m\angle x + 15^\circ = 65^\circ$, then $m\angle x + 15^\circ - 15^\circ = 65^\circ - 15^\circ$
Multiplication Property of Equality	If $a = b$, then $ac = bc$.	If $y = 8$, then $5 \cdot y = 5 \cdot 8$
Division Property of Equality	If $a = b$, then $\frac{a}{c} = \frac{b}{c}$.	If $3b = 18$, then $\frac{3b}{3} = \frac{18}{3}$
Distributive Property	$a(b+c) = ab+ac$	$5(2x-7) = 5(2x)-5(7) = 10x-35$

Properties of Congruence

Recall that $\overline{AB} \cong \overline{CD}$ if and only if $AB = CD$. $\overline{AB}$ and $\overline{CD}$ represent segments, while AB and CD are lengths of those segments, which means that AB and CD are numbers. The properties of equality apply to AB and CD.

This also holds true for angles and their measures. $\angle ABC \cong \angle DEF$ if and only if $m\angle ABC = m\angle DEF$. Therefore, the properties of equality apply to $m\angle ABC$ and $m\angle DEF$.

Just like the properties of equality, there are properties of congruence. These properties hold for figures and shapes.

TABLE 2.14:

	For Line Segments	*For Angles*
Reflexive Property of Congruence	$\overline{AB} \cong \overline{AB}$	$\angle ABC \cong \angle CBA$
Symmetric Property of Congruence	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$	If $\angle ABC \cong \angle DEF$, then $\angle DEF \cong \angle ABC$
Transitive Property of Congruence	If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$	If $\angle ABC \cong \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle GHI$

Using Properties of Equality with Equations

When you solve equations in algebra you use properties of equality. You might not write out the logical justification for each step in your solution, but you should know that there is an equality property that justifies that step. We will abbreviate "Property of Equality" "PoE" and "Property of Congruence" "PoC."

Example 1: Solve $2(3x-4)+11 = x-27$ and justify each step.

Solution:

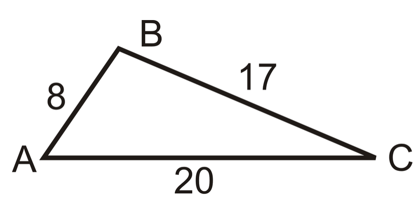
$$
\begin{aligned}
2(3x-4)+11 &= x-27 && \\
6x-8+11 &= x-27 && \text{Distributive Property} \\
6x+3 &= x-27 && \text{Combine like terms} \\
6x+3-3 &= x-27-3 && \text{Subtraction PoE} \\
6x &= x-30 && \text{Simplify} \\
6x-x &= x-x-30 && \text{Subtraction PoE} \\
5x &= -30 && \text{Simplify} \\
\frac{5x}{5} &= \frac{-30}{5} && \text{Division PoE} \\
x &= -6 && \text{Simplify}
\end{aligned}
$$

Example 2: Given points A, B, and C, with $AB = 8, BC = 17$, and $AC = 20$. Are A, B, and C collinear?

Solution: Set up an equation using the Segment Addition Postulate.

$$
\begin{aligned}
AB+BC &= AC && \text{Segment Addition Postulate} \\
8+17 &= 20 && \text{Substitution PoE} \\
25 &\neq 20 && \text{Combine like terms}
\end{aligned}
$$

Because the two sides are not equal, A, B and C are not collinear.



Example 3: If $m\angle A + m\angle B = 100^\circ$ and $m\angle B = 40^\circ$, prove that $\angle A$ is an acute angle.

Solution: We will use a 2-column format, with statements in one column and their corresponding reasons in the next. This is formally called a 2-column proof.

TABLE 2.15:

Statement	***Reason***
1. $m\angle A + m\angle B = 100^\circ$ and $m\angle B = 40^\circ$	***Given*** (always the reason for using facts that are told to us in the problem)
2. $m\angle A + 40^\circ = 100^\circ$	Substitution PoE
3. $m\angle A = 60^\circ$	Subtraction PoE
4. $\angle A$ is an acute angle	Definition of an acute angle, $m\angle A < 90^\circ$

Two-Column Proof

Example 4: Write a two-column proof for the following:

If A, B, C, and D are points on a line, in the given order, and $AB = CD$, then $AC = BD$.

Solution: First of all, when the statement is given in this way, the "if" part is the given and the "then" part is what we are trying to prove.

Always start with drawing a picture of what you are given.

Plot the points in the order A, B, C, D on a line.

Add the corresponding markings, $AB = CD$, to the line.

Draw the 2-column proof and start with the given information. From there, we can use deductive reasoning to reach the next statement and what we want to prove. **Reasons will be definitions, postulates, properties and previously proven theorems.**

TABLE 2.16:

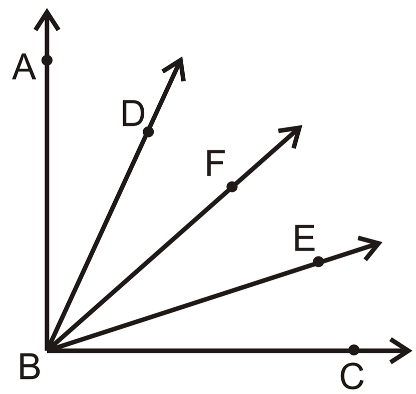
Statement	***Reason***
1. A, B, C, and D are collinear, in that order.	Given
2. $AB = CD$	Given
3. $BC = BC$	Reflexive PoE
4. $AB + BC = BC + CD$	Addition PoE
5. $AB + BC = AC$	
BC + CD = BD	Segment Addition Postulate
6. $AC = BD$	Substitution or Transitive PoE

When you reach what it is that you wanted to prove, you are done.

Prove Move: (A subsection that will help you with proofs throughout the book.) When completing a proof, a few things to keep in mind:

- **Number each step.**
- **Start with the given information.**
- **Statements with the same reason can (or cannot) be combined into one step.** It is up to you. For example, steps 1 and 2 above could have been one step. And, in step 5, the two statements could have been written separately.
- **Draw a picture and mark it with the given information.**
- **You must have a reason for EVERY statement.**
- **The order of the statements in the proof is not fixed.** For example, steps 3, 4, and 5 could have been interchanged and it would still make sense.

Example 5: Write a two-column proof.



Given: $\overrightarrow{BF}$ bisects $\angle ABC$; $\angle ABD \cong \angle CBE$

Prove: $\angle DBF \cong \angle EBF$

Solution: First, put the appropriate markings on the picture. Recall, that bisect means "to cut in half." Therefore, if $\overrightarrow{BF}$ bisects $\angle ABC$, then $m\angle ABF = m\angle FBC$. Also, because the word "bisect" was used in the given, the definition will probably be used in the proof.

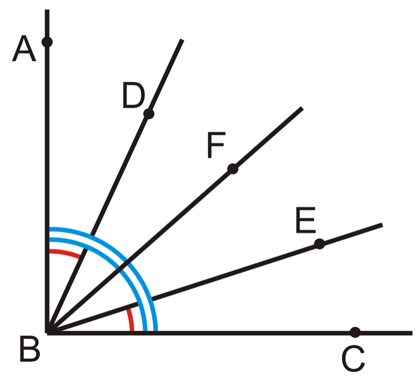


TABLE 2.17:

Statement	*Reason*
1. $\overrightarrow{BF}$ bisects $\angle ABC, \angle ABD \cong \angle CBE$	Given
2. $m\angle ABF = m\angle FBC$	Definition of an Angle Bisector
3. $m\angle ABD = m\angle CBE$	If angles are $\cong$, then their measures are equal.
4. $m\angle ABF = m\angle ABD + m\angle DBF$	
m $\angle FBC = m\angle EBF + m\angle CBE$	Angle Addition Postulate
5. $m\angle ABD + m\angle DBF = m\angle EBF + m\angle CBE$	Substitution PoE
6. $m\angle ABD + m\angle DBF = m\angle EBF + m\angle ABD$	Substitution PoE
7. $m\angle DBF = m\angle EBF$	Subtraction PoE
8. $\angle DBF \cong \angle EBF$	If measures are equal, the angles are $\cong$.

Prove Move: Use symbols and abbreviations for words within proofs. For example, $\cong$ was used in place of the word *congruent* above. You could also use $\angle$ for the word *angle.*

Know What? Revisited The sisters, in order are: Sally, Sue, Sara. The sister on the left couldn't have been Sara because that sister lied. The middle one could not be Sara for the same reason. So, the sister on the right must be Sara, which means she told Scott the truth and Sue is in the middle, leaving Sally to be the sister on the left.

Review Questions

For questions 1-8, solve each equation and justify each step.

1. $3x+11=-16$
2. $7x-3=3x-35$
3. $\frac{2}{3}g+1=19$
4. $\frac{1}{2}MN=5$
5. $5m\angle ABC=540^\circ$
6. $10b-2(b+3)=5b$
7. $\frac{1}{4}y+\frac{5}{6}=\frac{1}{3}$
8. $\frac{1}{4}AB+\frac{1}{3}AB=12+\frac{1}{2}AB$

For questions 9-14, use the given property or properties of equality to fill in the blank. x, y, and z are real numbers.

9. Symmetric: If $x=3$, then __________.
10. Distributive: If $4(3x-8)$, then __________.
11. Transitive: If $y=12$ and $x=y$, then __________.
12. Symmetric: If $x+y=y+z$, then __________.
13. Transitive: If $AB=5$ and $AB=CD$, then __________.
14. Substitution: If $x=y-7$ and $x=z+4$, then __________.
15. Given points E, F, and G and $EF=16$, $FG=7$ and $EG=23$. Determine if E, F and G are collinear.
16. Given points H, I and J and $HI=9, IJ=9$ and $HJ=16$. Are the three points collinear? Is I the midpoint?
17. If $m\angle KLM=56^\circ$ and $m\angle KLM+m\angle NOP=180^\circ$, explain how $\angle NOP$ must be an obtuse angle.

Fill in the blanks in the proofs below.

18. Given: $\angle ABC \cong DEF$
$\angle GHI \cong \angle JKL$ Prove: $m\angle ABC+m\angle GHI=m\angle DEF+m\angle JKL$

TABLE 2.18:

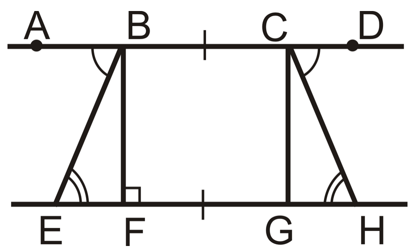
Statement	***Reason***
1.	Given
2. $m\angle ABC=m\angle DEF$ m $\angle GHI=m\angle JKL$	
3.	Addition PoE
4. $m\angle ABC+m\angle GHI=m\angle DEF+m\angle JKL$	

19. Given: M is the midpoint of $\overline{AN}$. N is the midpoint $\overline{MB}$ Prove: $AM=NB$

TABLE 2.19:

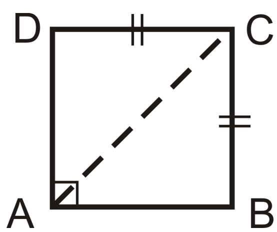
Statement	***Reason***
1.	Given
2.	Definition of a midpoint
3. $AM=NB$	

Use the diagram to answer questions 20-25.



20. Name a right angle.
21. Name two perpendicular lines.
22. Given that $EF = GH$, is $EG = FH$ true? Explain your answer.
23. Is $\angle CGH$ a right angle? Why or why not?
24. Using what is given in the picture AND $\angle EBF \cong \angle HCG$, prove $\angle ABF \cong \angle DCG$. Write a two-column proof.
25. Using what is given in the picture AND $AB = CD$, prove $AC = BD$. Write a two-column proof.

Use the diagram to answer questions 26-32.

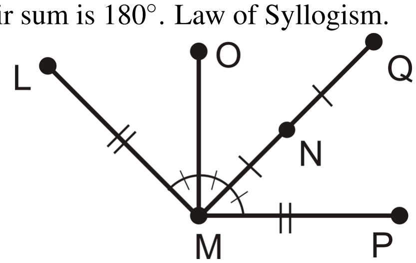


Which of the following must be true from the diagram?

Take each question separately, they do not build upon each other.

26. $\overline{AD} \cong \overline{BC}$
27. $\overline{AB} \cong \overline{CD}$
28. $\overline{CD} \cong \overline{BC}$
29. $\overline{AB} \perp \overline{AD}$
30. $ABCD$ is a square
31. $\overline{AC}$ bisects $\angle DAB$
32. Write a two-column proof. Given: Picture above and $\overline{AC}$ bisects $\angle DAB$Prove: $m\angle BAC = 45^\circ$
33. Draw a picture and write a two-column proof. Given: $\angle 1$ and $\angle 2$ form a linear pair and $m\angle 1 = m\angle 2$. Prove: $\angle 1$ is a right angle

Review Queue Answers

a. First, subtract 3 from both sides and then divide both sides by 2. $x = 3$
b. If 2 angles are a linear pair, then their sum is 180°. Law of Syllogism.
c. 

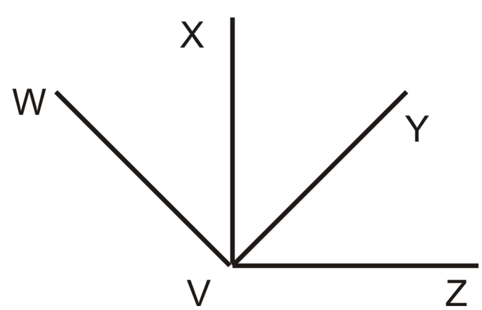
2.5 Proofs about Angle Pairs and Segments

Learning Objectives

- Use theorems about special pairs of angles.
- Use theorems about right angles and midpoints.

Review Queue

Write a 2-column proof

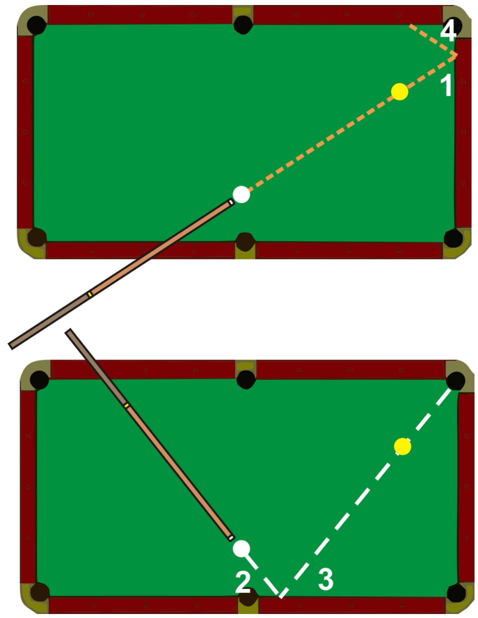


1. Given: $\overline{VX}$ is the angle bisector of $\angle WVY$.

$\overline{VY}$ is the angle bisector of $\angle XVZ$.

Prove: $\angle WVX \cong \angle YVZ$

Know What? The game of pool relies heavily on angles. The angle at which you hit the cue ball with your cue determines if a) you hit the yellow ball and b) if you can hit it into a pocket.

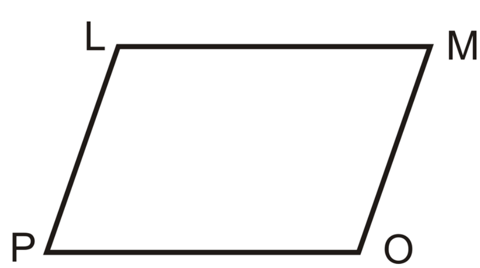


The top picture on the right illustrates if you were to hit the cue ball straight on and then hit the yellow ball. The orange line shows the path that the cue ball and then the yellow ball would take. You notice that $m\angle 1 = 56^\circ$. With a little focus, you notice that it makes more sense to approach the ball from the other side of the table and bank it off of the opposite side (see lower picture with the white path). You measure and need to hit the cue ball so that it hits the side of the table at a 50° angle (this would be $m\angle 2$). $\angle 3$ and $\angle 4$ are called the angles of reflection. Find the measures of these angles and how they relate to $\angle 1$ and $\angle 2$.

If you would like to play with the angles of pool, click the link for an interactive game. http://www.coolmath-games.com/0-poolgeometry/index.html

Naming Angles

As we learned in Chapter 1, angles can be addressed by numbers and three letters, where the letter in the middle is the vertex. We can shorten this label to one letter if there is only one angle with that vertex.



All of the angles in this parallelogram can be labeled by one letter, the vertex, instead of three.

$\angle MLP$ can be $\angle L$ $\quad$ $\angle LMO$ can be $\angle M$
$\angle MOP$ can be $\angle O$ $\quad$ $\angle OPL$ can be $\angle P$

This shortcut will now be used when applicable.

Right Angle Theorem: If two angles are right angles, then the angles are congruent.

Proof of the Right Angle Theorem

Given: $\angle A$ and $\angle B$ are right angles

Prove: $\angle A \cong \angle B$

TABLE 2.20:

Statement	*Reason*
1. $\angle A$ and $\angle B$ are right angles	Given
2. $m\angle A = 90^\circ$ and $m\angle B = 90^\circ$	Definition of right angles
3. $m\angle A = m\angle B$	Transitive PoE
4. $\angle A \cong \angle B$	$\cong$ angles have = measures

This theorem may seem redundant, but anytime right angles are mentioned, you need to use this theorem to say the angles are congruent.

Same Angle Supplements Theorem: If two angles are supplementary to the same angle then the angles are congruent.

So, if $m\angle A + m\angle B = 180^\circ$ and $m\angle C + m\angle B = 180^\circ$, then $m\angle A = m\angle C$. Using numbers to illustrate, we could say that if $\angle A$ is supplementary to an angle measuring 56°, then $m\angle A = 124^\circ$. $\angle C$ is also supplementary to 56°, so it too is 124°. Therefore, $m\angle A = m\angle C$. This example, however, does not constitute a proof.

Proof of the Same Angles Supplements Theorem

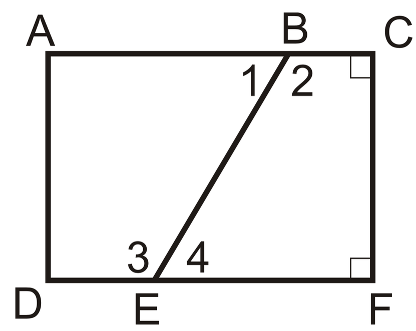
Given: $\angle A$ and $\angle B$ are supplementary angles. $\angle B$ and $\angle C$ are supplementary angles.

Prove: $\angle A \cong \angle C$

TABLE 2.21:

Statement	*Reason*
1. $\angle A$ and $\angle B$ are supplementary $\angle B$ and $\angle C$ are supplementary	Given
2. $m\angle A + m\angle B = 180^\circ$	
m $\angle B + m\angle C = 180^\circ$	Definition of supplementary angles
3. $m\angle A + m\angle B = m\angle B + m\angle C$	Substitution PoE
4. $m\angle A = m\angle C$	Subtraction PoE
5. $\angle A \cong \angle C$	$\cong$ angles have = measures

Example 1: Given that $\angle 1 \cong \angle 4$ and $\angle C$ and $\angle F$ are right angles, show which angles are congruent.



Solution: By the Right Angle Theorem, $\angle C \cong \angle F$. Also, $\angle 2 \cong \angle 3$ by the Same Angles Supplements Theorem. $\angle 1$ and $\angle 2$ are a linear pair, so they add up to 180°. $\angle 3$ and $\angle 4$ are also a linear pair and add up to 180°. Because $\angle 1 \cong \angle 4$, we can substitute $\angle 1$ in for $\angle 4$ and then $\angle 2$ and $\angle 3$ are supplementary to the same angle, making them congruent.

This is an example of a **paragraph proof**. Instead of organizing the proof in two columns, you explain everything in sentences.

Same Angle Complements Theorem: If two angles are complementary to the same angle then the angles are congruent.

So, if $m\angle A + m\angle B = 90^\circ$ and $m\angle C + m\angle B = 90^\circ$, then $m\angle A = m\angle C$. Using numbers, we could say that if $\angle A$ is supplementary to an angle measuring 56°, then $m\angle A = 34^\circ$. $\angle C$ is also supplementary to 56°, so it too is 34°. Therefore, $m\angle A = m\angle C$.

The proof of the Same Angles Complements Theorem is in the Review Questions. Use the proof of the Same Angles Supplements Theorem to help you.

Vertical Angles Theorem

Recall the Vertical Angles Theorem from Chapter 1. We will do a formal proof here.

Given: Lines k and m intersect.

Prove: $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$

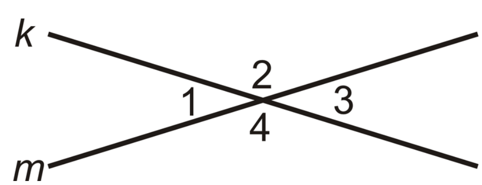


TABLE 2.22:

Statement	***Reason***
1. Lines k and m intersect	Given
2. $\angle 1$ and $\angle 2$ are a linear pair $\angle 2$ and $\angle 3$ are a linear pair $\angle 3$ and $\angle 4$ are a linear pair	Definition of a Linear Pair
3. $\angle 1$ and $\angle 2$ are supplementary $\angle 2$ and $\angle 3$ are supplementary $\angle 3$ and $\angle 4$ are supplementary	Linear Pair Postulate
4. $m\angle 1 + m\angle 2 = 180^\circ$ m $\angle 2 + m\angle 3 = 180^\circ$	

TABLE 2.22: (continued)

Statement	Reason
m $\angle 3+m\angle 4=180^\circ$	Definition of Supplementary Angles
5. $m\angle 1+m\angle 2=m\angle 2+m\angle 3$	
m $\angle 2+m\angle 3=m\angle 3+m\angle 4$	Substitution PoE
6. $m\angle 1=m\angle 3, m\angle 2=m\angle 4$	Subtraction PoE
7. $\angle 1\cong\angle 3, \angle 2\cong\angle 4$	$\cong$ angles have = measures

In this proof we combined everything. You could have done two separate proofs, one for $\angle 1\cong\angle 3$ and one for $\angle 2\cong\angle 4$.

Example 2: In the picture $\angle 2\cong\angle 3$ and $k\perp p$.

Each pair below is congruent. State why.

a) $\angle 1$ and $\angle 5$

b) $\angle 1$ and $\angle 4$

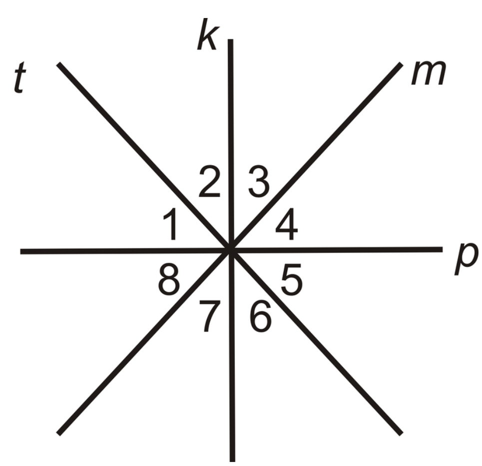
c) $\angle 2$ and $\angle 6$

d) $\angle 3$ and $\angle 7$

e) $\angle 6$ and $\angle 7$

f) $\angle 3$ and $\angle 6$

g) $\angle 4$ and $\angle 5$



Solution:

a), c) and d) Vertical Angles Theorem

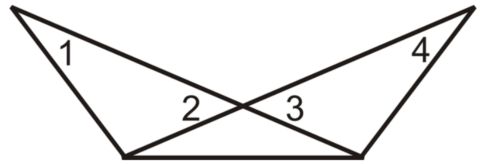
b) and g) Same Angles Complements Theorem

e) and f) Vertical Angles Theorem followed by the Transitive Property

Example 3: Write a two-column proof.

<u>Given:</u> $\angle 1\cong\angle 2$ and $\angle 3\cong\angle 4$

<u>Prove:</u> $\angle 1\cong\angle 4$



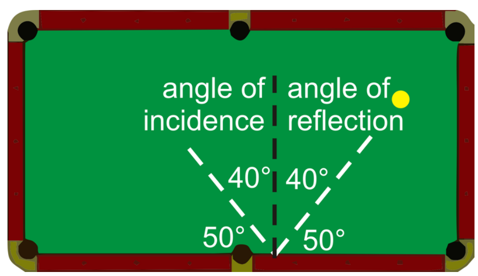
Solution:

TABLE 2.23:

Statement	*Reason*
1. $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$	Given
2. $\angle 2 \cong \angle 3$	Vertical Angles Theorem
3. $\angle 1 \cong \angle 4$	Transitive PoC

Know What? Revisited If $m\angle 2 = 50^\circ$, then

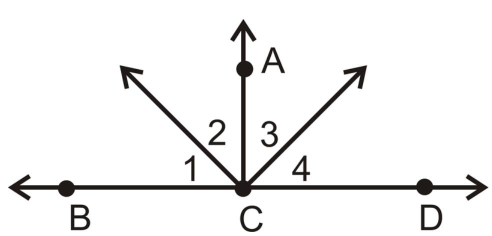
$m\angle 3 = 50^\circ$. Draw a perpendicular line at the point of reflection and the laws of reflection state that the angle of incidence is equal to the angle of reflection. So, this is a case of the Same Angles Complements Theorem. $\angle 2 \cong \angle 3$ because the angle of incidence and the angle of reflection are equal. We can also use this to find $m\angle 4$, which is 56°.



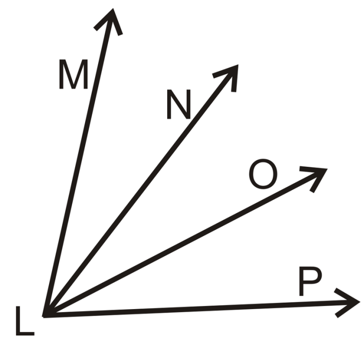
Review Questions

Write a two-column proof for questions 1-10.

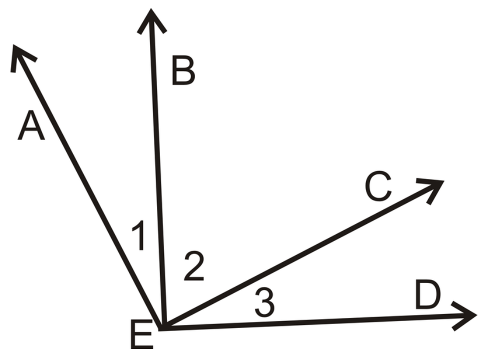
1. Given: $\overline{AC} \perp \overline{BD}$ and $\angle 1 \cong \angle 4$Prove: $\angle 2 \cong \angle 3$



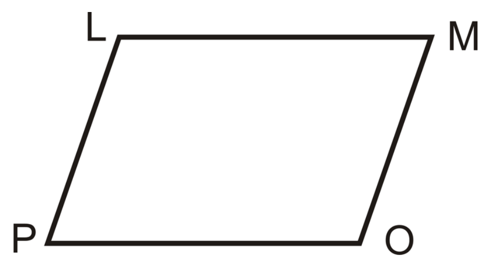
2. Given: $\angle MLN \cong \angle OLP$Prove: $\angle MLO \cong \angle NLP$



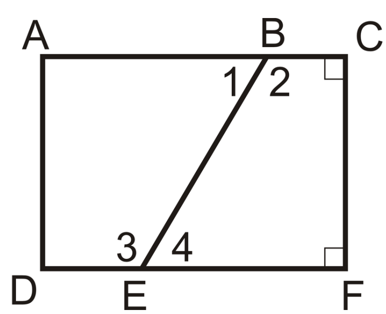
3. Given: $\overline{AE} \perp \overline{EC}$ and $\overline{BE} \perp \overline{ED}$Prove: $\angle 1 \cong \angle 3$



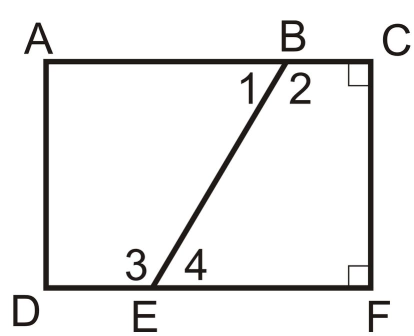
4. Given: $\angle L$ is supplementary to $\angle M \angle P$ is supplementary to $\angle O \angle L \cong \angle O$Prove: $\angle P \cong \angle M$



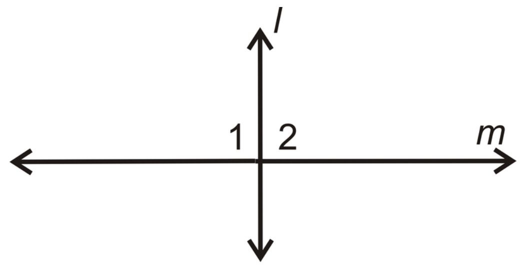
5. Given: $\angle 1 \cong \angle 4$Prove: $\angle 2 \cong \angle 3$



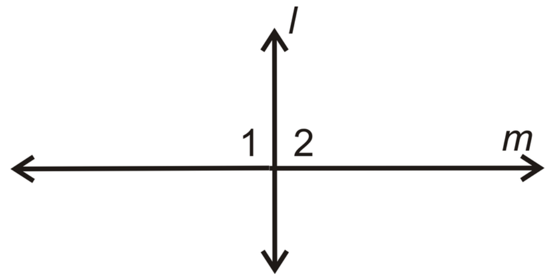
6. Given: $\angle C$ and $\angle F$ are right angles Prove: $m\angle C + m\angle F = 180^\circ$



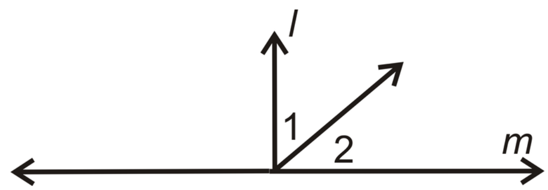
7. Given: $l \perp m$ Prove: $\angle 1 \cong \angle 2$



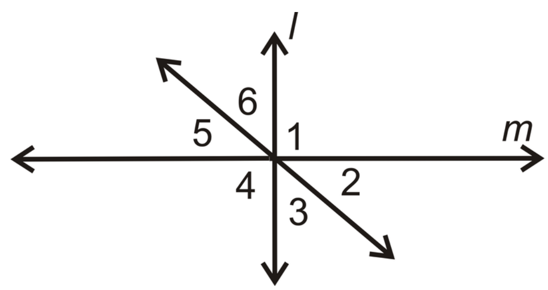
8. Given: $m\angle 1 = 90^\circ$ Prove: $m\angle 2 = 90^\circ$



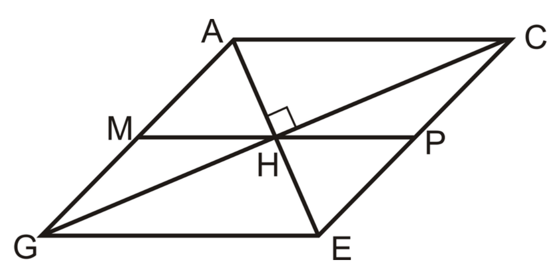
9. Given: $l \perp m$ Prove: $\angle 1$ and $\angle 2$ are complements



10. Given: $l \perp m \angle 2 \cong \angle 6$ Prove: $\angle 6 \cong \angle 5$



Use the picture for questions 11-20.



Given: H is the midpoint of $\overline{AE}, \overline{MP}$ and $\overline{GC}$

M is the midpoint of $\overline{GA}$

P is the midpoint of $\overline{CE}$

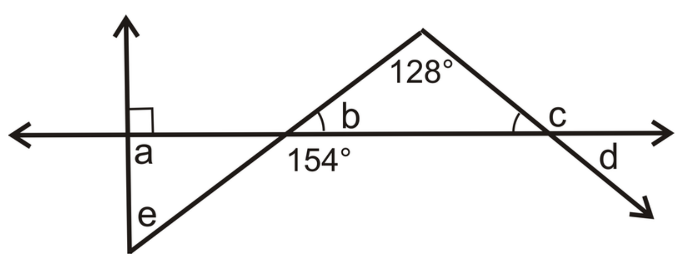
$\overline{AE} \perp \overline{GC}$

11. List two pairs of vertical angles.
12. List all the pairs of congruent segments.
13. List two linear pairs that do not have H as the vertex.
14. List a right angle.
15. List two pairs of adjacent angles that are NOT linear pairs.
16. What is the perpendicular bisector of $\overline{AE}$?
17. List two bisectors of $\overline{MP}$.
18. List a pair of complementary angles.
19. If $\overline{GC}$ is an angle bisector of $\angle AGE$, what two angles are congruent?
20. Fill in the blanks for the proof below. Given: Picture above and $\angle ACH \cong \angle ECH$Prove: $\overline{CH}$ is the angle bisector of $\angle ACE$

TABLE 2.24:

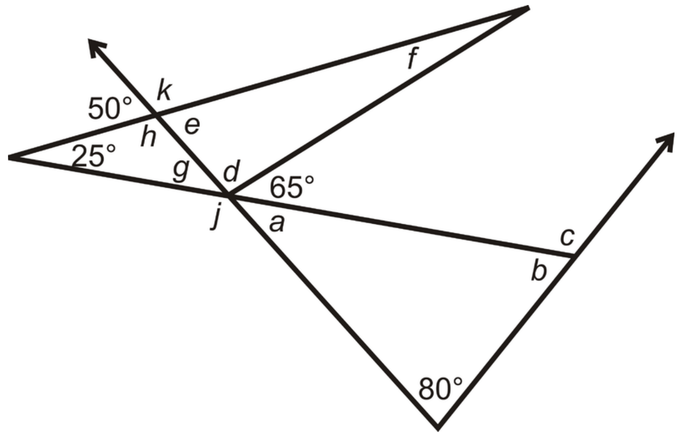
Statement	***Reason***
1. $\angle ACH \cong \angle ECH$	
$\overline{CH}$ is on the interior of $\angle ACE$	
2. $m\angle ACH = m\angle ECH$	
3.	Angle Addition Postulate
4.	Substitution
5. $m\angle ACE = 2m\angle ACH$	
6.	Division PoE
7.	

For questions 21-25, find the measure of the lettered angles in the picture below.



21. a
22. b
23. c
24. d
25. e (hint: e is complementary to b)

For questions 26-35, find the measure of the lettered angles in the picture below. *Hint: Recall the sum of the three angles in a triangle is* 180°.



26. a
27. b
28. c
29. d
30. e
31. f
32. g
33. h
34. j
35. k

Review Queue Answers

1.

TABLE 2.25:

Statement	*Reason*
1. $\overline{VX}$ is an $\angle$ bisector of $\angle WVY$	Given
$\overline{VY}$ is an $\angle$ bisector of $\angle XVZ$	
2. $\angle WVX \cong \angle XVY$	Definition of an angle bisector
$\angle XVY \cong \angle YVZ$	
3. $\angle WVX \cong \angle YVZ$	Transitive Property

2.6 Chapter 2 Review

Symbol Toolbox

$\rightarrow$ if-then

$\wedge$ and

$\therefore$ therefore

$\sim$ not

$\vee$ or

Keywords

Inductive Reasoning
The study of patterns and relationships is a part of mathematics. The conclusions made from looking at patterns are called **conjectures.** Looking for patterns and making conjectures is a part of **inductive reasoning,** where a rule or statement is assumed true because specific cases or examples are true.

Conjecture
The study of patterns and relationships is a part of mathematics. The conclusions made from looking at patterns are called **conjectures.**

Counterexample
We can disprove a conjecture or theory by coming up with a **counterexample.** Called proof by contradiction, only one counterexample is needed to disprove a conjecture or theory (no number of examples will prove a conjecture). The counterexample can be a drawing, statement, or number.

Conditional Statement (If-Then Statement)
Geometry uses **conditional statements** that can be symbolically written as $p \rightarrow q$ (read as "if p, then q"). "If" is the **hypothesis,** and "then" is the **conclusion.**

Hypothesis
The conditional statement is false when the hypothesis is true and the conclusion is false.

Conclusion
The second, or "then," part of a conditional statement. The conclusion is the result of a hypothesis.

Converse
A statement where the hypothesis and conclusion of a conditional statement are switched.

Inverse
A statement where the hypothesis and conclusion of a conditional statement are negated.

Contraposition

A statement where the hypothesis and conclusion of a conditional statement are exchanged and negated.

Biconditional Statement

If $p \rightarrow\ q$ is true and $q \rightarrow\ p$ is true, it can be written as $p \rightarrow\ q$.

If p is not true, then we cannot conclude q is true.

If we are given q, we cannot make a conculsion. We cannot conclude p is true.

Logic

The study of reasoning.

Deductive Reasoning

Uses logic and facts to prove the relationship is always true.

Law of Detachment

The *Law of Detachment* states: If $p\ q$ is true and p is true, then q is true.

If p is not true, then we cannot conclude q is true

If we are given q, we cannot make a conclusion. We cannot conclude p is true.

Law of Contrapositive

If the conditional statement is true, the converse and inverse may or may not be true. However, the contrapositive of a true statement is always true. The contrapositive is logically equivalent to the original conditional statement.

Law of Syllogism

The *Law of Syllogism* states: If $p \rightarrow\ q$ and $q \rightarrow\ r$ are true, then $p \rightarrow\ r$ is true.

Right Angle Theorem

If two angles are right angles, then the angles are congruent.

Same Angle Supplements Theorem

If two angles are supplementary to the same angle (or to congruent angles), then the angles are congruent.

Same Angle Complements Theorem

If two angles are complementary to the same angle (or to congruent angles), then the angles are congruent.

Reflexive Property of Equality

$a = a$.

Symmetric Property of Equality

$a = b$ and $b = a$.

Transitive Property of Equality

$a = b$ and $b = c$, then $a = c$.

Substitution Property of Equality

If $a = b$, then b can be used in place of a and vise versa.

Addition Property of Equality

If $a = b$, then $a + c = b + c$.

Subtraction Property of Equality

If $a = b$, then $a - c = b - c$.

Multiplication Property of Equality

If $a = b$, then $ac = bc$.

Division Property of Equality

If $a = b$, then $a \div c = b \div c$.

Distributive Property

$a(b + c) = ab + ac$.

Reflexive Property of Congruence

For Line Segments $\overline{AB} \cong \overline{AB}$ For Angles $\overline{AB} \cong \angle ABC \cong \angle CBA$

Symmetric Property of Congruence

For Line Segments If $\overline{AB} \cong CD$, then $\overline{CD} \cong \overline{AB}$ For Angles $\overline{CD} \cong \overline{AB}$ If $\angle ABC \cong \angle DFF \cong \angle ABC$

Transitive Property of Congruence

For Line Segments If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$ For Angles If $\angle ABC \cong \angle DEF$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle GHI$

Review

Match the definition or description with the correct word.

1. $5 = x$ and $y + 4 = x$, then $5 = y + 4$ — A. Law of Contrapositive
2. An educated guess — B. Inductive Reasoning
3. $6(2a + 1) = 12a + 12$ — C. Inverse
4. 2, 4, 8, 16, 32,... — D. Transitive Property of Equality
5. $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{AB}$ — E. Counterexample
6. $\sim p \rightarrow \sim q$ — F. Conjecture
7. Conclusions drawn from facts. — G. Deductive Reasoning
8. If I study, I will get an "A" on the test. I did not get an A. Therefore, I didn't study. — H. Distributive Property
9. $\angle A$ and $\angle B$ are right angles, therefore $\angle A \cong \angle B$. — I. Symmetric Property of Congruence
10. 2 disproves the statement: "All prime numbers are odd." — J. Right Angle Theorem — K. Definition of Right Angles

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9687 .

CHAPTER 3 Parallel and Perpendicular Lines

Chapter Outline

In this chapter, you will explore the different relationships formed by parallel and perpendicular lines and planes. Different angle relationships will also be explored and what happens to these angles when lines are parallel. You will continue to use proofs, to prove that lines are parallel or perpendicular. There will also be a review of equations of lines and slopes and how we show algebraically that lines are parallel and perpendicular.

3.1 Lines and Angles

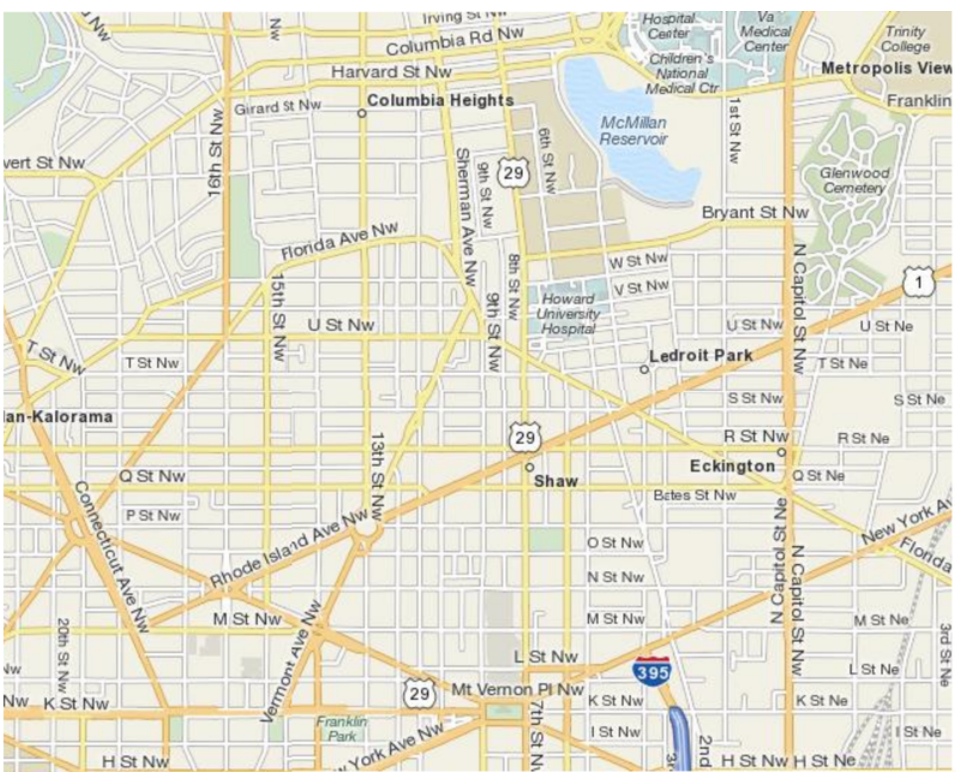
Learning Objectives

- Identify parallel lines, skew lines, and parallel planes.
- Use the Parallel Line Postulate and the Perpendicular Line Postulate.
- Identify angles made by transversals.

Review Queue

a. What is the equation of a line with slope -2 and passes through the point (0, 3)?
b. What is the equation of the line that passes through (3, 2) and (5, -6).
c. Change $4x - 3y = 12$ into slope-intercept form.
d. Are $y = \frac{1}{3}x$ and $y = -3x$ perpendicular? How do you know?

Know What? To the right is a partial map of Washington DC. The streets are designed on a grid system, where lettered streets, A through Z run east to west and numbered streets 1^{st} to 30^{th} run north to south. Just to mix things up a little, every state has its own street that runs diagonally through the city. There are, of course other street names, but we will focus on these three groups for this chapter. Can you explain which streets are parallel and perpendicular? Are any skew? How do you know these streets are parallel or perpendicular?

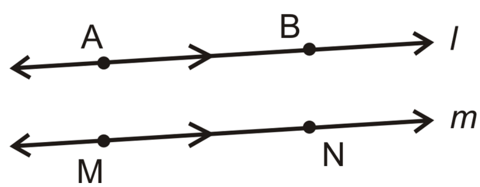


If you are having trouble viewing this map, check out the interactive map here: http://www.travelguide.tv/washington/map.html

Defining Parallel and Skew

Parallel: When two or more lines lie in the same plane and never intersect.

The symbol for parallel is $||$. To mark lines parallel, draw arrows ($>$) on each parallel line. If there are more than one pair of parallel lines, use two arrows ($>>$) for the second pair. The two lines to the right would be labeled $\overleftrightarrow{AB} \, || \, \overleftrightarrow{MN}$ or $l \, || \, m$.

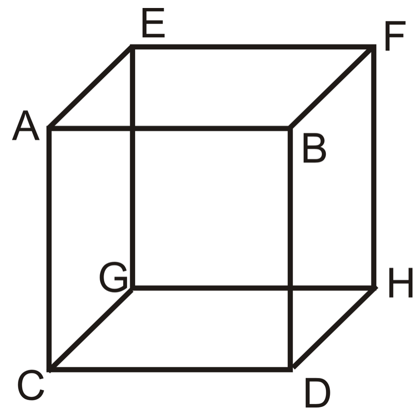


Planes can also be parallel or perpendicular. The image to the left shows two parallel planes, with a third blue plane that is perpendicular to both of them.

An example of parallel planes could be the top of a table and the floor. The legs would be in perpendicular planes to the table top and the floor.

Skew lines: Lines that are in different planes and never intersect.

Example 1: In the cube above, list:



a) 3 pairs of parallel planes

b) 2 pairs of perpendicular planes

c) 3 pairs of skew line segments

Solution:

a) Planes ABC and EFG, Planes AEG and FBH, Planes AEB and CDH

b) Planes ABC and CDH, Planes AEB and FBH (there are others, too)

c) $\overline{BD}$ and $\overline{CG}$, $\overline{BF}$ and $\overline{EG}$, $\overline{GH}$ and $\overline{AE}$ (there are others, too)

Parallel Line Postulate

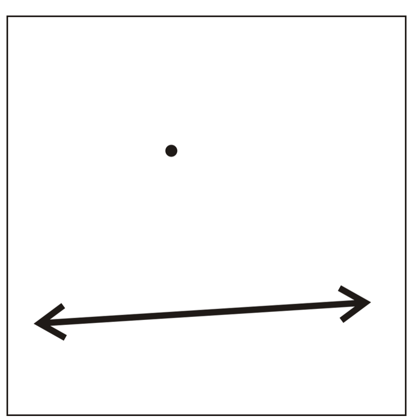
Parallel Line Postulate: Fora line and a point not on the line, there is exactly one line parallel to this line through the point.

There are infinitely many lines that pass through A, but only one is parallel to l.

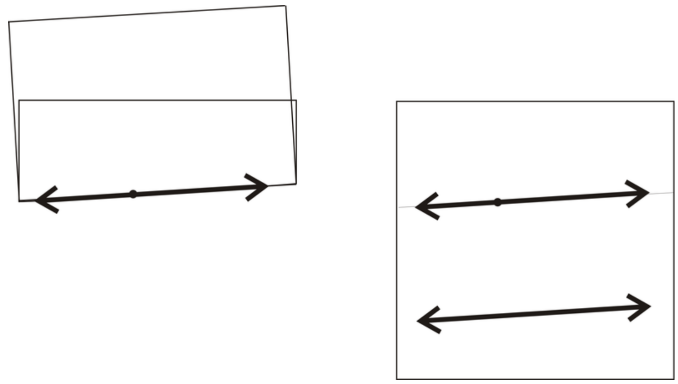
Investigation 3-1: Patty Paper and Parallel Lines

1. Get a piece of patty paper (a translucent square piece of paper).

Draw a line and a point above the line.



2. Fold up the paper so that the line is over the point. Crease the paper and unfold.



3. Are the lines parallel? Yes, by design, this investigation replicates the line we drew in #1 over the point. Therefore, there is only one line parallel through this point to this line.

Perpendicular Line Postulate

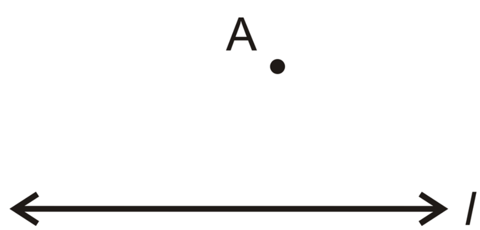
Perpendicular Line Postulate: For a line and a point not on the line, there is exactly one line perpendicular to the line that passes through the point.

There are infinitely many lines that pass through A, but only one that is perpendicular to l.

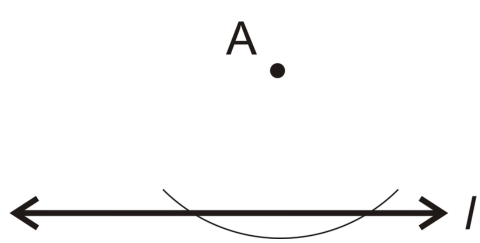
Investigation 3-2: Perpendicular Line Construction; through a Point NOT on the Line

1. Draw a horizontal line and a point above that line.

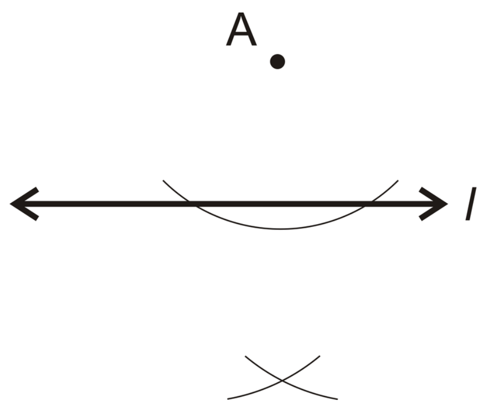
Label the line l and the point A.



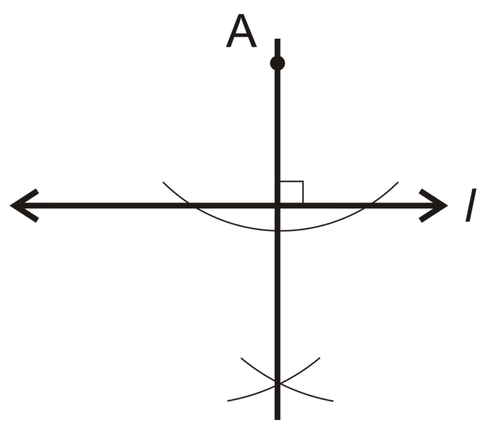
2. Take the compass and put the pointer on A. Open the compass so that it reaches beyond line l. Draw an arc that intersects the line twice.



3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc below the line. Repeat this on the other side so that the two arc marks intersect.



4. Take your straightedge and draw a line from point A to the arc intersections below the line. This line is perpendicular to l and passes through A.



Notice that this is a different construction from a perpendicular bisector.

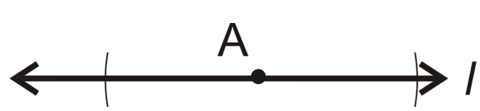
To see a demonstration of this construction, go to: http://www.mathsisfun.com/geometry/construct-perpnotline.html

Investigation 3-3: Perpendicular Line Construction; through a Point on the Line

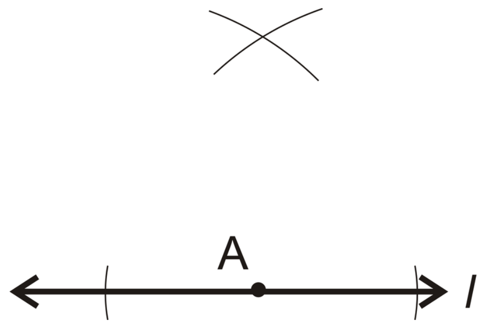
1. Draw a horizontal line and a point on that line.

Label the line l and the point A.

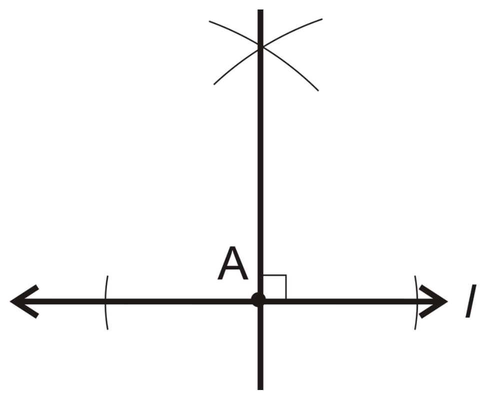
2. Take the compass and put the pointer on A. Open the compass so that it reaches out horizontally along the line. Draw two arcs that intersect the line on either side of the point.



3. Move the pointer to one of the arc intersections. Widen the compass a little and draw an arc above or below the line. Repeat this on the other side so that the two arc marks intersect.



4. Take your straightedge and draw a line from point A to the arc intersections above the line. This line is perpendicular to l and passes through A.



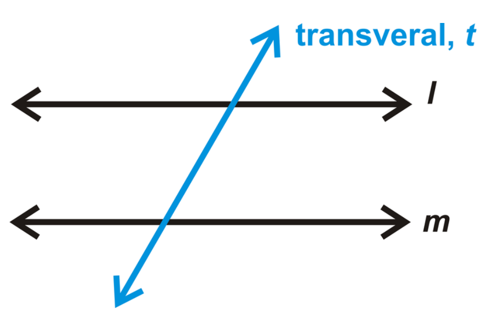
Notice that this is a different construction from a perpendicular bisector.

To see a demonstration of this construction, go to: http://www.mathsisfun.com/geometry/construct-perponline.html

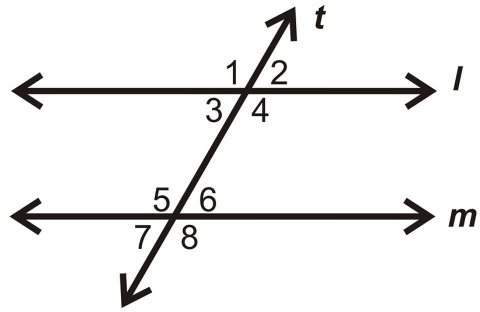
Angles and Transversals

Transversal: A line that intersects two distinct lines. These two lines may or may not be parallel.

The area *between* l and m is the called the *interior*. The area *outside* l and m is called the *exterior*.



Looking at t, l and m, there are 8 angles formed and several linear pairs vertical angle pairs. There are also 4 new angle relationships, defined here:



Corresponding Angles: Two angles that are in the "same place" with respect to the transversal, but on different lines. Imagine sliding the four angles formed with line *l* down to line *m*. The angles which match up are corresponding. $\angle 2$ ***and*** $\angle 6$ ***are corresponding angles.***

Alternate Interior Angles: Two angles that are on the interior of *l* and *m*, but on opposite sides of the transversal. $\angle 3$ ***and*** $\angle 6$ ***are alternate interior angles.***

Alternate Exterior Angles: Two angles that are on the exterior of *l* and *m*, but on opposite sides of the transversal. $\angle 1$ ***and*** $\angle 8$ ***are alternate exterior angles.***

Same Side Interior Angles: Two angles that are on the same side of the transversal and on the interior of the two lines. $\angle 3$ ***and*** $\angle 5$ ***are same side interior angles.***

Example 2: Using the picture above, list all the other pairs of each of the newly defined angle relationships.

Solution:

Corresponding Angles: $\angle 3$ and $\angle 7$, $\angle 1$ and $\angle 5$, $\angle 4$ and $\angle 8$

Alternate Interior Angles: $\angle 4$ and $\angle 5$

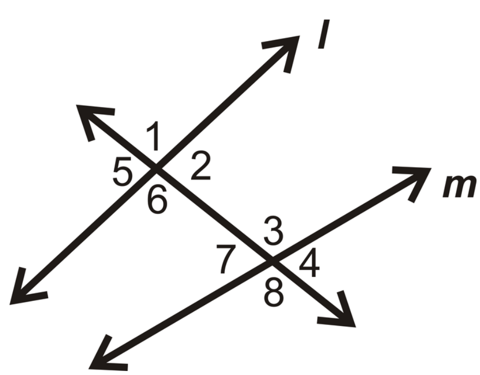
Alternate Exterior Angles: $\angle 2$ and $\angle 7$

Same Side Interior Angles: $\angle 4$ and $\angle 6$

Example 3: If $\angle 2 = 48^\circ$ (in the picture above), what other angles do you know?

Solution: $\angle 2 \cong \angle 3$ by the Vertical Angles Theorem, so $m\angle 3 = 48^\circ$. $\angle 2$ is also a linear pair with $\angle 1$ and $\angle 4$, so it is supplementary to those two. They are both 132°. We do not know the measures of $\angle 5$, $\angle 6$, $\angle 7$, or $\angle 8$ because we do not have enough information.

Example 4: For the picture to the right, determine:



a) A corresponding angle to $\angle 3$?

b) An alternate interior angle to $\angle 7$?

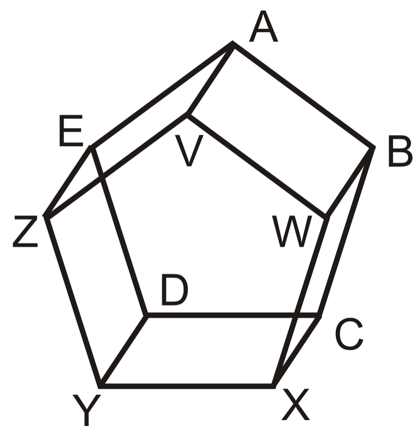
c) An alternate exterior angle to $\angle 4$?

Solution: The corresponding angle to $\angle 3$ is $\angle 1$. The alternate interior angle to $\angle 7$ is $\angle 2$. And, the alternate exterior angle to $\angle 4$ is $\angle 5$.

Know What? Revisited For Washington DC, all of the lettered streets are parallel, as are all of the numbered streets. The lettered streets are perpendicular to the numbered streets. There are no skew streets because all of the streets are in the same plane. We also do not know if any of the state-named streets are parallel or perpendicular.

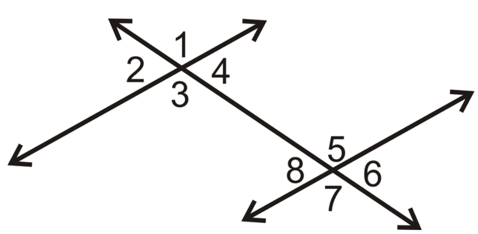
Review Questions

Use the figure below to answer questions 1-5. The two pentagons are parallel and all of the rectangular sides are perpendicular to both of them.



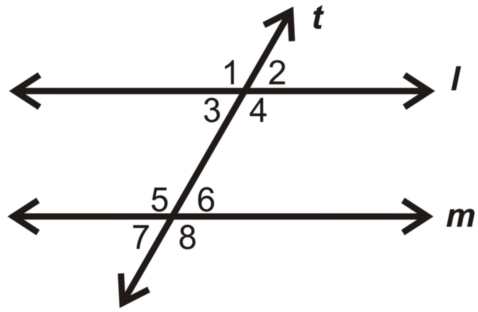
1. Find two pairs of skew lines.
2. List a pair of parallel lines.
3. List a pair of perpendicular lines.
4. For $\overline{AB}$, how many perpendicular lines pass through point V? What line is this?
5. For $\overline{XY}$, how many parallel lines passes through point D? What line is this?

For questions 6-12, use the picture below.



6. What is the corresponding angle to $\angle 4$?
7. What is the alternate interior angle with $\angle 5$?
8. What is the corresponding angle to $\angle 8$?
9. What is the alternate exterior angle with $\angle 7$?
10. What is the alternate interior angle with $\angle 4$?
11. What is the same side interior angle with $\angle 3$?
12. What is the corresponding angle to $\angle 1$?

Use the picture below for questions 13-16.



13. If $m\angle 2 = 55^\circ$, what other angles do you know?
14. If $m\angle 5 = 123^\circ$, what other angles do you know?
15. If $t \perp l$, is $t \perp m$? Why or why not?
16. Is $l \mid\mid m$? Why or why not?
17. ***Construction*** Draw a line and a point not on the line. Construct a perpendicular line to your original line through your point.
18. ***Construction*** Construct a perpendicular line to the line you constructed in #12. Use the point you originally drew, so that you will be constructing a perpendicular line through a point on the line.
19. Can you use patty paper to do the construction in number 17? Draw a line and a point not on the line on a piece of patty paper (or any thin white paper or tracing paper). Think about how you could make a crease in the paper that would be a line perpendicular to your original line through your point.
20. Using what you discovered in number 19, use patty paper to construct a line perpendicular to a given line through a point on the given line.
21. Draw a pair of parallel lines using your ruler. Describe how you did this.
22. Draw a pair of perpendicular lines using your ruler. Describe your method.

Geometry is often apparent in nature. Think of examples of each of the following in nature.

23. Parallel Lines or Planes
24. Perpendicular Lines or Planes
25. Skew Lines

Algebra Connection In questions 26-35 we will begin to explore the concepts of parallel and perpendicular lines in the coordinate plane.

26. Write the equations of two lines parallel to $y = 3$.
27. Write the equations of two lines perpendicular to $y = 5$.
28. What is the relationship between the two lines you found for number 27?
29. Plot the points $A(2,-5),\ B(-3,1),\ C(0,4),\ D(-5,10)$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. What are the slopes of these lines? What is the geometric relationship between these lines?
30. Plot the points $A(2,1),\ B(7,-2),\ C(2,-2),\ D(5,3)$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. What are the slopes of these lines? What is the geometric relationship between these lines?
31. Based on what you discovered in numbers 29 and 30, can you make a conjecture about the slopes of parallel and perpendicular lines?

Find the equation of the line that is <u>parallel</u> to the given line and passes through (5, -1).

32. $y = 2x - 7$
33. $y = -\frac{3}{5}x + 1$

Find the equation of the line that is <u>*perpendicular*</u> to the given line and passes through (2, 3).

34. $y = \frac{2}{3}x - 5$
35. $y = -\frac{1}{4}x + 9$

Review Queue Answers

a. $y = -2x + 3$
b. $y = -4x + 14$
c. $y = \frac{4}{3}x - 4$
d. Yes, the lines are perpendicular. The slopes are reciprocals and opposite signs.

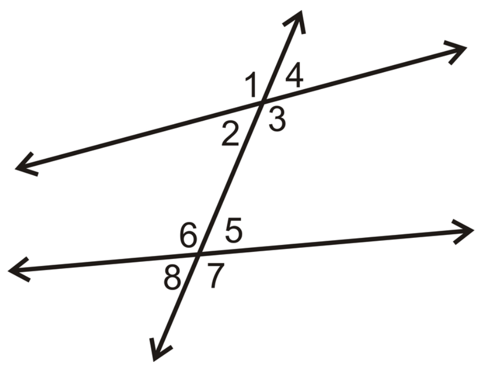
3.2 Properties of Parallel Lines

Learning Objectives

- Use the Corresponding Angles Postulate.
- Use the Alternate Interior Angles Theorem.
- Use the Alternate Exterior Angles Theorem.
- Use Same Side Interior Angles Theorem.

Review Queue

Use the picture below to determine:



a. A pair of corresponding angles.
b. A pair of alternate interior angles.
c. A pair of same side interior angles.
d. If $m\angle 4 = 37^\circ$, what other angles do you know?

Know What? The streets below are in Washington DC. The red street is R St. and the blue street is Q St. These two streets are parallel. The transversals are: Rhode Island Ave. (green) and Florida Ave. (orange).

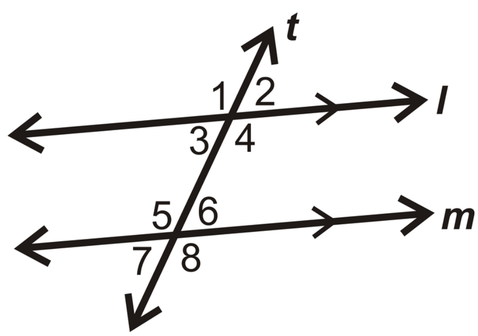
a. If $m\angle FTS = 35^\circ$, determine the other angles that are 35°.
b. If $m\angle SQV = 160^\circ$, determine the other angles that are 160°.
c. Why do you think the "State Streets" exists? Why aren't all the streets parallel or perpendicular?

In this section, we are going to discuss a specific case of two lines cut by a transversal. The two lines are now going to be parallel. If the two lines are parallel, all of the angles, corresponding, alternate interior, alternate exterior and same side interior have new properties. We will begin with corresponding angles.

Corresponding Angles Postulate

Corresponding Angles Postulate: If two parallel lines are cut by a transversal, then the corresponding angles are congruent.

If $l \parallel m$ and both are cut by t, then $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$.

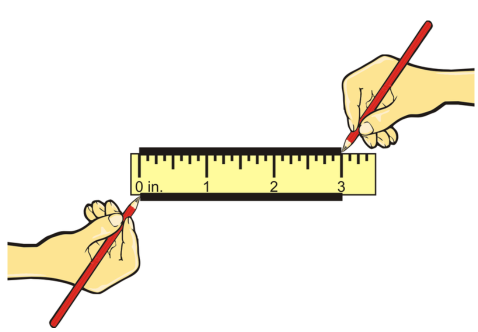


l **must be parallel** to m in order to use this postulate. Recall that a postulate is just like a theorem, but does not need to be proven. We can take it as true and use it just like a theorem from this point.

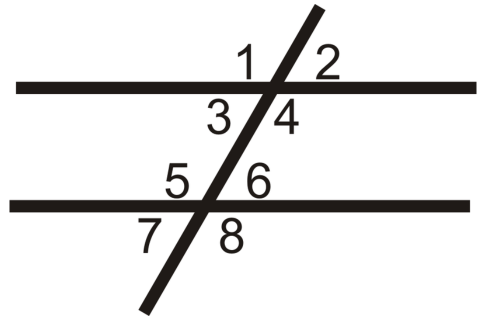
Investigation 3-4: Corresponding Angles Exploration

You will need: paper, ruler, protractor

a. Place your ruler on the paper. On either side of the ruler, draw lines, 3 inches long. This is the easiest way to ensure that the lines are parallel.



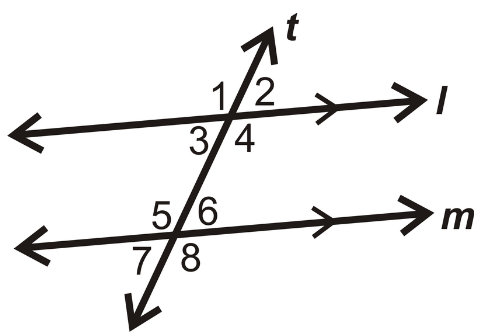
b. Remove the ruler and draw a transversal. Label the eight angles as shown.



c. Using your protractor, measure all of the angles. What do you notice?

In this investigation, you should see that $m\angle 1 = m\angle 4 = m\angle 5 = m\angle 8$ and $m\angle 2 = m\angle 3 = m\angle 6 = m\angle 7$. $\angle 1 \cong \angle 4$, $\angle 5 \cong \angle 8$ by the Vertical Angles Theorem. By the Corresponding Angles Postulate, we can say $\angle 1 \cong \angle 5$ and therefore $\angle 1 \cong \angle 8$ by the Transitive Property. You can use this reasoning for the other set of congruent angles as well.

Example 1: If $m\angle 2 = 76^\circ$, what is $m\angle 6$?



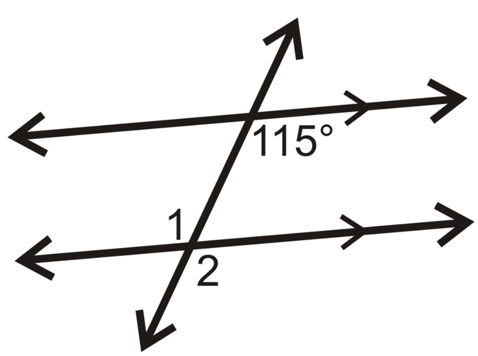
Solution: $\angle 2$ and $\angle 6$ are corresponding angles and $l \parallel m$, from the markings in the picture. By the Corresponding Angles Postulate the two angles are equal, so $m\angle 6 = 76^\circ$.

Example 2: Using the measures of $\angle 2$ and $\angle 6$ from Example 2, find all the other angle measures.

Solution: If $m\angle 2 = 76^\circ$, then $m\angle 1 = 180^\circ - 76^\circ = 104^\circ$ because they are a linear pair. $\angle 3$ is a vertical angle with $\angle 2$, so $m\angle 3 = 76^\circ$. $\angle 1$ and $\angle 4$ are vertical angles, so $m\angle 4 = 104^\circ$. By the Corresponding Angles Postulate, we know $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$, $\angle 3 \cong \angle 7$, and $\angle 4 \cong \angle 8$, so $m\angle 5 = 104^\circ$, $m\angle 6 = 76^\circ$, $m\angle 7 = 76^\circ$, and $m\angle 104^\circ$.

Alternate Interior Angles Theorem

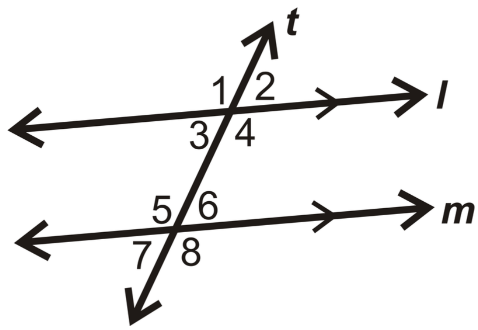
Example 3: Find $m\angle 1$.



Solution: $m\angle 2 = 115^\circ$ because they are corresponding angles and the lines are parallel. $\angle 1$ and $\angle 2$ are vertical angles, so $m\angle 1 = 115^\circ$ also.

$\angle 1$ and the 115° angle are alternate interior angles.

Alternate Interior Angles Theorem: If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.



Proof of Alternate Interior Angles Theorem

Given: $l \parallel m$

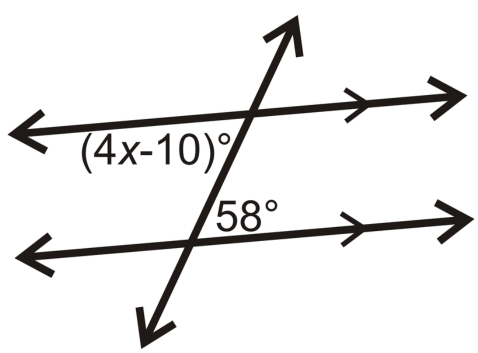
Prove: $\angle 3 \cong \angle 6$

TABLE 3.1:

Statement	*Reason*
1. $l \parallel m$	Given
2. $\angle 3 \cong \angle 7$	Corresponding Angles Postulate
3. $\angle 7 \cong \angle 6$	Vertical Angles Theorem
4. $\angle 3 \cong \angle 6$	Transitive PoC

There are several ways we could have done this proof. For example, Step 2 could have been $\angle 2 \cong \angle 6$ for the same reason, followed by $\angle 2 \cong \angle 3$. We could have also proved that $\angle 4 \cong \angle 5$.

Example 4: ***Algebra Connection*** Find the measure of the angle and x.

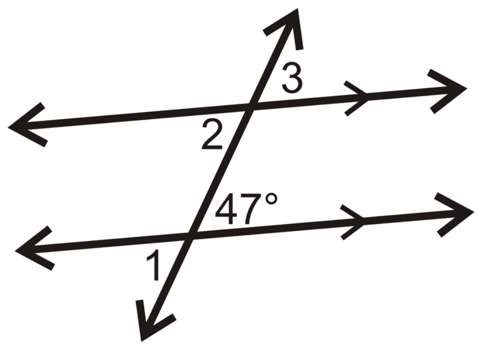


Solution: The two given angles are alternate interior angles so, they are equal. Set the two expressions equal to each other and solve for x.

$$\begin{aligned}(4x-10)^\circ &= 58^\circ \\ 4x &= 68^\circ \\ x &= 17^\circ\end{aligned}$$

Alternate Exterior Angles Theorem

Example 5: Find $m\angle 1$ and $m\angle 3$.



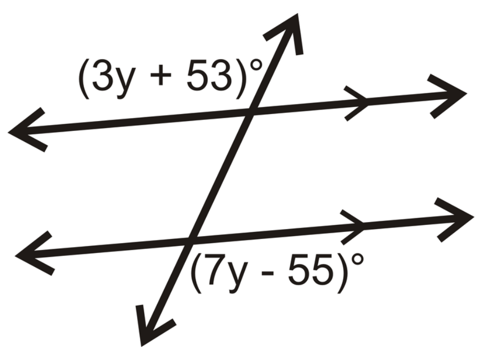
Solution: $m\angle 1 = 47^\circ$ because they are vertical angles. Because the lines are parallel, $m\angle 3 = 47^\circ$ by the Corresponding Angles Theorem. Therefore, $m\angle 2 = 47^\circ$.

$\angle 1$ and $\angle 3$ are alternate exterior angles.

Alternate Exterior Angles Theorem: If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent.

The proof of this theorem is very similar to that of the Alternate Interior Angles Theorem and you will be asked to do in the exercises at the end of this section.

Example 6: ***Algebra Connection*** Find the measure of each angle and the value of y.



Solution: The given angles are alternate exterior angles. Because the lines are parallel, we can set the expressions equal to each other to solve the problem.

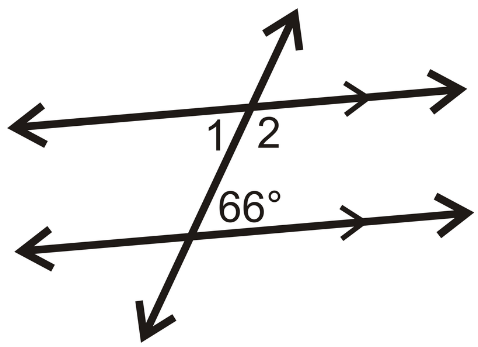
$$\begin{aligned}(3y+53)^\circ &= (7y-55)^\circ \\ 108^\circ &= 4y \\ 27^\circ &= y\end{aligned}$$

If $y = 27^\circ$, then each angle is $3(27^\circ) + 53^\circ$, or 134°.

Same Side Interior Angles Theorem

Same side interior angles have a different relationship that the previously discussed angle pairs.

Example 7: Find $m\angle 2$.



Solution: Here, $m\angle 1 = 66^\circ$ because they are alternate interior angles. $\angle 1$ and $\angle 2$ are a linear pair, so they are supplementary.

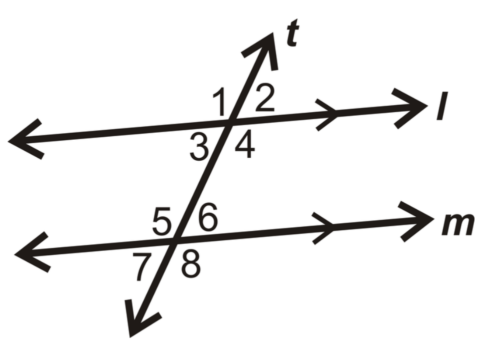
$$\begin{aligned} m\angle 1 + m\angle 2 &= 180^\circ \\ 66^\circ + m\angle 2 &= 180^\circ \\ m\angle 2 &= 114^\circ \end{aligned}$$

This example shows that if two parallel lines are cut by a transversal, the same side interior angles are supplementary.

Same Side Interior Angles Theorem: If two parallel lines are cut by a transversal, then the same side interior angles are supplementary.

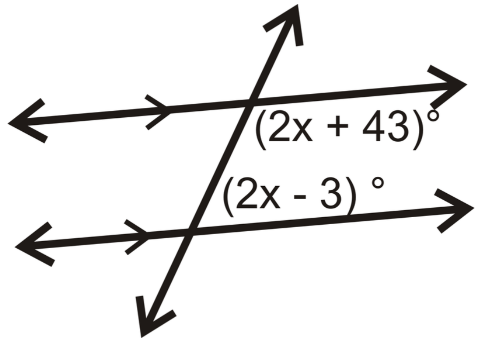
If $l \parallel m$ and both are cut by t, then

$m\angle 3 + m\angle 5 = 180^\circ$ and $m\angle 4 + m\angle 6 = 180^\circ$.



You will be asked to do the proof of this theorem in the review questions.

Example 8: ***Algebra Connection*** Find the measure of x.

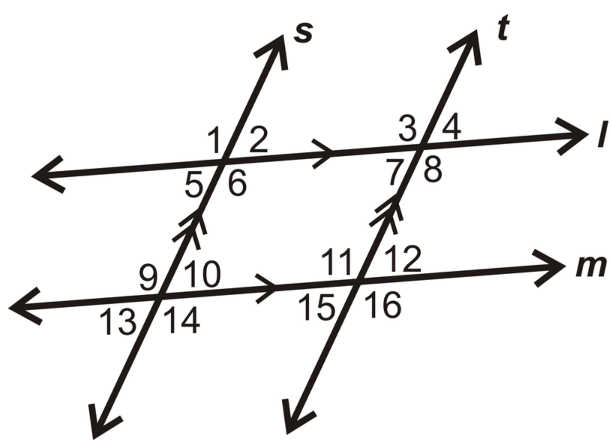


Solution: The given angles are same side interior angles. The lines are parallel, therefore the angles add up to 180°. Write an equation.

$$\begin{aligned}(2x+43)^\circ + (2x-3)^\circ &= 180^\circ\\ (4x+40)^\circ &= 180^\circ\\ 4x &= 140^\circ\\ x &= 35^\circ\end{aligned}$$

While you might notice other angle relationships, there are no more theorems to worry about. However, we will continue to explore these other angle relationships. For example, same side exterior angles are also supplementary. You will prove this in the review questions.

Example 9: $l \parallel m$ and $s \parallel t$. Prove $\angle 1 \cong \angle 16$.



Solution:

TABLE 3.2:

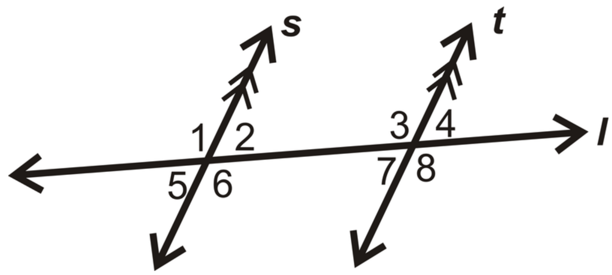
Statement	*Reason*
1. $l \parallel m$ and $s \parallel t$	Given
2. $\angle 1 \cong \angle 3$	Corresponding Angles Postulate
3. $\angle 3 \cong \angle 16$	Alternate Exterior Angles Theorem
4. $\angle 1 \cong \angle 16$	Transitive PoC

Know What? Revisited Using what we have learned in this lesson, the other angles that are 35° are $\angle TLQ$, $\angle ETL$, and the vertical angle with $\angle TLQ$. The other angles that are 160° are $\angle FSR$, $\angle TSQ$, and the vertical angle with

$\angle SQV$. You could argue that the "State Streets" exist to help traffic move faster and more efficiently through the city.

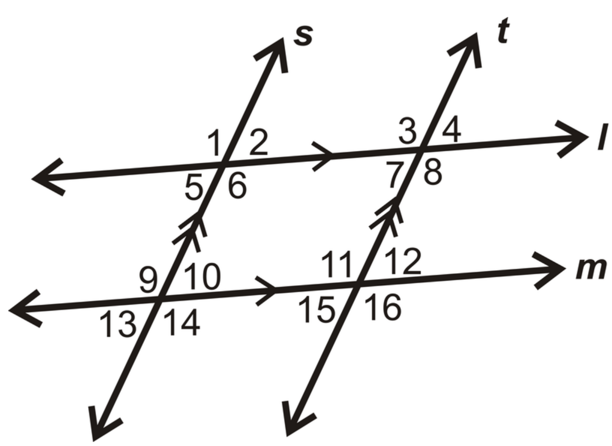
Review Questions

For questions 1-7, determine if each angle pair below is congruent, supplementary or neither.



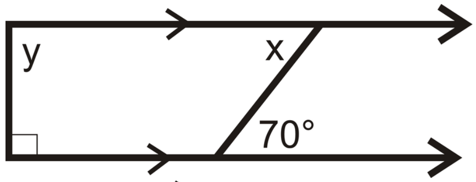
1. $\angle 1$ and $\angle 7$
2. $\angle 4$ and $\angle 2$
3. $\angle 6$ and $\angle 3$
4. $\angle 5$ and $\angle 8$
5. $\angle 1$ and $\angle 6$
6. $\angle 4$ and $\angle 6$
7. $\angle 2$ and $\angle 3$

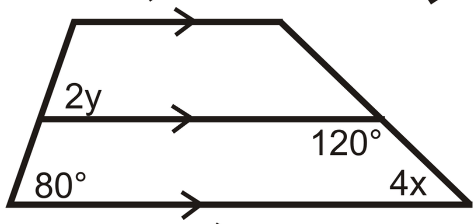
For questions 8-16, determine if the angle pairs below are: Corresponding Angles, Alternate Interior Angles, Alternate Exterior Angles, Same Side Interior Angles, Vertical Angles, Linear Pair or None.

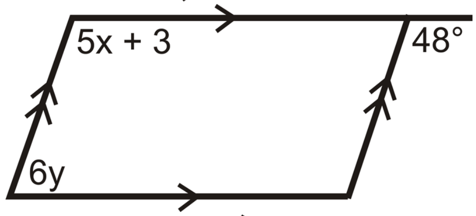


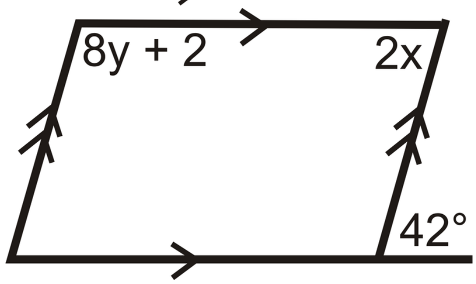
8. $\angle 2$ and $\angle 13$
9. $\angle 7$ and $\angle 12$
10. $\angle 1$ and $\angle 11$
11. $\angle 6$ and $\angle 10$
12. $\angle 14$ and $\angle 9$
13. $\angle 3$ and $\angle 11$
14. $\angle 4$ and $\angle 15$
15. $\angle 5$ and $\angle 16$
16. List all angles congruent to $\angle 8$.

For 17-20, find the values of x and y.

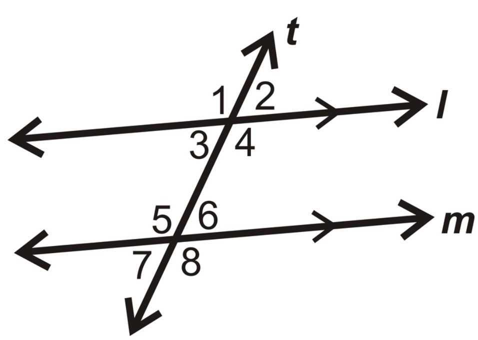
17. 

18. 

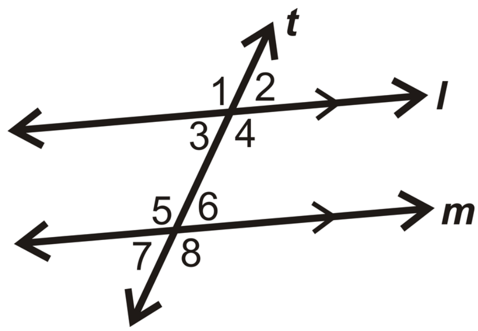
19. 

20. 

Algebra Connection For questions 21-25, use thepicture to the right. Find the value of x and/or y.



21. $m\angle 1 = (4x+35)^\circ$, $m\angle 8 = (7x-40)^\circ$
22. $m\angle 2 = (3y+14)^\circ$, $m\angle 6 = (8x-76)^\circ$
23. $m\angle 3 = (3x+12)^\circ$, $m\angle 5 = (5x+8)^\circ$
24. $m\angle 4 = (5x-33)^\circ$, $m\angle 5 = (2x+60)^\circ$
25. $m\angle 1 = (11y-15)^\circ$, $m\angle 7 = (5y+3)^\circ$
26. Fill in the blanks in the proof below.

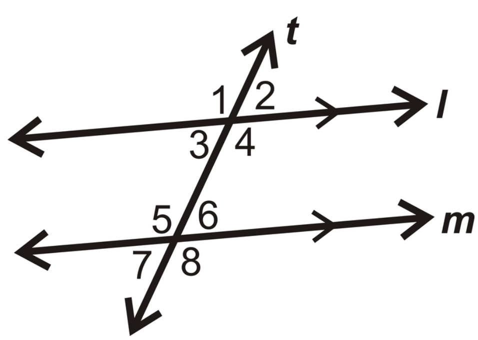


Given: $l \mid\mid m$Prove: $\angle 3$ and $\angle 5$ are supplementary (Same Side Interior Angles Theorem)

TABLE 3.3:

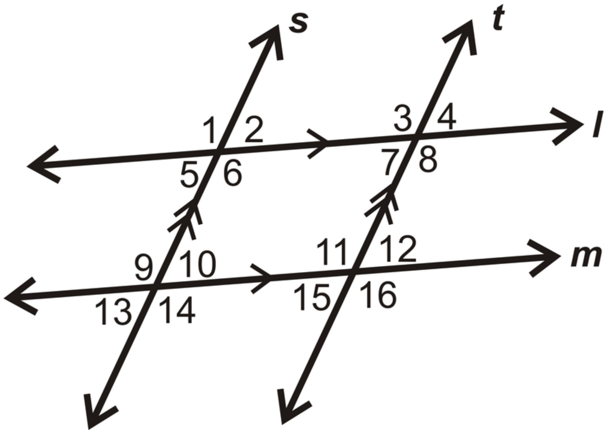
Statement	*Reason*
1.	Given
2. $\angle 1 \cong \angle 5$	
3.	$\cong$ angles have = measures
4.	Linear Pair Postulate
5.	Definition of Supplementary Angles
6. $m\angle 3 + m\angle 5 = 180^\circ$	
7. $\angle 3$ and $\angle 5$ are supplementary	

For 27 and 28, use the picture to the right to complete each proof.

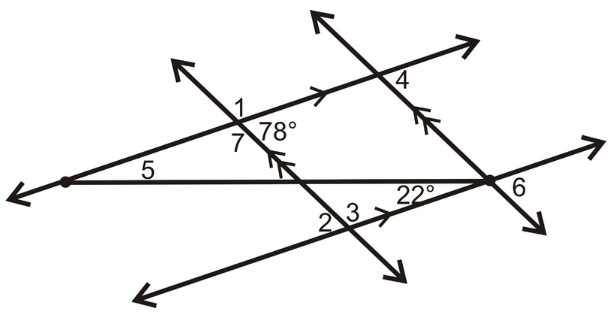


27. Given: $l \mid\mid m$Prove: $\angle 1 \cong \angle 8$ (Alternate Exterior Angles Theorem)
28. Given: $l \mid\mid m$Prove: $\angle 2$ and $\angle 8$ are supplementary

For 29-31, use the picture to the right to complete each proof.

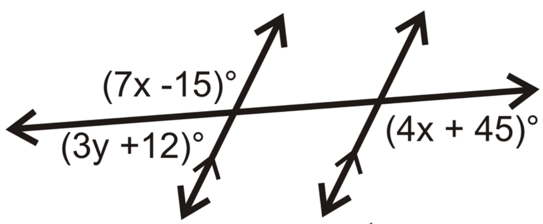


29. Given: $l \parallel m$, $s \parallel t$ Prove: $\angle 4 \cong \angle 10$
30. Given: $l \parallel m$, $s \parallel t$ Prove: $\angle 2 \cong \angle 15$
31. Given: $l \parallel m$, $s \parallel t$ Prove: $\angle 4$ and $\angle 9$ are supplementary
32. Find the measures of all the numbered angles in the figure below.

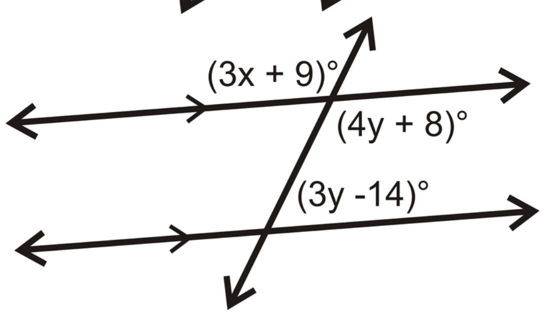


Algebra Connection For 32 and 33, find the values of x and y.

33.



34.



35. ***Error Analysis*** Nadia is working on Problem 31. Here is her proof:

TABLE 3.4:

Statement	***Reason***
1. $l \parallel m$, $s \parallel t$	Given
2. $\angle 4 \cong \angle 15$	Alternate Exterior Angles Theorem

TABLE 3.4: (continued)

Statement	***Reason***
3. $\angle 15 \cong \angle 14$	Same Side Interior Angles Theorem
4. $\angle 14 \cong \angle 9$	Vertical Angles Theorem
5. $\angle 4 \cong \angle 9$	Transitive PoC

What happened? Explain what is needed to be done to make the proof correct.

Review Queue Answers

a. $\angle 1$ and $\angle 6, \angle 2$ and $\angle 8, \angle 3$ and $\angle 7$, or $\angle 4$ and $\angle 5$
b. $\angle 2$ and $\angle 5$ or $\angle 3$ and $\angle 6$
c. $\angle 1$ and $\angle 7$ or $\angle 4$ and $\angle 8$
d. $\angle 3$ and $\angle 5$ or $\angle 2$ and $\angle 6$

3.3 Proving Lines Parallel

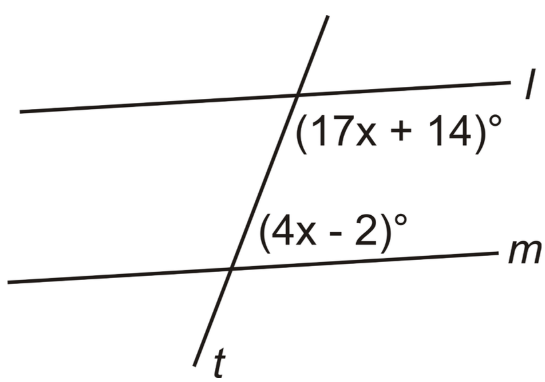
Learning Objectives

- Use the *converses* of the Corresponding Angles Postulate, Alternate Interior Angles Theorem, Alternate Exterior Angles Theorem, and the Same Side Interior Angles Theorem to show that lines are parallel.
- Construct parallel lines using the above converses.
- Use the Parallel Lines Property.

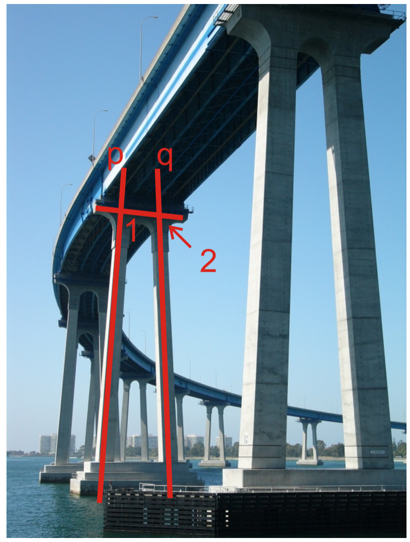
Review Queue

Answer the following questions.

a. Write the converse of the following statements:
   a. *If it is summer, then I am out of school.*
   b. *I will go to the mall when I am done with my homework.*
   c. *If two parallel lines are cut by a transversal, then the corresponding angles are congruent.*
b. Are any of the three converses from #1 true? Why or why not? Give a counterexample.
c. Determine the value of x if $l \parallel m$.



Know What? Here is a picture of the support beams for the Coronado Bridge in San Diego. This particular bridge, called a girder bridge, is usually used in straight, horizontal situations. The Coronado Bridge is diagonal, so the beams are subject to twisting forces (called torque). This can be fixed by building a curved bridge deck. To aid the curved bridge deck, the support beams should not be parallel. If they are, the bridge would be too fragile and susceptible to damage.



This bridge was designed so that $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$. Are the support beams parallel?

Corresponding Angles Converse

Recall that the converse of a statement switches the conclusion and the hypothesis. So, if a, then b becomes if b, then a. We will find the converse of all the theorems from the last section and will determine if they are true.

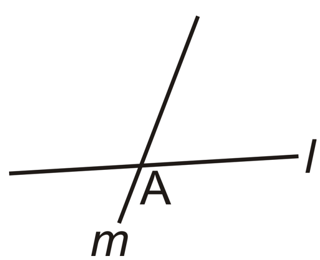
The Corresponding Angles Postulate says: *If two lines are parallel, then the corresponding angles are congruent.* The converse is:

Converse of Corresponding Angles Postulate: If corresponding angles are congruent when two lines are cut by a transversal, then the lines are parallel.

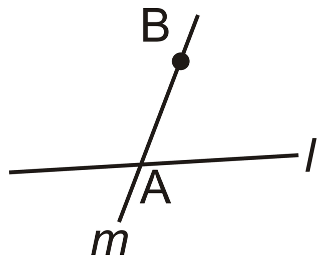
Is this true? For example, if the corresponding angles both measured 60°, would the lines be parallel? YES. All eight angles created by l, m and the transversal are either 60° or 120°, making the slopes of l and m the same which makes them parallel. This can also be seen by using a construction.

Investigation 3-5: Creating Parallel Lines using Corresponding Angles

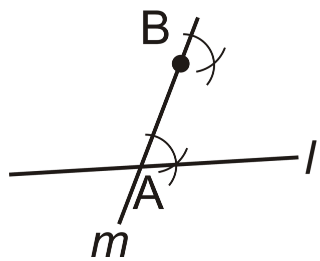
a. Draw two intersecting lines. Make sure they are not perpendicular. Label them l and m, and the point of intersection, A, as shown.



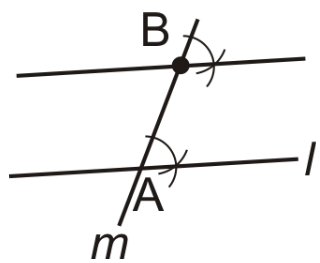
b. Create a point, B, on line m, above A.



c. Copy the acute angle at A (the angle to the right of m) at point B. See Investigation 2-2 in Chapter 2 for the directions on how to copy an angle.

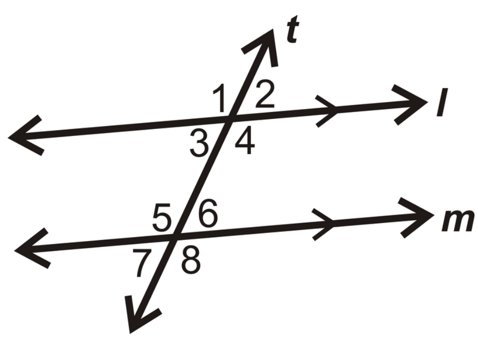


d. Draw the line from the arc intersections to point B.



From this construction, we can see that the lines are parallel.

Example 1: If $m\angle 8 = 110^\circ$ and $m\angle 4 = 110^\circ$, then what do we know about lines l and m?



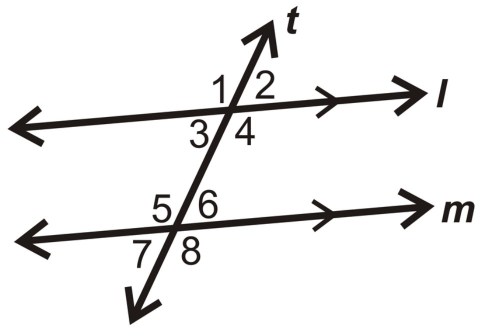
Solution: $\angle 8$ and $\angle 4$ are corresponding angles. Since $m\angle 8 = m\angle 4$, we can conclude that $l \mid\mid m$.

Alternate Interior Angles Converse

We also know, from the last lesson, that when parallel lines are cut by a transversal, the alternate interior angles are congruent. The converse of this theorem is also true:

Converse of Alternate Interior Angles Theorem: If two lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel.

Example 3: Prove the Converse of the Alternate Interior Angles Theorem.



Given: l and m and transversal t

$\angle 3 \cong \angle 6$

Prove: $l \,||\, m$

Solution:

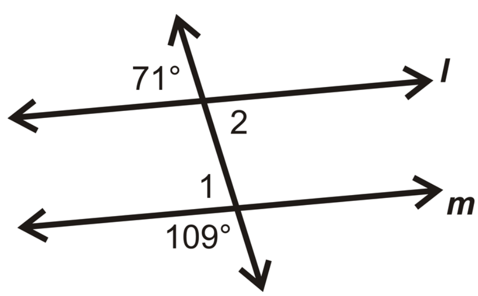
TABLE 3.5:

Statement	*Reason*
1. l and m and transversal $t \angle 3 \cong \angle 6$	Given
2. $\angle 3 \cong \angle 2$	Vertical Angles Theorem
3. $\angle 2 \cong \angle 6$	Transitive PoC
4. $l \,||\, m$	Converse of the Corresponding Angles Postulate

Prove Move: Shorten the names of these theorems. Discuss with your teacher an appropriate abbreviations. For example, the Converse of the Corresponding Angles Theorem could be "Converse CA Thm" or "ConvCA."

Notice that the Corresponding Angles Postulate was not used in this proof. The Transitive Property is the reason for Step 3 because we do not know if l is parallel to m until we are done with the proof. You could conclude that if we are trying to prove two lines are parallel, the converse theorems will be used. And, if we are proving two angles are congruent, we must be given that the two lines are parallel.

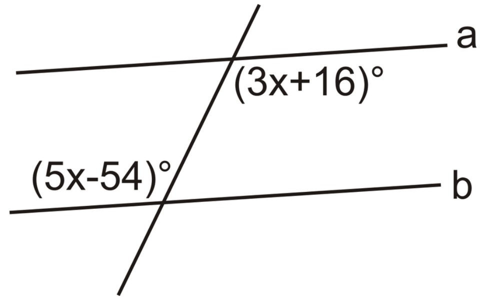
Example 4: Is $l \,||\, m$?



Solution: First, find $m\angle 1$. We know its linear pair is 109°. By the Linear Pair Postulate, these two angles add up to 180°, so $m\angle 1 = 180^\circ - 109^\circ = 71^\circ$. This means that $l \,||\, m$, by the Converse of the Corresponding Angles Postulate.

Example 5: ***Algebra Connection*** What does x have to be to make $a \,||\, b$?

Solution: Because these are alternate interior angles, they must be equal for $a \,||\, b$. Set the expressions equal to each other and solve.



$$
\begin{aligned}
3x+16^\circ &= 5x-54^\circ \\
70^\circ &= 2x \\
35^\circ &= x \qquad \text{To make } a \,||\, b,\ x=35^\circ.
\end{aligned}
$$

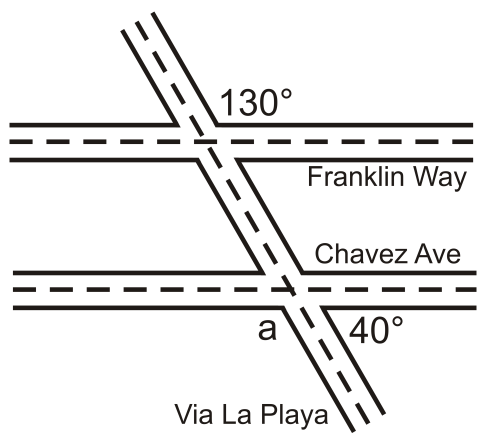
Converse of Alternate Exterior Angles & Consecutive Interior Angles

You have probably guessed that the converse of the Alternate Exterior Angles Theorem and the Consecutive Interior Angles Theorem areal so true.

Converse of the Alternate Exterior Angles Theorem: If two lines are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel.

Example 6: ***Real-World Situation*** The map below shows three roads in Julio's town.

Julio used a surveying tool to measure two angles at the intersections in this picture he drew (NOT to scale). ***Julio wants to know if Franklin Way is parallel to Chavez Avenue.***



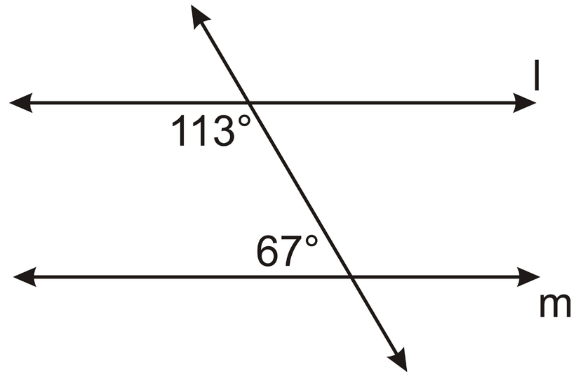
Solution: The labeled 130° angle and $\angle a$ are alternate exterior angles. If $m\angle a = 130^\circ$, then the lines are parallel. To find $m\angle a$, use the other labeled angle which is 40°, and its linear pair. Therefore, $\angle a + 40^\circ = 180^\circ$ and $\angle a = 140^\circ$. $140^\circ \neq 130^\circ$, so Franklin Way and Chavez Avenue are <u>not</u> parallel streets.

The final converse theorem is of the Same Side Interior Angles Theorem. Remember that these angles are not congruent when lines are parallel, they are **<u>supplementary</u>**.

Converse of the Same Side Interior Angles Theorem: If two lines are cut by a transversal and the consecutive interior angles are supplementary, then the lines are parallel.

Example 7: Is $l \,||\, m$? How do you know?

Solution: These are Same Side Interior Angles. So, if they add up to 180°, then $l \,||\, m$. $113^\circ + 67^\circ = 180^\circ$, therefore $l \,||\, m$.

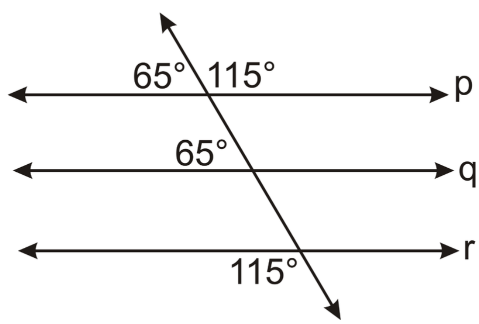


Parallel Lines Property

The Parallel Lines Property is a transitive property that can be applied to parallel lines. Remember the Transitive Property of Equality is: If $a = b$ and $b = c$, then $a = c$. The Parallel Lines Property changes = to $||$.

Parallel Lines Property: If lines $l \,||\, m$ and $m \,||\, n$, then $l \,||\, n$.

Example 8: Are lines q and r parallel?



Solution: First find if $p \,||\, q$, followed by $p \,||\, r$. If so, then $q \,||\, r$.

$p \,||\, q$ by the Converse of the Corresponding Angles Postulate, the corresponding angles are 65°. $p \,||\, r$ by the Converse of the Alternate Exterior Angles Theorem, the alternate exterior angles are 115°. Therefore, by the Parallel Lines Property, $q \,||\, r$.

Know What? Revisited: The CoronadoBridge has $\angle 1$ and $\angle 2$, which are corresponding angles. These angles must be equal for the beams to be parallel. $\angle 1 = 92^\circ$ and $\angle 2 = 88^\circ$ and $92^\circ \neq 88^\circ$, so the beams are not parallel, therefore a sturdy and safe girder bridge.

Review Questions

1. ***Construction*** Using Investigation 3-1 to help you, show that two lines are parallel by constructing congruent alternate interior angles. HINT: Steps 1 and 2 will be exactly the same, but at step 3, you will copy the angle in a different location.

2. ***Construction*** Using Investigation 3-1 to help you, show that two lines are parallel by constructing supplementary consecutive interior angles. HINT: Steps 1 and 2 will be exactly the same, but at step 3, you will copy a different angle.

For Questions 3-5, fill in the blanks in the proofs below.

3. Given: $l \parallel m,\ p \parallel q$ Prove: $\angle 1 \cong \angle 2$

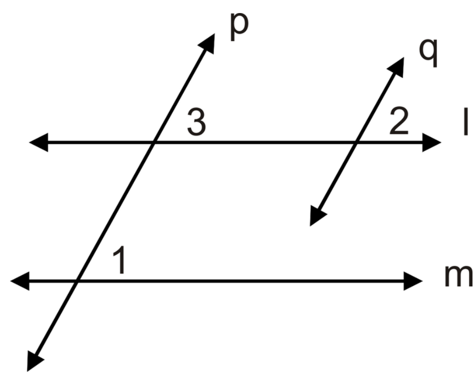


TABLE 3.6:

Statement	***Reason***
1. $l \parallel m$	1.
2.	2. Corresponding Angles Postulate
3. $p \parallel q$	3.
4.	4.
5. $\angle 1 \cong \angle 2$	5.

4. Given: $p \parallel q,\ \angle 1 \cong \angle 2$ Prove: $l \parallel m$

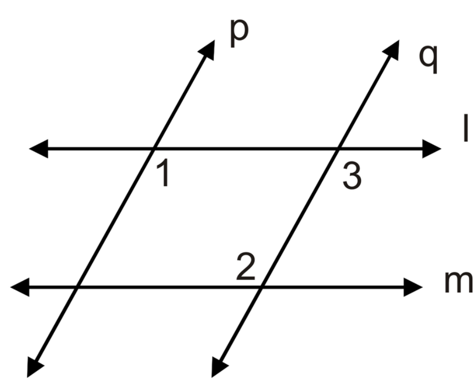


TABLE 3.7:

Statement	***Reason***
1. $p \parallel q$	1.
2.	2. Corresponding Angles Postulate
3. $\angle 1 \cong \angle 2$	3.
4.	4. Transitive PoC
5.	5. Converse of Alternate Interior Angles Theorem

5. Given: $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4$Prove: $l \,||\, m$

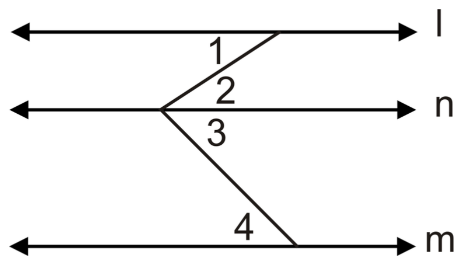
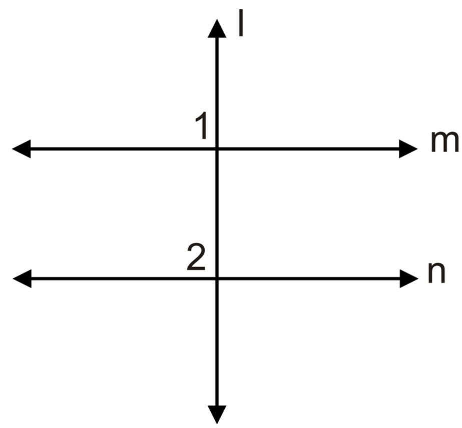


TABLE 3.8:

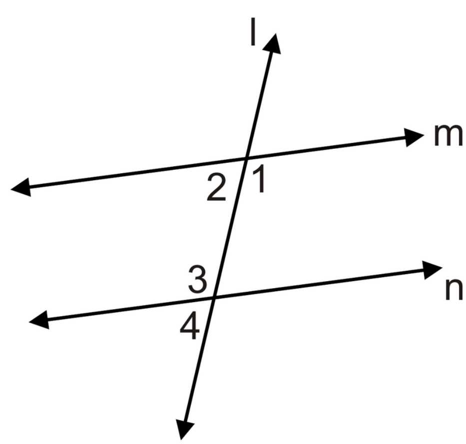
Statement	*Reason*
1. $\angle 1 \cong \angle 2$	1.
2. $l \,\|\, n$	2.
3. $\angle 3 \cong \angle 4$	3.
4.	4. Converse of Alternate Interior Angles Theorem
5. $l \,\|\, m$	5.

For Questions 6-9, create your own two column proof.

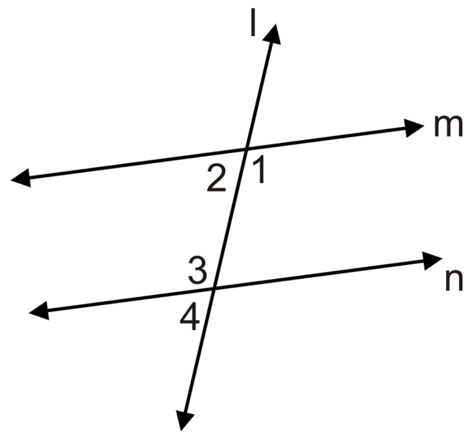
6. Given: $m \perp l,\ n \perp l$Prove: $m \,||\, n$



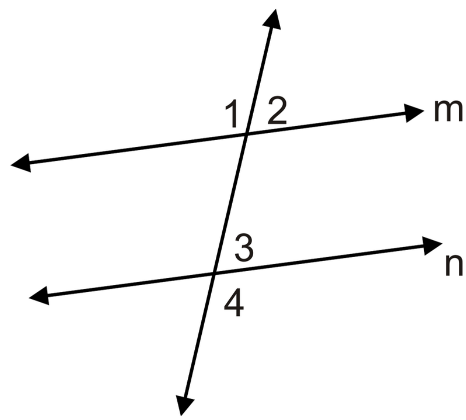
7. Given: $\angle 1 \cong \angle 3$Prove: $\angle 1$ and $\angle 4$ are supplementary



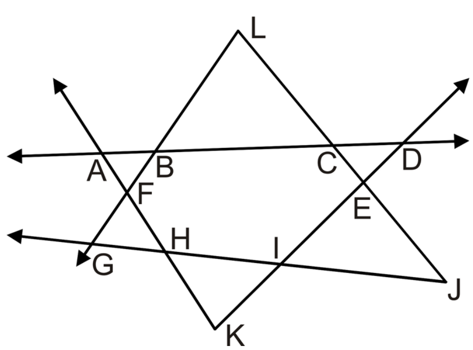
8. Given: $\angle 2 \cong \angle 4$Prove: $\angle 1 \cong \angle 3$



9. Given: $\angle 2 \cong \angle 3$Prove: $\angle 1 \cong \angle 4$

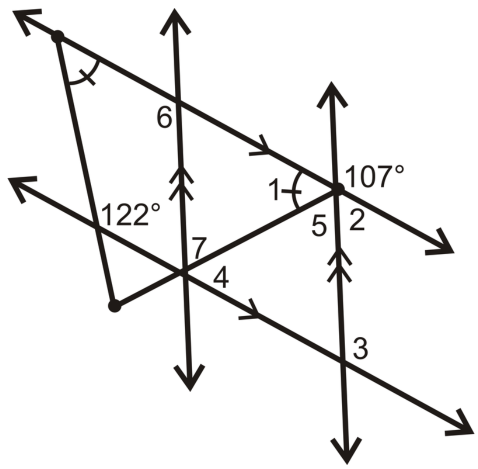


In 10-15, use the given information to determine which lines are parallel. If there are none, write *none*. Consider each question individually.



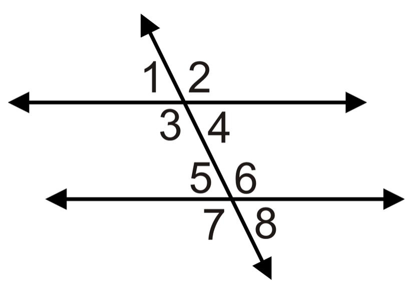
10. $\angle LCD \cong \angle CJI$
11. $\angle BCE$ *and* $\angle BAF$ are supplementary
12. $\angle FGH \cong \angle EIJ$
13. $\angle BFH \cong \angle CEI$
14. $\angle LBA \cong \angle IHK$
15. $\angle ABG \cong \angle BGH$

In 16-22, find the measure of the lettered angles below.

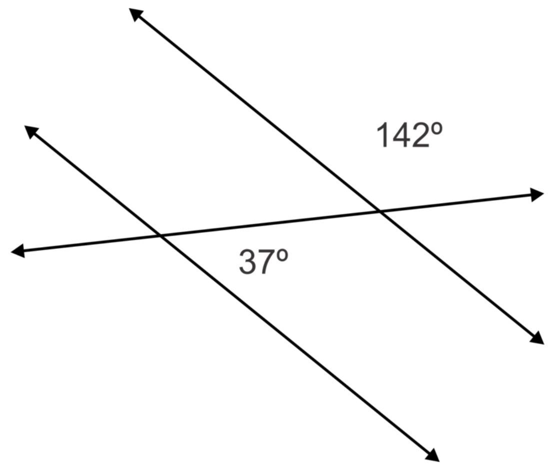


16. $m\angle 1$
17. $m\angle 2$
18. $m\angle 3$
19. $m\angle 4$
20. $m\angle 5$
21. $m\angle 6$
22. $m\angle 7$

For 23-27, what does x have to measure to make the lines parallel?



23. $m\angle 3 = (3x+25)^\circ$ and $m\angle 5 = (4x-55)^\circ$
24. $m\angle 2 = (8x)^\circ$ and $m\angle 7 = (11x-36)^\circ$
25. $m\angle 1 = (6x-5)^\circ$ and $m\angle 5 = (5x+7)^\circ$
26. $m\angle 4 = (3x-7)^\circ$ and $m\angle 7 = (5x-21)^\circ$
27. $m\angle 1 = (9x)^\circ$ and $m\angle 6 = (37x)^\circ$
28. ***Construction*** Draw a straight line. Construct a line perpendicular to this line through a point on the line. Now, construct a perpendicular line to this new line. What can you conclude about the original line and this final line?
29. How could you prove your conjecture from problem 28?
30. What is wrong in the following diagram, given that $j \,||\, k$?



Review Queue Answers

a.
   a. *If I am out of school, then it is summer.*
   b. *If I go to the mall, then I am done with my homework.*
   c. *If corresponding angles created by two lines cut by a transversal are congruent, then the two lines are parallel.*

   a. Not true, I could be out of school on any school holiday or weekend during the school year.
   b. Not true, I don't have to be done with my homework to go to the mall.
   c. Yes, because if two corresponding angles are congruent, then the slopes of these two lines have to be the same, making the lines parallel.

b. The two angles are supplementary.

$$\begin{aligned}(17x+14)^\circ+(4x-2)^\circ &= 180^\circ\\ 21x+12^\circ &= 180^\circ\\ 21x &= 168^\circ\\ x &= 8^\circ\end{aligned}$$

3.4 Properties of Perpendicular Lines

Learning Objectives

- Understand the properties of perpendicular lines.
- Explore problems with parallel lines and a perpendicular transversal.
- Solve problems involving complementary adjacent angles.

Review Queue

Determine if the following statements are true or false. If they are true, write the converse. If they are false, find a counter example.

1. Perpendicular lines form four right angles.

2. A right angle is greater than or equal to 90°.

Find the slope between the two given points.

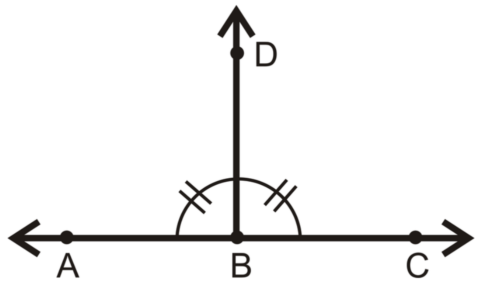
3. (-3, 4) and (-3, 1)

4. (6, 7) and (-5, 7)

Know What? There are several examples of slope in nature. To the right are pictures of Half Dome, in Yosemite-National Park and the horizon over the Pacific Ocean. These are examples of horizontal and vertical lines in real life. Can you determine the slope of these lines?

Congruent Linear Pairs

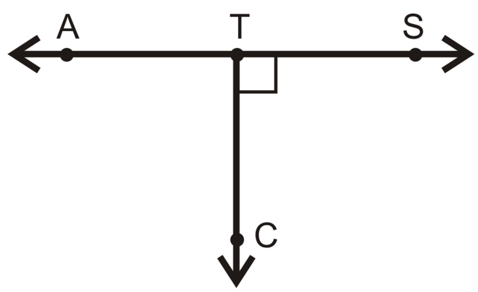
Recall that a linear pair is a pair of adjacent angles whose outer sides form a straight line. The Linear Pair Postulate says that the angles in a linear pair are supplementary. What happens when the angles in a linear pair are congruent?



$$\begin{aligned} m\angle ABD + m\angle DBC &= 180^\circ && \text{Linear Pair Postulate} \\ m\angle ABD &= m\angle DBC && \text{The two angles are congruent} \\ m\angle ABD + m\angle ABD &= 180^\circ && \text{Substitution PoE} \\ 2m\angle ABD &= 180^\circ && \text{Combine like terms} \\ m\angle ABD &= 90^\circ && \text{Division PoE} \end{aligned}$$

So, anytime a linear pair is congruent, the angles are both 90°.

Example 1: Find $m\angle CTA$.

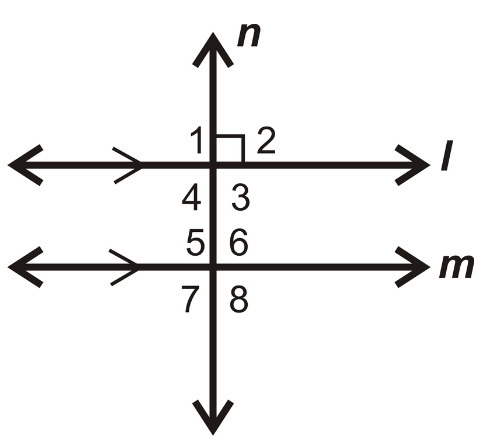


Solution: First, these two angles form a linear pair. Second, from the marking, we know that $\angle STC$ is a right angle. Therefore, $m\angle STC = 90^\circ$. So, $m\angle CTA$ is also 90°.

Perpendicular Transversals

Recall that when two lines intersect, four angles are created. If the two lines are perpendicular, then all four angles are right angles, even though only one needs to be marked with the square. Therefore, all four angles are 90°.

When a parallel line is added, then there are eight angles formed. If $l \parallel m$ and $n \perp l$, is $n \perp m$? Let's prove it here.



Given: $l \parallel m$, $l \perp n$

Prove: $n \perp m$

TABLE 3.9:

Statement	*Reason*
1. $l \parallel m$, $l \perp n$	Given
2. $\angle 1$, $\angle 2$, $\angle 3$, and $\angle 4$ are right angles	Definition of perpendicular lines
3. $m\angle 1 = 90^\circ$	Definition of a right angle
4. $m\angle 1 = m\angle 5$	Corresponding Angles Postulate
5. $m\angle 5 = 90^\circ$	Transitive PoE
6. $m\angle 6 = m\angle 7 = 90^\circ$	Congruent Linear Pairs
7. $m\angle 8 = 90^\circ$	Vertical Angles Theorem
8. $\angle 5$, $\angle 6$, $\angle 7$, and $\angle 8$ are right angles	Definition of right angle
9. $n \perp m$	Definition of perpendicular lines

Theorem 3-1: If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

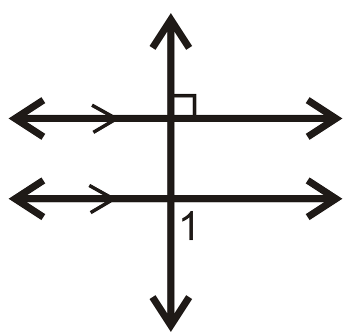
Or, if $l \parallel m$ and $l \perp n$, then $n \perp m$.

Theorem 3-2: If two lines are perpendicular to the same line, they are parallel to each other.

Or, if $l \perp n$ and $n \perp m$, then $l \parallel m$. You will prove this theorem in the review questions.

From these two theorems, we can now assume that any angle formed by two parallel lines and a perpendicular transversal will always be 90°.

Example 2: Determine the measure of $\angle 1$.



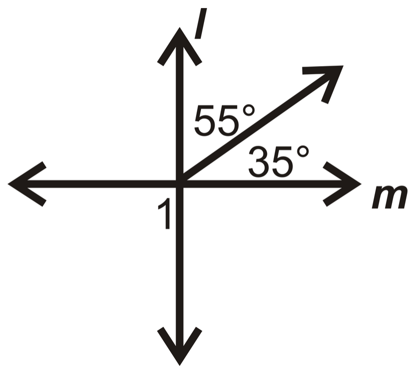
Solution: From Theorem 3-1, we know that the lower parallel line is also perpendicular to the transversal. Therefore, $m\angle 1 = 90^\circ$.

Adjacent Complementary Angles

Recall that complementary angles add up to 90°. If complementary angles are adjacent, their nonadjacent sides are perpendicular rays. What you have learned about perpendicular lines can be applied to this situation.

Example 3: Find $m\angle 1$.

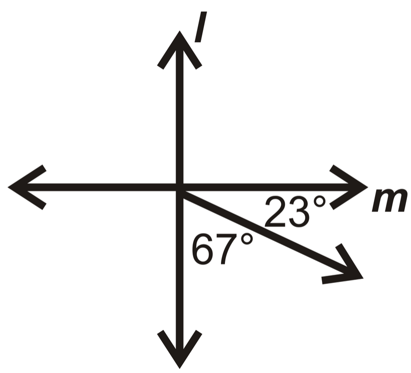
Solution: The two adjacent angles add up to 90°, so $l \perp m$. Therefore, $m\angle 1 = 90^\circ$.



Example 4: Is $l \perp m$? Explain why or why not.

Solution: If the two adjacent angles add up to 90°, then l and m are perpendicular.

$23^\circ + 67^\circ = 90^\circ$. Therefore, $l \perp m$.



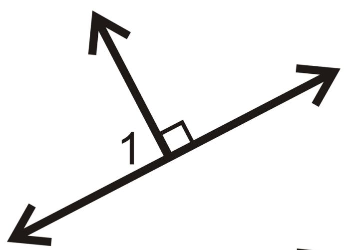
Know What? Revisited

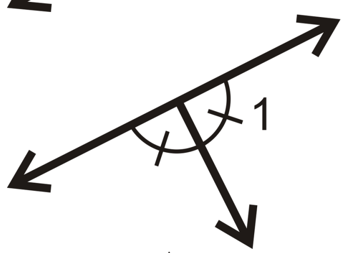
Half Dome is vertical and the slope of any vertical line is undefined. Thousands of people flock to Half Dome to attempt to scale the rock. This front side is very difficult to climb because it is vertical. The only way to scale the front side is to use the provided cables at the base of the rock. http://www.nps.gov/yose/index.htm

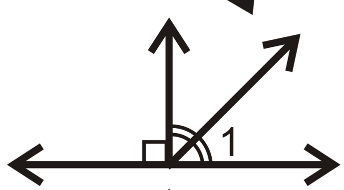
Any horizon over an ocean is horizontal, which has a slope of zero, or no slope. There is no steepness, so no incline or decline. The complete opposite of Half Dome. Actually, if Half Dome was placed on top of an ocean or flat ground, the two would be perpendicular!

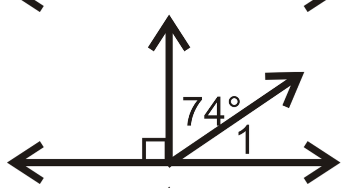
Review Questions

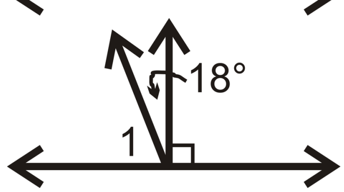
Find the measure of $\angle 1$ for each problem below.

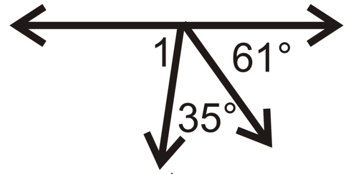
1. 

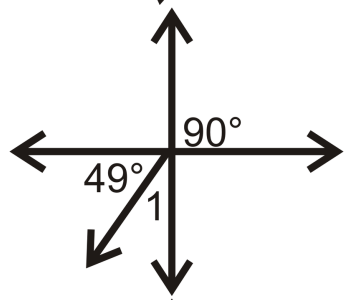
2. 

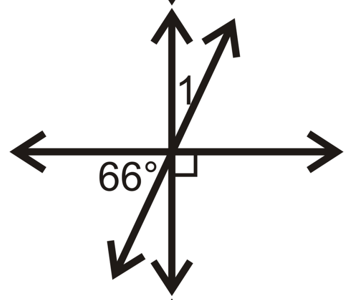
3. 

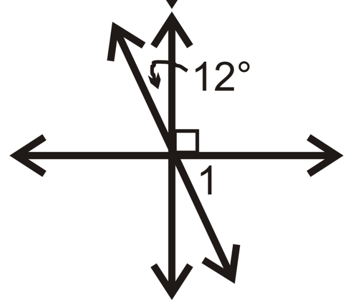
4. 

5. 

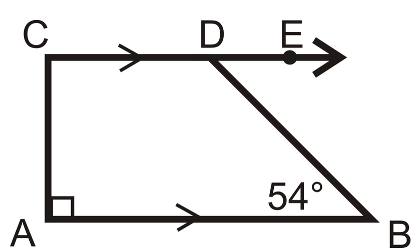
6. 

7. 

8. 

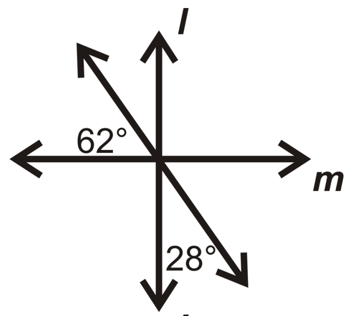
9. 

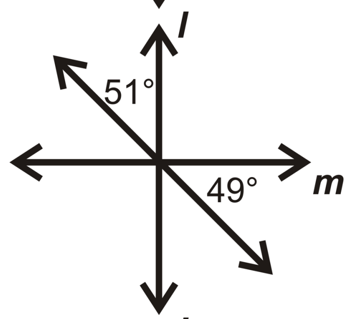
For questions 10-13, use the picture below.

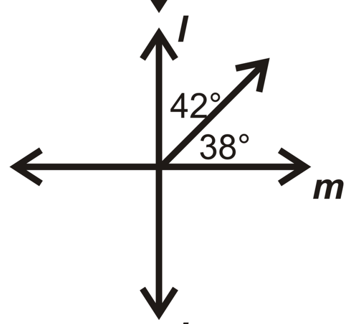


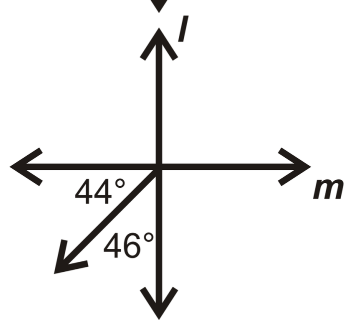
10. Find $m\angle ACD$.
11. Find $m\angle CDB$.
12. Find $m\angle EDB$.
13. Find $m\angle CDE$.

In questions 14-17, determine if $l \perp m$.

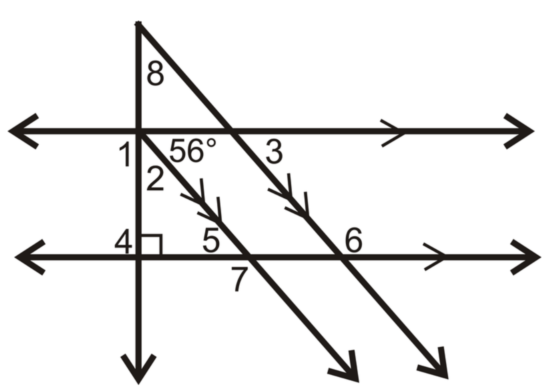
14. 

15. 

16. 

17. 

For questions 18-25, use the picture below.

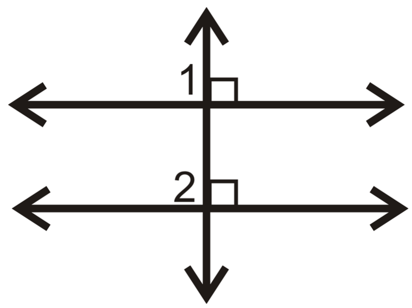


18. Find $m\angle 1$.
19. Find $m\angle 2$.
20. Find $m\angle 3$.

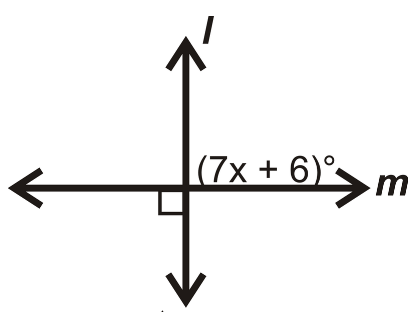
21. Find $m\angle 4$.
22. Find $m\angle 5$.
23. Find $m\angle 6$.
24. Find $m\angle 7$.
25. Find $m\angle 8$.

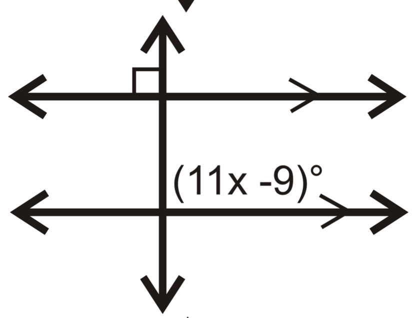
Complete the proof.

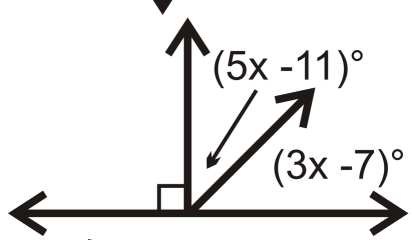
26. Given: $l \perp m$, $l \perp n$Prove: $m \parallel n$



Algebra Connection Find the value of x.

27.


28.


29.


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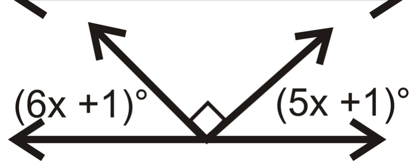
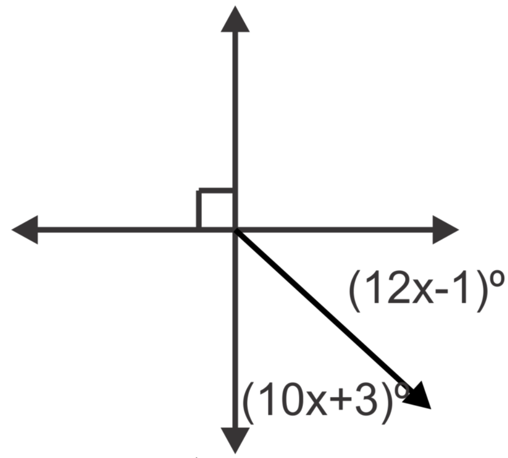
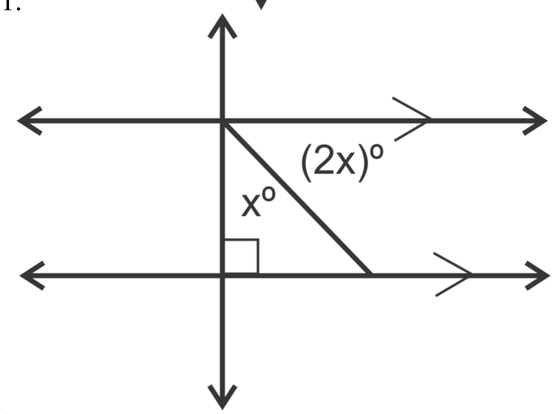


31. 

32. 

Review Queue Answers

a. True; If four right angles are formed by two intersecting lines, then the lines are perpendicular.
b. False; 95° is not a right angle.
c. Undefined slope; this is a vertical line.
d. Zero slope; this would be a horizontal line.

3.5 Midpoints and Bisectors

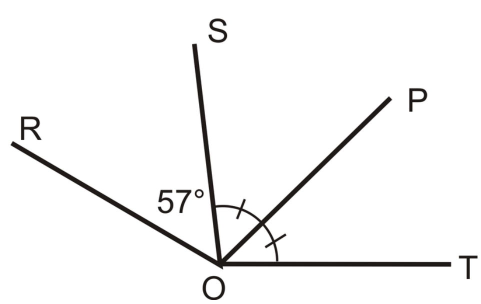
Learning Objectives

- Identify the midpoint of line segments.
- Identify the bisector of a line segment.
- Understand and the Angle Bisector Postulate.

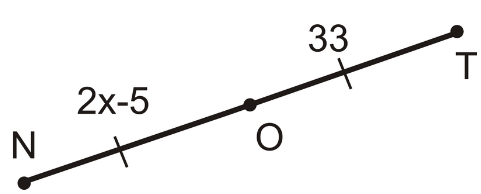
Review Queue

Answer the following questions.

a. $m\angle ROT = 165^\circ$, find $m\angle POT$



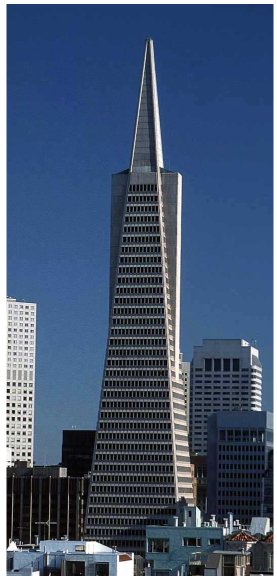
b. Find x.



c. Use the Angle Addition Postulate to write an equation for the angles in #1.

Know What? The building to the right is the TransamericaBuilding in San Francisco. This building was completed in 1972 and, at that time was one of the tallest buildings in the world. It is a pyramid with two "wings" on either side, to accommodate elevators. Because San Francisco has problems with earthquakes, there are regulations on how a building can be designed. In order to make this building as tall as it is and still abide by the building codes, the designer used this pyramid shape.

It is very important in designing buildings that the angles and parts of the building are equal. What components of this building look equal? Analyze angles, windows, and the sides of the building.



Congruence

You could argue that another word for ***equal*** is ***congruent***. However, the two differ slightly.

Congruent: When two geometric figures have the same shape and size.

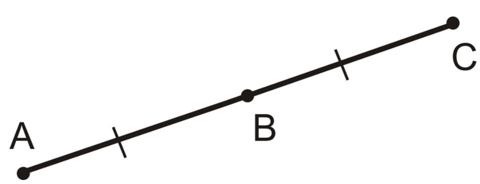
We label congruence with a $\cong$ sign. Notice the $\sim$ above the = sign. $\overline{AB} \cong \overline{BA}$ means that $\overline{AB}$ is congruent to $\overline{BA}$. If we know two segments or angles are congruent, then their measures are also equal. If two segments or angles have the same measure, then, they are also congruent.

TABLE 3.10:

Equal	***Congruent***
$=$	$\cong$
used with ***measurement***	used with ***figures***
$m\overline{AB} = AB = 5\ cm$	$\overline{AB} \cong \overline{BA}$
$m\angle ABC = 60^\circ$	$\angle ABC \cong \angle CBA$

Midpoints

Midpoint: A point on a line segment that divides it into two congruent segments.

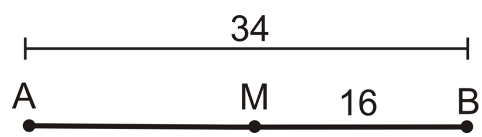


Because $AB = BC$, B is the midpoint of $\overline{AC}$.

Midpoint Postulate: Any line segment will have exactly one midpoint.

This might seem self-explanatory. However, be careful, this postulate is referring to the *midpoint,* not the lines that pass through the midpoint, which is infinitely many.

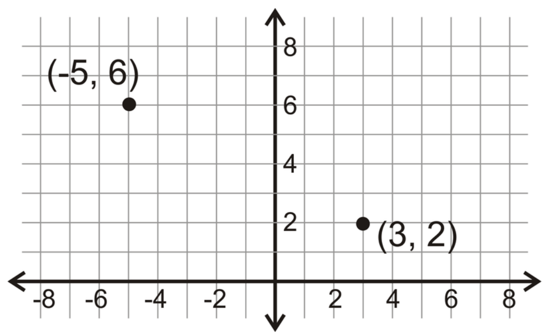
Example 1: Is M a midpoint of $\overline{AB}$?



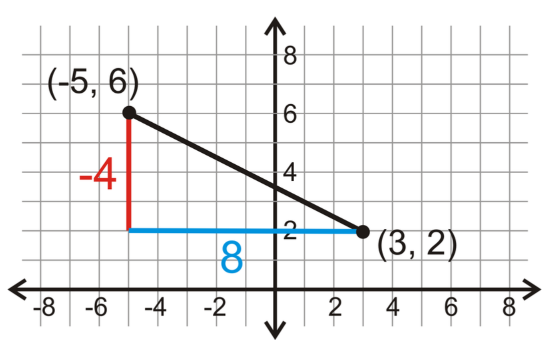
Solution: No, it is not because $MB = 16$ and $AM = 34 - 16 = 18$.

Midpoint Formula

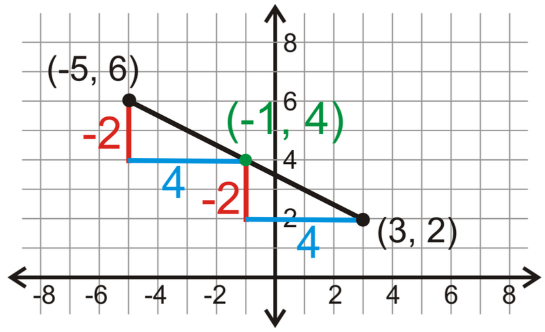
When points are plotted in the coordinate plane, you can use slope to find the midpoint between then. We will generate a formula here.



Here are two points, (-5, 6) and (3, 4). Draw a line between the two points and determine the vertical distance and the horizontal distance.



So, it follows that the midpoint is down and over half of each distance. The midpoint would then be down 2 (or -2) from (-5, 6) and over positively 4. If we do that we find that the midpoint is (-1, 4).



Let's create a formula from this. If the two endpoints are (-5, 6) and (3, 4), then the midpoint is (-1, 4). -1 is ***halfway*** between -5 and 3 and 4 is ***halfway*** between 6 and 2. Therefore, the formula for the midpoint is the average of the $x-$values and the average of the $y-$values.

Midpoint Formula: For two points, $(x_1,\ y_1)$ and $(x_2,\ y_2)$, the midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Example 2: Find the midpoint between (9, -2) and (-5, 14).

Solution: Plug the points into the formula.

$$\left(\frac{9+(-5)}{2}, \frac{-2+14}{2}\right) = \left(\frac{4}{2}, \frac{12}{2}\right) = (2,6)$$

Example 3: If $M(3,-1)$ is the midpoint of $\overline{AB}$ and $B(7,-6)$, find A.

Solution: Plug what you know into the midpoint formula.

$$\begin{aligned}\left(\frac{7+x_A}{2}, \frac{-6+y_A}{2}\right) &= (3,-1) \\ \frac{7+x_A}{2} = 3 \text{ and } \frac{-6+y_A}{2} &= -1 \qquad A \text{ is } (-1,4). \\ 7+x_A = 6 \text{ and } -6+y_A &= -2 \\ x_A = -1 \text{ and } y_A &= 4\end{aligned}$$

Another way to find the other endpoint is to find the difference between M and B and then duplicate it on the other side of M.

$x-$ *values*: $7-3=4$, so 4 on the other side of 3 is $3-4=-1$

$y-$ *values*: $-6-(-1)=-5$, so -5 on the other side of -1 is $-1-(-5)=4$

A is still (-1, 4). You may use either method.

Segment Bisectors

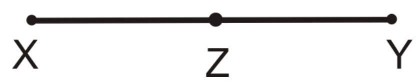
Segment Bisector: A line, segment, or ray that passes through a midpoint of another segment.

A bisector cuts a line segment into two congruent parts.

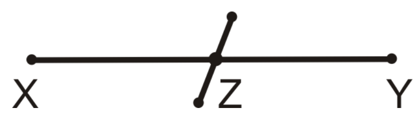
Example 4: Use a ruler to draw a bisector of the segment below.

Solution: The first step in identifying a bisector is finding the midpoint. Measure the line segment and it is 4 cm long. To find the midpoint, divide 4 by 2.

So, the midpoint will be 2 cm from either endpoint, or halfway between. Measure 2 cm from one endpoint and draw the midpoint.

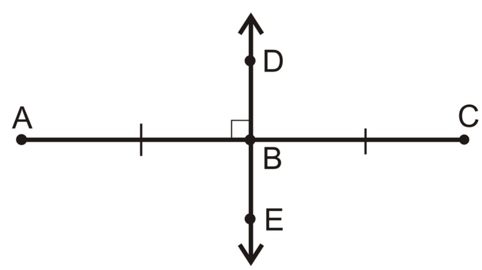


To finish, draw a line that passes through the midpoint. It doesn't matter how the line intersects $\overline{XY}$, as long as it passes through Z.



A specific type of segment bisector is called a perpendicular bisector.

Perpendicular Bisector: A line, ray or segment that passes through the midpoint of another segment and intersects the segment at a right angle.

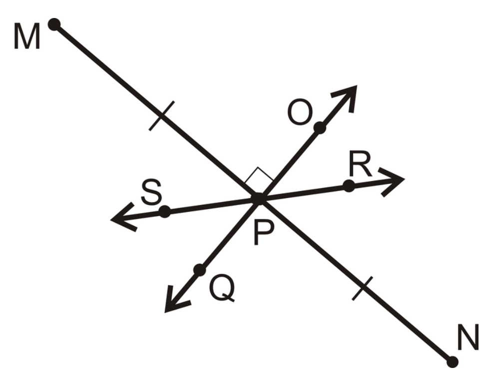


$\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AC}$, so $\overline{AB} \cong \overline{BC}$ and $\overline{AC} \perp \overleftrightarrow{DE}$.

Perpendicular Bisector Postulate: For every line segment, there is one perpendicular bisector that passes through the midpoint.

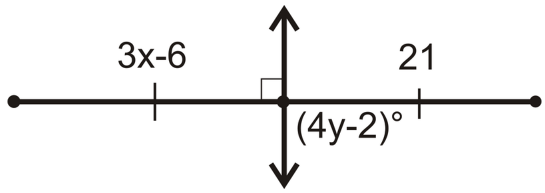
There are *infinitely many bisectors*, but only *one perpendicular bisector* for any segment.

Example 5: Which line is the perpendicular bisector of $\overline{MN}$?



Solution: The perpendicular bisector must bisect $\overline{MN}$ and be perpendicular to it. Only $\overleftrightarrow{OQ}$ satisfies both requirements. $\overleftrightarrow{SR}$ is just a bisector.

Example 6: ***Algebra Connection*** Find x and y.



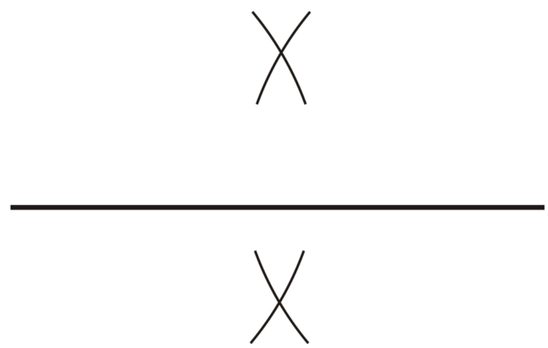
Solution: The line shown is the perpendicular bisector. So, $3x-6=21,\ 3x=27,\ x=9$.

And, $(4y-2)^\circ = 90^\circ, 4y^\circ = 92^\circ, y = 23^\circ$.

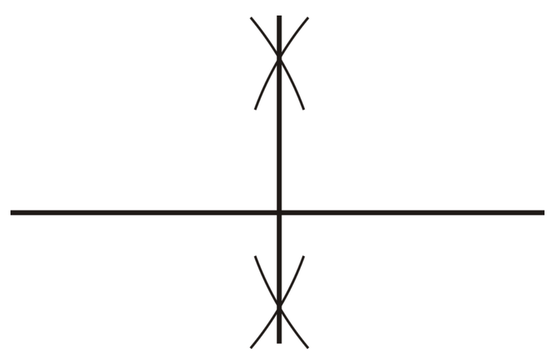
Investigation 1-3: Constructing a Perpendicular Bisector

a. Draw a line that is at least 6 cm long, about halfway down your page.

b. Place the pointer of the compass at an endpoint. Open the compass to be greater than half of the segment. Make arc marks above and below the segment. Repeat on the other endpoint. Make sure the arc marks intersect.



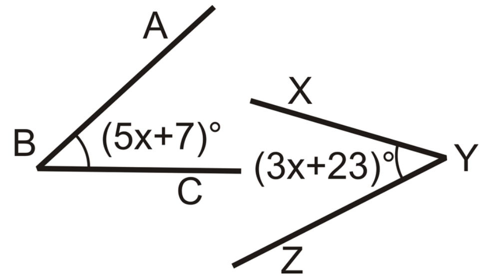
c. Use your straight edge to draw a line connecting the arc intersections.



This constructed line bisects the line you drew in #1 and intersects it at 90°. So, this construction also works to create a right angle. To see an animation of this investigation, go to http://www.mathsisfun.com/geometry/construct-linebisect.html .

Congruent Angles

Example 7: ***Algebra Connection*** What is the measure of each angle?



Solution: From the picture, we see that the angles are congruent, so the given measures are equal.

$$\begin{aligned}(5x+7)^\circ &= (3x+23)^\circ \\ 2x^\circ &= 16^\circ \\ x &= 8^\circ\end{aligned}$$

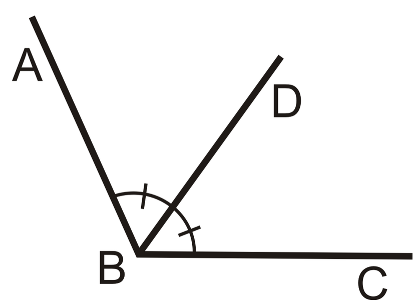
To find the measure of $\angle ABC$, plug in $x = 8^\circ$ to $(5x+7)^\circ$.

$$\begin{aligned}&(5(8)+7)^\circ \\ &(40+7)^\circ \\ &47^\circ\end{aligned}$$

Because $m\angle ABC = m\angle XYZ$, $m\angle XYZ = 47^\circ$ too.

Angle Bisectors

Angle Bisector: A ray that divides an angle into two congruent angles, each having a measure exactly half of the original angle.

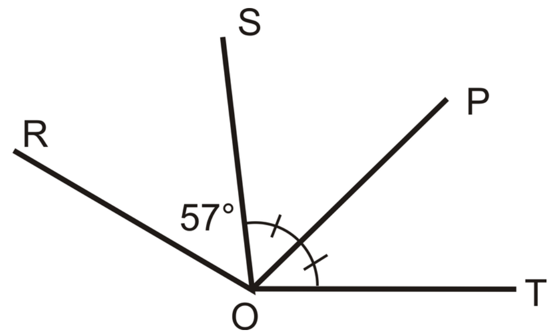


$\overline{BD}$ is the angle bisector of $\angle ABC$

$$\begin{aligned}\angle ABD &\cong \angle DBC \\ m\angle ABD &= \frac{1}{2} m\angle ABC\end{aligned}$$

Angle Bisector Postulate: Every angle has exactly one angle bisector.

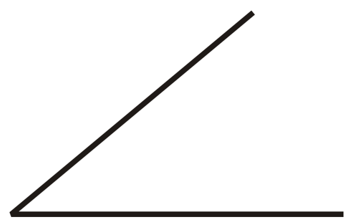
Example 8: Let's take a look at Review Queue #1 again. Is $\overline{OP}$ the angle bisector of $\angle SOT$? Recall, that $m\angle ROT = 165^\circ$, what is $m\angle SOP$ and $m\angle POT$?



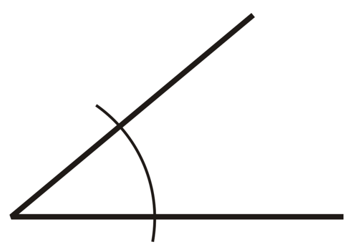
Solution: Yes, $\overline{OP}$ is the angle bisector of $\angle SOT$ according to the markings in the picture. If $m\angle ROT = 165^\circ$ and $m\angle ROS = 57^\circ$, then $m\angle SOT = 165^\circ - 57^\circ = 108^\circ$. The $m\angle SOP$ and $m\angle POT$ are each half of 108° or 54°.

Investigation 1-4: Constructing an Angle Bisector

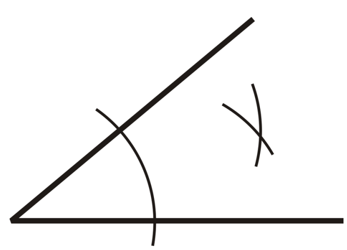
a. Draw an angle on your paper. Make sure one side is horizontal.



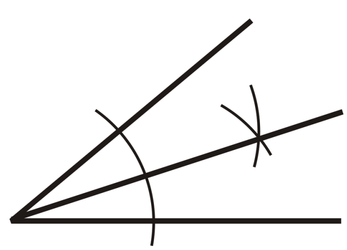
b. Place the pointer on the vertex. Draw an arc that intersects both sides.



c. Move the pointer to the arc intersection with the horizontal side. Make a second arc mark on the interior of the angle. Repeat on the other side. Make sure they intersect.



d. Connect the arc intersections from #3 with the vertex of the angle.

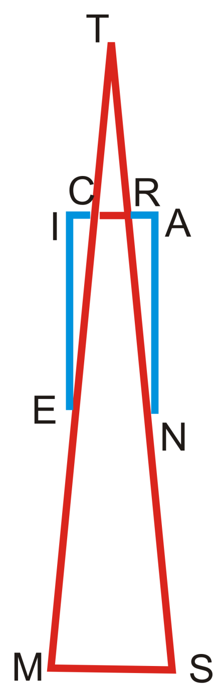


To see an animation of this construction, view http://www.mathsisfun.com/geometry/construct-anglebisect.html .

Know What? Revisited The image to the right is an outline of the Transamerica Building from earlier in the lesson. From this outline, we can see the following parts are congruent:

$$\overline{TR} \cong \overline{TC} \qquad \angle TCR \cong \angle TRC$$
$$\overline{RS} \cong \overline{CM} \qquad \angle CIE \cong \angle RAN$$
$$\overline{CI} \cong \overline{RA} \quad \text{and} \quad \angle TMS \cong \angle TSM$$
$$\overline{AN} \cong \overline{IE} \qquad \angle IEC \cong \angle ANR$$
$$\overline{TS} \cong \overline{TM} \qquad \angle TCI \cong \angle TRA$$

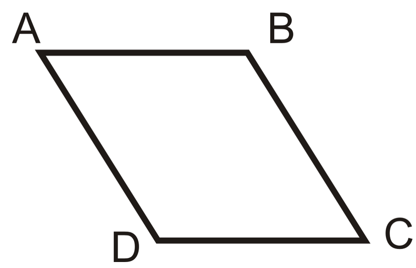
As well at these components, there are certain windows that are congruent and all four triangular sides of the building are congruent to each other.



Review Questions

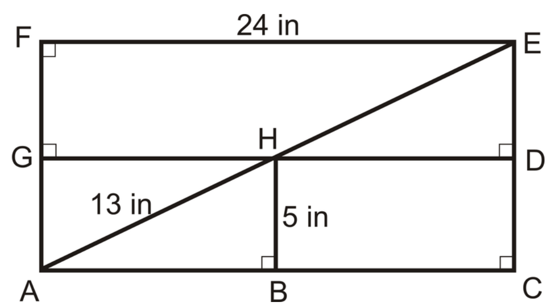
1. Copy the figure below and label it with the following information:

$$\angle A \cong \angle C$$
$$\angle B \cong \angle D$$
$$\overline{AB} \cong \overline{CD}$$
$$\overline{AD} \cong \overline{BC}$$



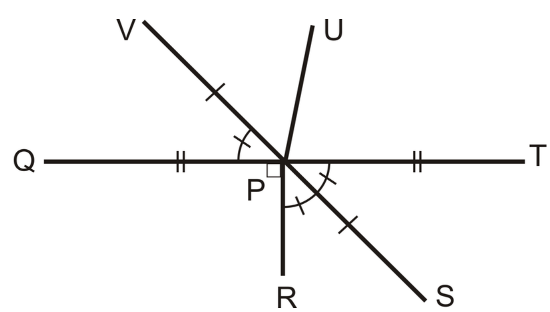
For 2-9, find the lengths, given: H is the midpoint of $\overline{AE}$ and $\overline{DG}$, B is the midpoint of $\overline{AC}$, $\overline{GD}$ is the perpendicular bisector of $\overline{FA}$ and $\overline{EC}$, $\overline{AC} \cong \overline{FE}$, and $\overline{FA} \cong \overline{EC}$.

2. AB
3. GA
4. ED
5. HE
6. $m\angle HDC$
7. FA
8. GD
9. $m\angle FED$



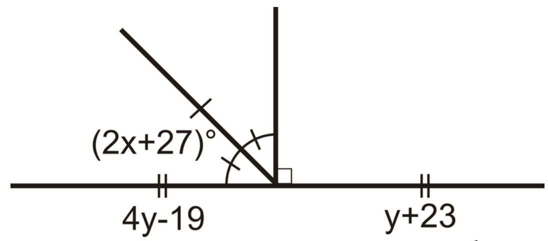
10. How many copies of triangle AHB can fit inside rectangle $FECA$ without overlapping?

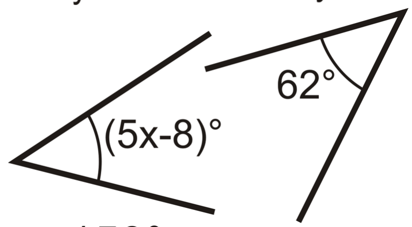
For 11-18, use the following picture to answer the questions.

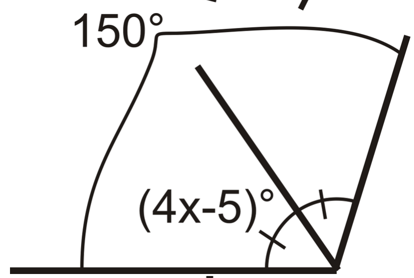


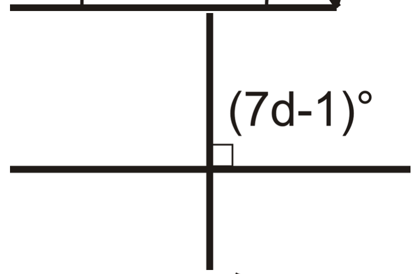
11. What is the angle bisector of $\angle TPR$?
12. P is the midpoint of what two segments?
13. What is $m\angle QPR$?
14. What is $m\angle TPS$?
15. How does $\overline{VS}$ relate to $\overline{QT}$?
16. How does $\overline{QT}$ relate to $\overline{VS}$?
17. Is $\overline{PU}$ a bisector? If so, of what?
18. What is $m\angle QPV$?

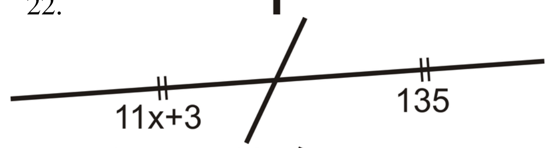
Algebra Connection For 19-24, use algebra to determine the value of variable(s) in each problem.

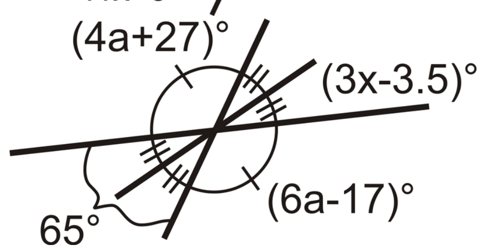
19. 

20. 

21. 

22. 

23. 

24. 

25. ***Construction*** Using your protractor, draw an angle that is 110°. Then, use your compass to construct the angle bisector. What is the measure of each angle?
26. ***Construction*** Using your protractor, draw an angle that is 75°. Then, use your compass to construct the angle bisector. What is the measure of each angle?
27. ***Construction*** Using your ruler, draw a line segment that is 7 cm long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?
28. ***Construction*** Using your ruler, draw a line segment that is 4 in long. Then use your compass to construct the perpendicular bisector, What is the measure of each segment?
29. ***Construction*** Draw a straight angle (180°). Then, use your compass to construct the angle bisector. What kind of angle did you just construct?

For questions 30-33, find the midpoint between each pair of points.

30. (-2, -3) and (8, -7)
31. (9, -1) and (-6, -11)
32. (-4, 10) and (14, 0)
33. (0, -5) and (-9, 9)

Given the midpoint (M) and either endpoint of $\overline{AB}$, find the other endpoint.

34. $A(-1,2)$ and $M(3,6)$
35. $B(-10,-7)$ and $M(-2,1)$
36. ***Error Analysis*** Erica is looking at a geometric figure and trying to determine which parts are congruent. She wrote $\overline{AB} = \overline{CD}$. Is this correct? Why or why not?
37. ***Challenge*** Use the Midpoint Formula to solve for the $x-$value of the midpoint and the $y-$value of the midpoint. Then, use this formula to solve #34. Do you get the same answer?
38. ***Construction Challenge*** Use construction tools and the constructions you have learned in this section to construct a 45° angle.
39. ***Construction Challenge*** Use construction tools and the constructions you have learned in this section to construct two 2 in segments that bisect each other. Now connect all four endpoints with segments. What figure have you constructed?
40. Describe an example of how the concept of midpoint (or the midpoint formula) could be used in the real world.

Review Queue Answers

a. See Example 6
b. $2x - 5 = 33$
   $2x = 38$
   $x = 19$
c. $m\angle ROT = m\angle ROS + m\angle SOP + m\angle POT$

3.6 Finding the Slope and Equation of a Line

Objective

To review how to find the slope and equation of a line.

Review Queue

1. Plot the following points on the same graph.

a) (4, -2)

b) (-2, -7)

c) (6, 1)

d) (0, 8)

Solve the following equations for the indicated variable.

2. $3x-4y=12; x$

3. $3x-4y=12; y$

4. $2b+5c=-10; b$

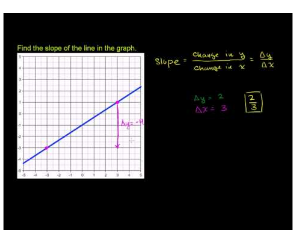
5. $2b+5c=-10; c$

Finding Slope

Objective

To find the slope of a line and between two points.

Watch This



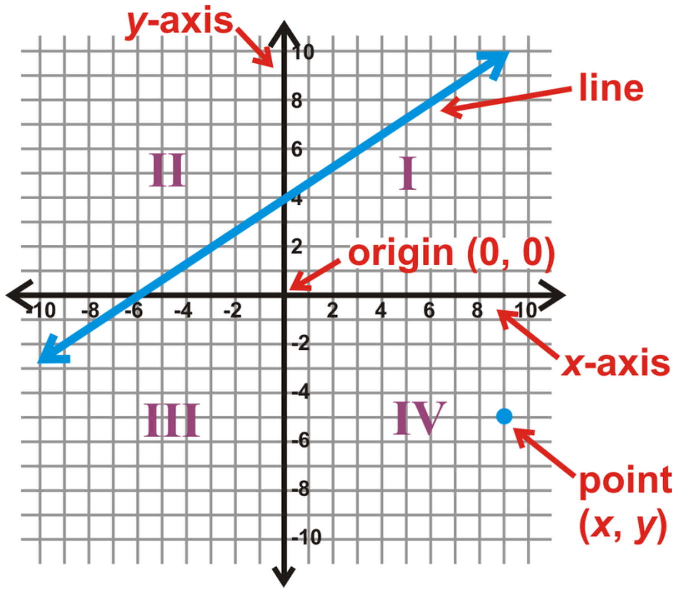
MEDIA

Click image to the left for more content.

Khan Academy: Slope of a line

Guidance

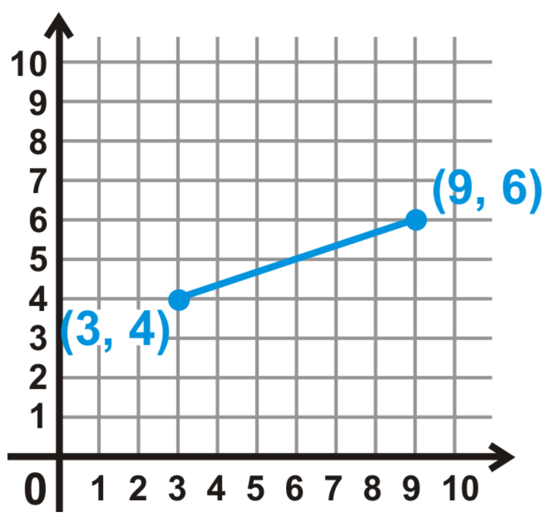
The slope of a line determines how steep or flat it is. When we place a line in the coordinate plane, we can measure the slope, or steepness, of a line. Recall the parts of the coordinate plane, also called an $x-y$ plane and the Cartesian plane, after the mathematician Descartes.



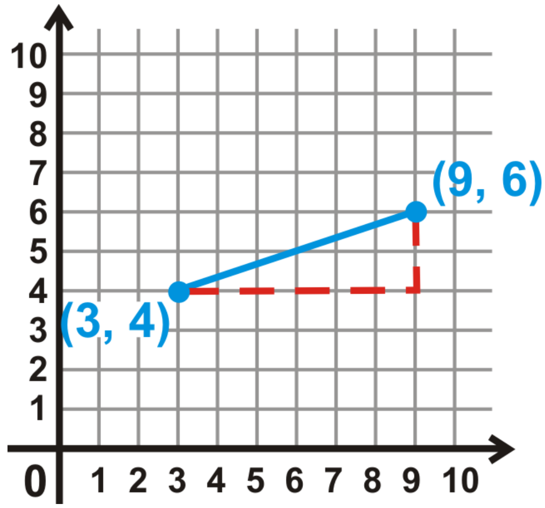
To plot a point, order matters. First, every point is written (x, y), where x is the movement in the $x-$direction and y is the movement in the $y-$direction. If x is negative, the point will be in the 2^{nd} or 3^{rd} quadrants. If y is negative, the point will be in the 3^{rd} or 4^{th} quadrants. The quadrants are always labeled in a counter-clockwise direction and using Roman numerals.

The point in the 4^{th} quadrant would be (9, -5).

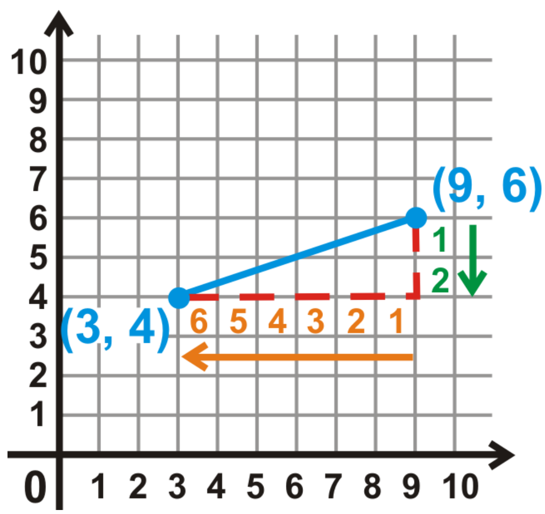
To find the slope of a line or between two points, first, we start with right triangles. Let's take the two points (9, 6) and (3, 4). Plotting them on an $x-y$ plane, we have:



To turn this segment into a right triangle, draw a vertical line down from the higher point, and a horizontal line from the lower point, towards the vertical line. Where the two lines intersect is the third vertex of the slope triangle.



Now, count the vertical and horizontal units along the horizontal and vertical sides (**red** dotted lines).

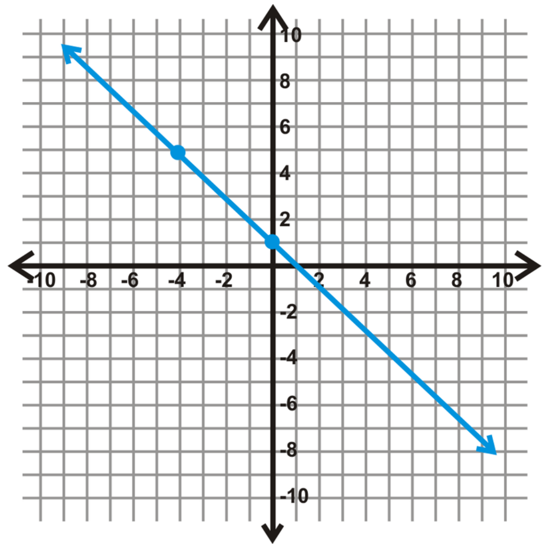


The slope is a fraction with the vertical distance over the horizontal distance, also called the "rise over run." Because the vertical distance goes down, we say that it is -2. The horizontal distance moves towards the negative direction (the left), so we would say that it is -6. So, for slope between these two points, the slope would be $\frac{-2}{-6}$ or $\frac{1}{3}$.

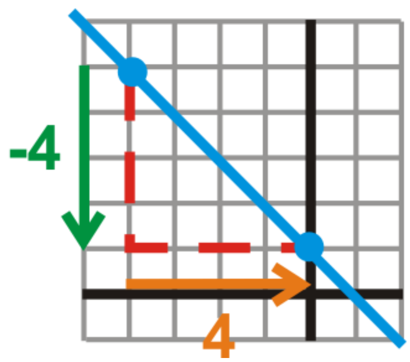
Note: You can also draw the right triangle above the line segment.

Example A

Use a slope triangle to find the slope of the line below.



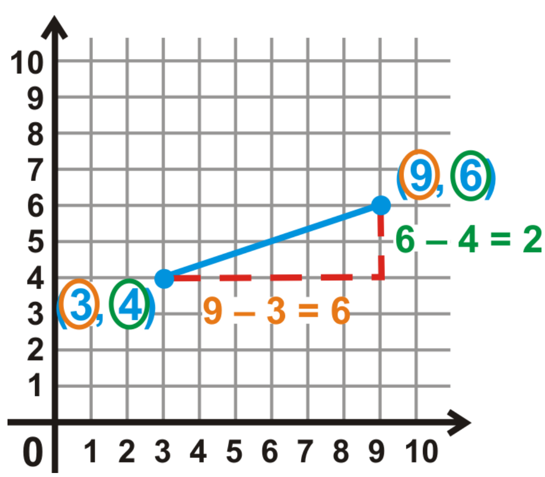
Solution: Notice the two points that are drawn on the line. These are given to help you find the slope. Draw a triangle between these points and find the slope.



From the slope triangle above, we see that the slope is $\frac{-4}{4} = -1$.

Whenever a slope reduces to a whole number, the "run" will always be positive 1. Also, notice that this line points in the opposite direction as the line segment above. We say this line has a *negative* slope because the slope is a negative number and points from the 2^{nd} to 4^{th} quadrants. A line with positive slope will point in the opposite direction and point between the 1^{st} and 3^{rd} quadrants.

If we go back to our previous example with points (9, 6) and (3, 4), we can find the vertical distance and horizontal distance another way.



From the picture, we see that the vertical distance is the same as the difference between the $y-$values and the horizontal distance is the difference between the $x-$values. Therefore, the slope is $\frac{6-4}{9-3}$. We can extend this idea to any two points, (x_1, y_1) and (x_2, y_2).

Slope Formula: For two points (x_1, y_1) and (x_2, y_2), the slope between them is $\frac{y_2-y_1}{x_2-x_1}$. The symbol for slope is m.

It does not matter which point you choose as (x_1, y_1) or (x_2, y_2).

Example B

Find the slope between (-4, 1) and (6, -5).

Solution: Use the Slope Formula above. Set $(x_1, y_1) = (-4, 1)$ and $(x_2, y_2) = (6, -5)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-4)}{-5-1} = \frac{10}{-6} = -\frac{5}{3}$$

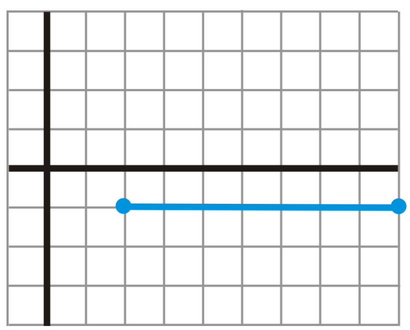
Example C

Find the slope between (9, -1) and (2, -1).

Solution: Use the Slope Formula. Set $(x_1, y_1) = (9, -1)$ and $(x_2, y_2) = (2, -1)$.

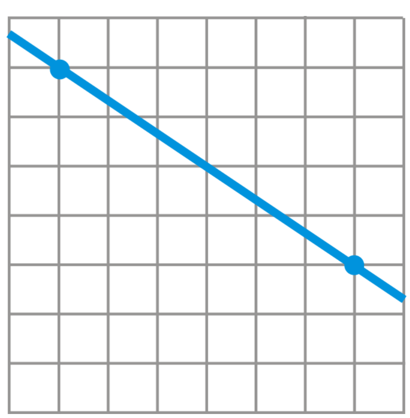
$$m = \frac{-1-(-1)}{2-9} = \frac{0}{-7} = 0$$

Here, we have zero slope. Plotting these two points we have a horizontal line. This is because the $y-$values are the same. Anytime the $y-$values are the same we will have a horizontal line and the slope will be zero.



Guided Practice

1. Use a slope triangle to find the slope of the line below.



2. Find the slope between (2, 7) and (-3, -3).

3. Find the slope between (-4, 5) and (-4, -1).

Answers

1. Counting the squares, the vertical distance is down 6, or -6, and the horizontal distance is to the right 8, or +8. The slope is then $\frac{-6}{8}$ or $-\frac{2}{3}$.

2. Use the Slope Formula. Set $(x_1, y_1) = (2, 7)$ and $(x_2, y_2) = (-3, -3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-7}{-3-2} = \frac{-10}{-5} = 2$$

3. Again, use the Slope Formula. Set $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (-4, -1)$.

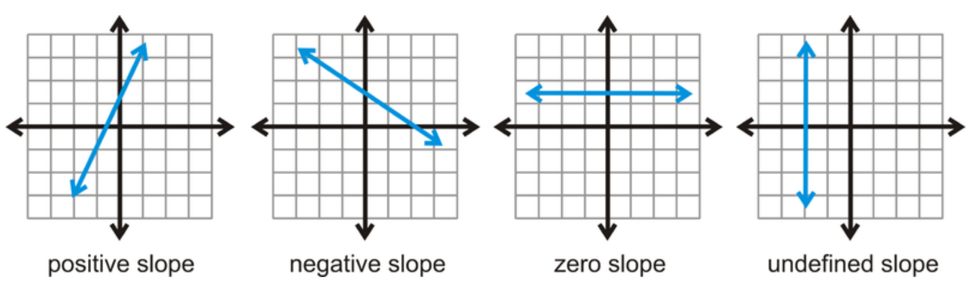
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{-4-(-4)} = \frac{-6}{0}$$

You cannot divide by zero. Therefore, this slope is undefined. If you were to plot these points, you would find they form a vertical line. *All vertical lines have an undefined slope.*

Important Note: Always reduce your slope fractions. Also, if the numerator or denominator of a slope is negative, then the slope is negative. If they are both negative, then we have a negative number divided by a negative number, which is positive, thus a positive slope.

Vocabulary

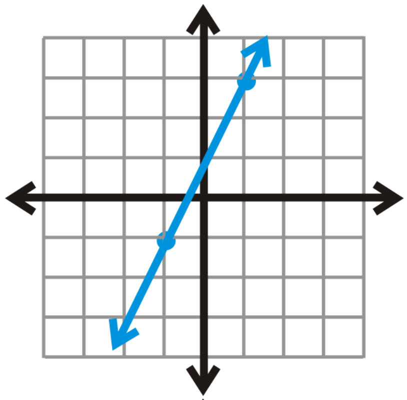
Slope
The steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. Slope can also be called "rise over run" or "the change in the $y-$values over the change in the $x-$values." The symbol for slope is m.

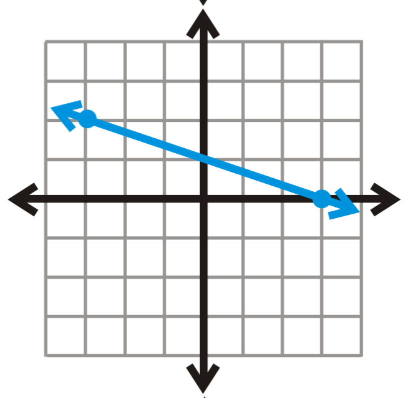


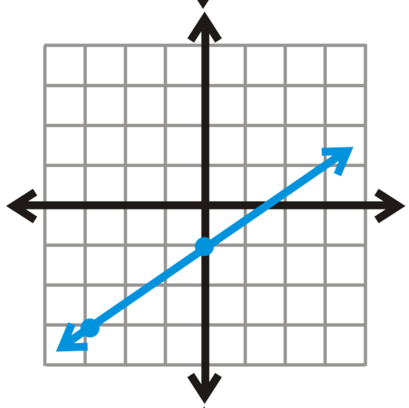
Slope Formula
For two points (x_1, y_1) and (x_2, y_2), the slope between them is $\frac{y_2 - y_1}{x_2 - x_1}$.

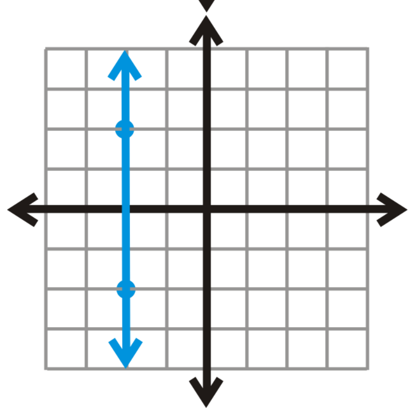
Problem Set

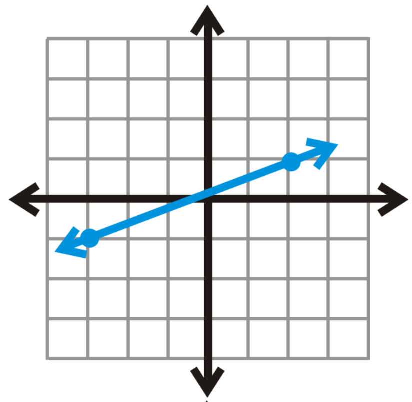
Find the slope of each line by using slope triangles.

1. 

2. 

3. 

4. 



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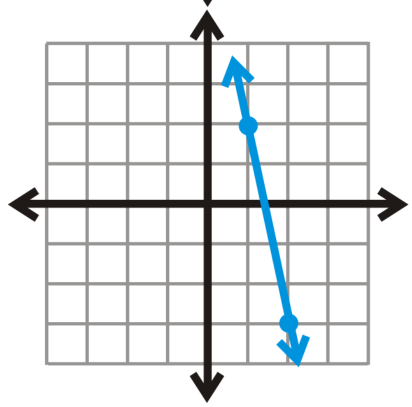


6.

Find the slope between each pair of points using the Slope Formula.

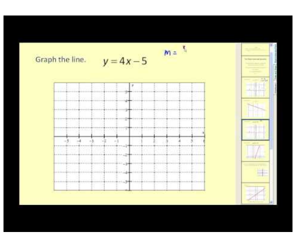
7. (-5, 6) and (-3, 0)
8. (1, -1) and (6, -1)
9. (3, 2) and (-9, -2)
10. (8, -4) and (8, 1)
11. (10, 2) and (4, 3)
12. (-3, -7) and (-6, -3)
13. (4, -5) and (0, -13)
14. (4, -15) and (-6, -11)
15. (12, 7) and (10, -1)
16. **Challenge** The slope between two points (a,b) and (1, -2) is $\frac{1}{2}$. Find a and b.

Finding the Equation of a Line in Slope-Intercept Form

Objective

To find the equation of a line (the slope and $y-$intercept) in slope-intercept form.

Watch This



MEDIA

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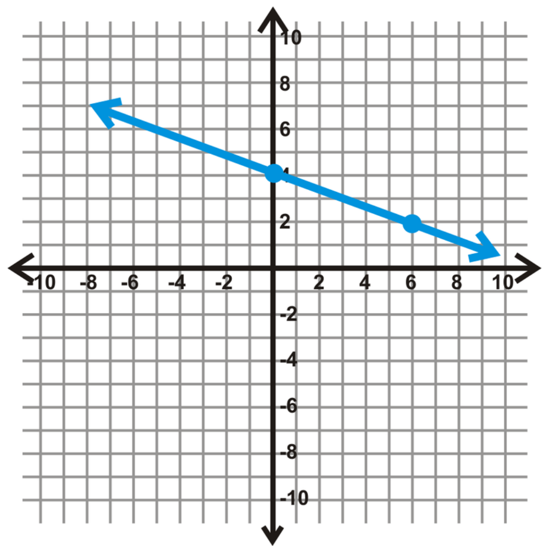
James Sousa: Slope Intercept Form of a Line

Guidance

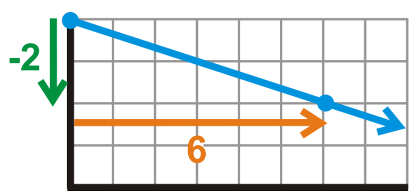
In the previous concept, we found the slope between two points. We will now find the entire equation of a line. Recall from Algebra I that the equation of a line in slope-intercept form is $y = mx + b$, where m is the slope and b is the $y-$intercept. You can find the slope either by using slope triangles or the Slope Formula. To find the $y-$intercept, or b, you can either locate where the line crosses the $y-$axis (if given the graph) or by using algebra.

Example A

Find the equation of the line below.



Solution: Analyze the line. We are given two points on the line, one of which is the $y-$intercept. From the graph, it looks like the line passes through the $y-$axis at (0, 4), making $b = 4$. Now, we need to find the slope. You can use slope triangles or the Slope Formula. Using slope triangles, we have:



From this, we see that the slope is $-\frac{2}{6}$ or $-\frac{1}{3}$.

Plugging our found information into the slope-intercept equation, the equation of this line is $y = -\frac{1}{3}x + 4$.

Alternate Method: If we had used the Slope Formula, we would use (0, 4) and (6, 2), which are the values of the given points.

$$m = \frac{2-4}{6-0} = \frac{-2}{6} = -\frac{1}{3}$$

Example B

The slope of a line is -4 and the $y-$intercept is (0, 3). What is the equation of the line?

Solution: This problem explicitly tells us the slope and $y-$intercept. The slope is -4, meaning $m = -4$. The $y-$intercept is (0, 3), meaning $b = 3$. Therefore, the equation of the line is $y = -4x + 3$.

Example C

The slope of a line is $\frac{1}{2}$ and it passes through the point (4, -7). What is the equation of the line?

Solution: In this problem, we are given m and a point on the line. The point, (4, -7) can be substituted in for x and y in the equation. We need to solve for the $y-$intercept, or b. Plug in what you know to the slope-intercept equation.

$$\begin{aligned}
y &= mx + b \\
-7 &= \frac{1}{2}(4) + b \\
-7 &= 2 + b \\
-9 &= b
\end{aligned}$$

From this, the equation of the line is $y = \frac{1}{2}x - 9$.

We can test if a point is on a line or not by plugging it into the equation. If the equation holds true, the point is on the line. If not, then the point is not on the line.

Example D

Find the equation of the line that passes through (12, 7) and (10, -1).

Solution: In this example, we are not given the slope or the $y-$intercept. First, we need to find the slope using the Slope Formula.

$$m = \frac{-1-7}{10-12} = \frac{-8}{-2} = 4$$

Now, plug in one of the points for x and y. It does not matter which point you choose because they are both on the line.

$$\begin{aligned}
7 &= 4(12) + b \\
7 &= 48 + b \\
-41 &= b
\end{aligned}$$

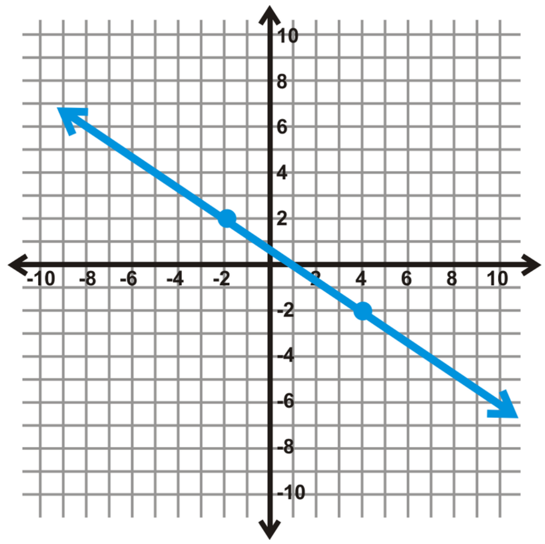
The equation of the line is $y = 4x - 41$.

Guided Practice

1. What is the equation of the line where the slope is 1 and passes through (5, 3)?

2. Find the equation of the line that passes through (9, -4) and (-1, -8).

3. Find the equation of the line below.



Answers

1. We are told that $m = 1, x = 5$, and $y = 3$. Plug this into the slope-intercept equation and solve for b.

$$\begin{aligned} 3 &= 1(5) + b \\ 3 &= 5 + b \\ -2 &= b \end{aligned}$$

The equation of the line is $y = x - 2$

2. First, find the slope.

$$m = \frac{-8-(-4)}{-1-9} = \frac{-4}{-10} = \frac{2}{5}$$

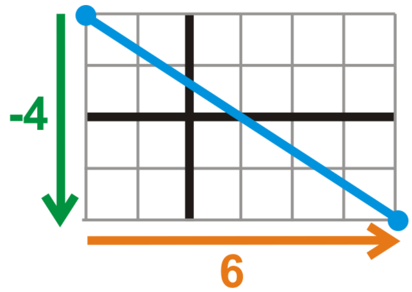
Now, find the $y-$intercept. We will use the second point. Remember, it does not matter which point you use.

$$\begin{aligned} -8 &= \frac{2}{5}(-1) + b \\ -8 &= -\frac{2}{5} + b \\ -7\frac{3}{5} &= b \end{aligned}$$

The equation of the line is $y = \frac{2}{5}x - 7\frac{3}{5}$ or $y = \frac{2}{5}x - \frac{38}{5}$.

When your $y-$intercept is a fraction, make sure it is reduced. Double-check with your teacher on how s/he wants you to leave your answer.

3. We can find the slope one of two ways: using slope triangles or by using the Slope Formula. We are given (by the drawn points in the picture) that (-2, 2) and (4, -2) are on the line. Drawing a slope triangle, we have:



We have that the slope is $-\frac{4}{6}$ or $-\frac{2}{3}$. To find the $y-$intercept, it looks like it is somewhere between 0 and 1. Take one of the points and plug in what you know to the slope-intercept equation.

$$\begin{aligned}2 &= -\frac{2}{3}(-2)+b \\ 2 &= \frac{4}{3}+b \\ \frac{2}{3} &= b\end{aligned}$$

The equation of the line is $y=-\frac{2}{3}x+\frac{2}{3}$.

Vocabulary

Slope-Intercept Form
The equation of a line in the form $y=mx+b$, where m is the slope and b is the $y-$intercept.

$y-$**intercept**
The point where a line crosses the $y-$axis. This point will always have the form $(0,y)$.

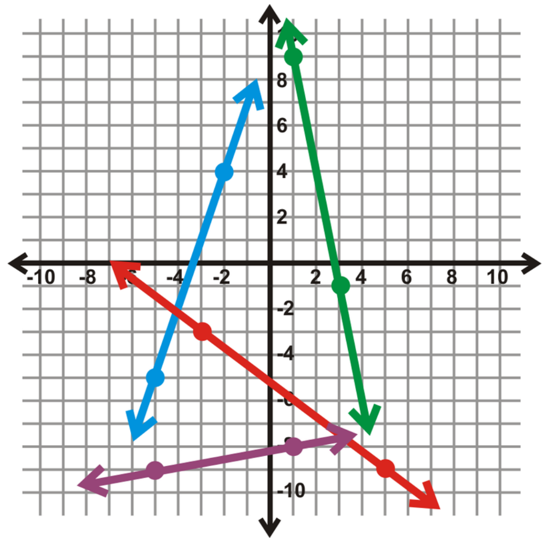
$x-$**intercept**
The point where a line crosses the $x-$axis. This point will always have the form $(x,0)$.

Problem Set

Find the equation of each line with the given information below.

1. slope = 2, $y-$intercept = (0, 3)
2. $m=-\frac{1}{4}$, $b=2.6$
3. slope = -1, $y-$intercept = (0, 2)
4. $x-$intercept = (-2, 0), $y-$intercept = (0, -5)
5. slope $=\frac{2}{3}$ and passes through (6, -4)
6. slope $=-\frac{3}{4}$ and passes through (-2, 5)
7. slope = -3 and passes through (-1, -7)
8. slope = 1 and passes through (2, 4)
9. passes through (-5, 4) and (1, 1)
10. passes through (5, -1) and (-10, -10)
11. passes through (-3, 8) and (6, 5)
12. passes through (-4, -21) and (2, 9)

For problems 13-16, find the equation of the lines in the graph below.



13. Green Line
14. Blue Line
15. Red Line
16. Purple Line
17. Find the equation of the line with zero slope and passes through (8, -3).
18. Find the equation of the line with zero slope and passes through the point (-4, 5).
19. Find the equation of the line with zero slope and passes through the point (a, b).
20. **Challenge** Find the equation of the line with an *undefined* slope that passes through (a, b).

3.7 Standard Form of a Line

Objective

To familiarize students with the standard form of a line, as well as finding the equations of lines that are parallel or perpendicular to a given line.

Review Queue

1. Solve $4x - y = 6$ for y.
2. Solve $4x - 8y = 12$ for y.
3. What are the slope and $y-$intercept of $y = -\frac{2}{3}x + 5$?
4. Define *parallel* and *perpendicular* in your own words.

Standard Form

Objective

To manipulate and use the standard form of a line.

Guidance

Slope-intercept form is one way to write the equation of a line. Another way is called standard form. Standard form looks like $Ax + By = C$, where A, B, and C are all integers. In the Review Queue above, the equations from problems 1 and 2 are in standard form. Once they are solved for y, they will be in slope-intercept form.

Example A

Find the equation of a line, in standard form, where the slope is $\frac{3}{4}$ and passes through (4, -1).

Solution: To find the equation in standard form, you need to determine what A, B, and C are. Let's start this example by finding the equation in slope-intercept form.

$$\begin{aligned}
-1 &= \frac{3}{4}(4) + b \\
-1 &= 3 + b \\
-4 &= b
\end{aligned}$$

In slope-intercept form, the equation is $y = \frac{3}{4}x - 4$.

To change this to standard form we need to subtract the $x-$term from both sides of the equation.

$$-\frac{3}{4}x + y = -4$$

However, we are not done. In the definition, A, B, and C are all integers. At the moment, A is a fraction. To undo the fraction, we must multiply all the terms by the denominator, 4. We also will multiply by a negative so that the $x-$coefficient will be positive.

$$-4\left(-\frac{3}{4}x+y=-4\right)$$
$$3x-4y=16$$

Example B

The equation of a line is $5x-2y=12$. What are the slope and $y-$intercept?

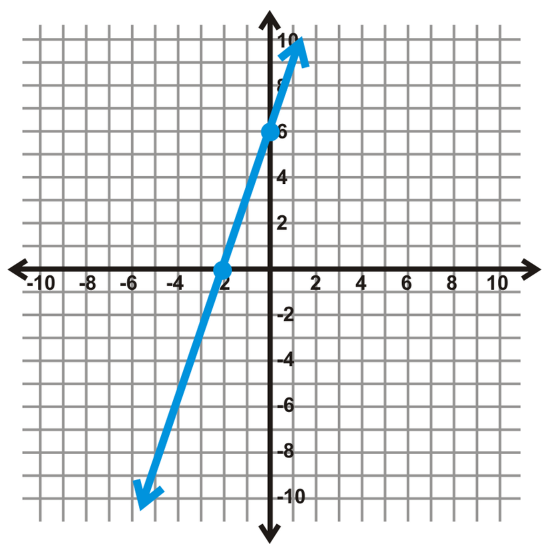
Solution: To find the slope and $y-$intercept of a line in standard form, we need to switch it to slope-intercept form. This means, we need to solve the equation for y.

$$\begin{aligned}5x-2y&=12\\-2y&=-5x+12\\y&=\frac{5}{2}x-6\end{aligned}$$

From this, the slope is $\frac{5}{2}$ and the $y-$intercept is (0, -6).

Example C

Find the equation of the line below, in standard form.



Solution: Here, we are given the intercepts. The slope triangle is drawn by the axes, $\frac{-6}{-2}=3$. And, the $y-$intercept is (0, 6). The equation of the line, in slope-intercept form, is $y=3x+6$. To change the equation to standard form, subtract the $x-$term to move it over to the other side.

$$-3x+y=6 \textit{ or } 3x-y=-6$$

Example D

The equation of a line is $6x-5y=45$. What are the intercepts?

Solution: For the $x-$intercept, the $y-$value is zero. Plug in zero for y and solve for x.

$$\begin{aligned}6x-5y&=45\\6x-5(0)&=45\\6x&=45\\x&=\frac{45}{6}\ or\ \frac{15}{2}\end{aligned}$$

The $x-$intercept is $\left(\frac{15}{2},0\right)$.

For the $y-$intercept, the $x-$value is zero. Plug in zero for x and solve for y.

$$\begin{aligned}6x-5y&=45\\6(0)-5y&=45\\5y&=45\\y&=9\end{aligned}$$

The $y-$intercept is (0, 9).

Guided Practice

1. Find the equation of the line, in standard form that passes through (8, -1) and (-4, 2).

2. Change $2x+3y=9$ to slope-intercept form.

3. What are the intercepts of $3x-4y=-24$?

Answers

1. Like with Example A, we need to first find the equation of this line in $y-$intercept form and then change it to standard form. First, find the slope.

$$\frac{2-(-1)}{-4-8}=\frac{3}{-12}=-\frac{1}{4}$$

Find the $y-$intercept using slope-intercept form.

$$\begin{aligned}2&=-\frac{1}{4}(-4)+b\\2&=1+b\\1&=b\end{aligned}$$

The equation of the line is $y=-\frac{1}{4}x+1$.

To change this equation into standard form, add the $x-$term to both sides and multiply by 4 to get rid of the fraction.

$$\begin{aligned}\frac{1}{4}x+y&=1\\4\left(\frac{1}{4}x+y=1\right)\\x+4y&=1\end{aligned}$$

2. To change $2x+3y=9$ into slope-intercept form, solve for y.

$$\begin{aligned} 2x+3y&=9\\ 3y&=-2x+9\\ y&=-\frac{2}{3}x+3 \end{aligned}$$

3. Copy Example D to find the intercepts of $3x-4y=-24$. First, plug in zero for y and solve for x.

$$\begin{aligned} 3x-4(0)&=-24\\ 3x&=-24\\ x&=-8 \end{aligned}$$

$x-$intercept is (-8, 0)

Now, start over and plug in zero for x and solve for y.

$$\begin{aligned} 3(0)-4y&=-24\\ -4y&=-24\\ y&=6 \end{aligned}$$

$y-$intercept is (6, 0)

Vocabulary

Standard Form (of a line)
When a line is in the form $Ax+By=C$ and A,B, and C are integers.

Problem Set

Change the following equations into standard form.

1. $y=-\frac{2}{3}x+4$
2. $y=x-5$
3. $y=\frac{1}{5}x-1$

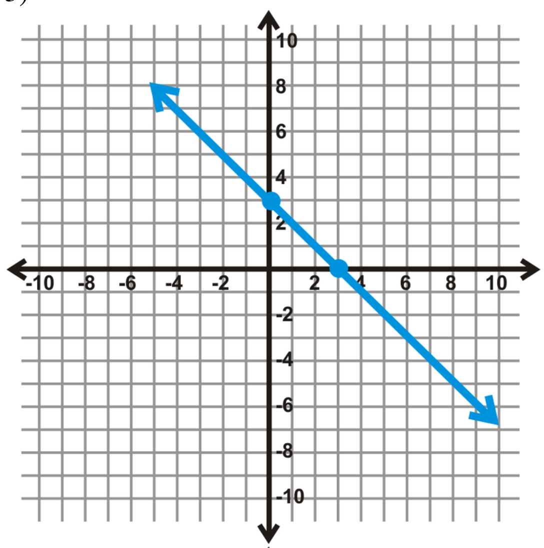
Change the following equations into slope-intercept form.

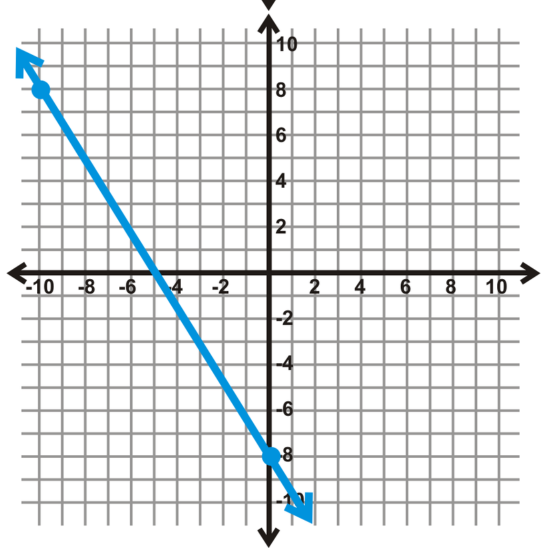
4. $4x+5y=20$
5. $x-2y=9$
6. $2x-3y=15$

Find the x and $y-$intercepts of the following equations.

7. $3x+4y=12$
8. $6x-y=8$
9. $3x+8y=-16$

Find the equation of the lines below, in standard form.

10. slope = 2 and passes through (3, -5)
11. slope $= -\frac{1}{2}$ and passes through (6, -3).
12. passes through (5, -7) and (-1, 2)
13. passes through (-5, -5) and (5, -3)
14. 

15. 

16. Change $Ax + By = C$ into slope-intercept form.
17. From #16, what are the slope and $y-$intercept equal to (in terms of A, B, and/or C)?
18. Using #16 and #17, find one possible combination of A, B, and C for $y = \frac{1}{2}x - 4$. Write your answer in standard form.
19. The measure of a road's slope is called the *grade*. The grade of a road is measured in a percentage, for how many vertical feet the road rises or declines over 100 feet. For example, a road with a grade incline of 5% means that for every 100 horizontal feet the road rises 5 vertical feet. What is the slope of a road with a grade decline of 8%?
20. The population of a small town in northern California gradually increases by about 50 people a year. In 2010, the population was 8500 people. Write an equation for the population of this city and find its estimated population in 2017.

Finding the Equation of Parallel Lines

Objective

To find the equation of a line that is parallel to a given line.

Guidance

When two lines are parallel, they have the same slope and never intersect. So, if a given line has a slope of -2, then any line that is parallel to that line will also have a slope of -2, but it will have a different $y-$intercept.

Example A

Find the equation of the line that is parallel to $y = \frac{2}{3}x - 5$ and passes through (-12, 1).

Solution: We know that the slopes will be the same; however we need to find the $y-$intercept for this new line. Use the point you were given, (-12, 1) and plug it in for x and y to solve for b.

$$\begin{aligned} y &= \frac{2}{3}x + b \\ 1 &= \frac{2}{3}(-12) + b \\ 1 &= -8 + b \\ 9 &= b \end{aligned}$$

The equation of the parallel line is $y = \frac{2}{3}x + 9$.

Example B

Write the equation of the line that passes through (4, -7) and is parallel to $y = -2$.

Solution: The line $y = -2$ does not have an $x-$term, meaning it has no slope. This is a horizontal line. Therefore, to find the horizontal line that passes through (4, -7), we only need the $y-$coordinate. The line would be $y = -7$.

The same would be true for vertical lines, but all vertical line equations are in the form $x = a$. The $x-$coordinate of a given point would be what is needed to determine the equation of the parallel vertical line.

Example C

Write the equation of the line that passes through (6, -10) and is parallel to the line that passes through (4, -6) and (3, -4).

Solution: First, we need to find the slope of the line that our line will be parallel to. Use the points (4, -6) and (3, -4) to find the slope.

$$m = \frac{-4-(-6)}{3-4} = \frac{2}{-1} = -2$$

This is the slope of our given line as well as the parallel line. Use the point (6, -10) to find the $y-$intercept of the line that we are trying to find the equation for.

$$\begin{aligned} -10 &= -2(6) + b \\ -10 &= -12 + b \\ 2 &= b \end{aligned}$$

The equation of the line is $y = -2x + 2$.

Guided Practice

1. Find the equation of the line that is parallel to $x - 2y = 8$ and passes through (4, -3).

2. Find the equation of the line that is parallel to $x = 9$ and passes through (-1, 3).

3. Find the equation of the line that passes through (-5, 2) and is parallel to the line that passes through (6, -1) and (1, 3).

Answers

1. First, we need to change this line from standard form to slope-intercept form.

$$\begin{aligned}
x - 2y &= 8 \\
-2y &= -x + 8 \quad \text{Now, we know the slope is } \frac{1}{2}\text{. Let's find the new } y- \text{ intercept.} \\
y &= \frac{1}{2}x - 4
\end{aligned}$$

$$\begin{aligned}
-3 &= \frac{1}{2}(4) + b \\
-3 &= 2 + b \\
-5 &= b
\end{aligned}$$

The equation of the parallel line is $y = \frac{1}{2}x - 5$ or $x - 2y = 10$.

2. $x = 9$ is a vertical line that passes through the $x-$axis at 9. Therefore, we only need to $x-$coordinate of the point to determine the equation of the parallel vertical line. The parallel line through (-1, 3) would be $x = -1$.

3. First, find the slope between (6, -1) and (1, 3).

$$m = \frac{-1-3}{6-1} = \frac{-4}{5} = -\frac{4}{5}$$

This will also be the slope of the parallel line. Use this slope with the given point, (-5, 2).

$$\begin{aligned}
2 &= -\frac{4}{5}(-5) + b \\
2 &= 1 + b \\
1 &= b
\end{aligned}$$

The equation of the parallel line is $y = -\frac{4}{5}x + 1$.

Vocabulary

Parallel
: When two or more lines are in the same plane and never intersect. These lines will always have the same slope.

Problem Set

Find the equation of the line, given the following information. You may leave your answer in slope-intercept form.

1. Passes through (4, 7) and is parallel to $x - y = -5$.
2. Passes through (-6, -2) and is parallel to $y = 4$.
3. Passes through (-3, 5) and is parallel to $y = -\frac{1}{3}x - 1$.
4. Passes through (1, -9) and is parallel to $x = 8$.
5. Passes through the $y-$intercept of $2x - 3y = 6$ and parallel to $x - 4y = 10$.
6. Passes through (-12, 4) and is parallel to $y = -3x + 5$.
7. Passes through the $x-$intercept of $2x - 3y = 6$ and parallel to $x + 4y = -3$.
8. Passes through (7, -8) and is parallel to $2x + 5y = 14$.
9. Passes through (1, 3) and is parallel to the line that passes through (-6, 2) and (-4, 6).
10. Passes through (-18, -10) and is parallel to the line that passes through (-2, 2) and (-8, 1).
11. Passes through (-4, -1) and is parallel to the line that passes through (15, 7) and (-1, -1).

Are the pairs of lines parallel? Briefly explain how you know.

12. $x - 2y = 4$ and $-5x + 10y = 16$
13. $3x + 4y = -8$ and $6x + 12y = -1$
14. $5x - 5y = 20$ and $x + y = 7$
15. $8x - 12y = 36$ and $10x - 15y = -15$

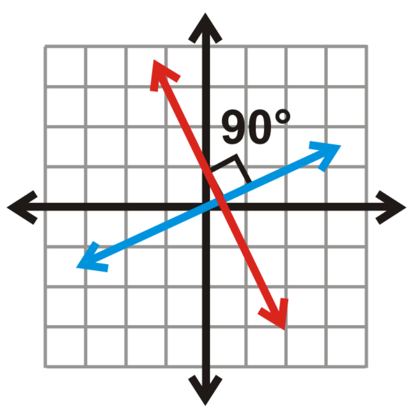
Finding the Equation of Perpendicular Lines

Objective

To find the equation of a line that is perpendicular to a given line and determine if pairs of lines are parallel, perpendicular, or neither.

Guidance

When two lines are perpendicular, they intersect at a 90°, or right, angle. The slopes of two perpendicular lines, are therefore, not the same. Let's investigate the relationship of perpendicular lines.



Investigation: Slopes of Perpendicular Lines

Tools Needed: Pencil, ruler, protractor, and graph paper

1. Draw an $x - y$ plane that goes from -5 to 5 in both the x and y directions.
2. Plot (0, 0) and (1, 3). Connect these to form a line.
3. Plot (0, 0) and (-3, 1). Connect these to form a second line.
4. Using a protractor, measure the angle formed by the two lines. What is it?

5. Use slope triangles to find the slope of both lines. What are they?
6. Multiply the slope of the first line times the slope of the second line. What do you get?

From this investigation, the lines from #2 and #3 are perpendicular because they form a 90° angle. The slopes are 3 and $-\frac{1}{3}$, respectively. When multiplied together, the product is -1. This is true of all perpendicular lines.

The product of the slopes of two perpendicular lines is -1.

If a line has a slope of m, then the perpendicular slope is $-\frac{1}{m}$.

Example A

Find the equation of the line that is perpendicular to $2x-3y=15$ and passes through (6, 5).

Solution: First, we need to change the line from standard to slope-intercept form.

$$\begin{aligned} 2x-3y &= 15 \\ -3y &= -2x+15 \\ y &= \frac{2}{3}x-5 \end{aligned}$$

Now, let's find the perpendicular slope. From the investigation above, we know that the slopes must multiply together to equal -1.

$$\begin{aligned} \frac{2}{3}\cdot m &= -1 \\ \frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot m &= -1\cdot\frac{3}{2} \\ m &= -\frac{3}{2} \end{aligned}$$

Notice that the perpendicular slope is the *opposite sign and reciprocals* with the original slope. Now, we need to use the given point to find the $y-$intercept.

$$\begin{aligned} 5 &= -\frac{3}{2}(6)+b \\ 5 &= -9+b \\ 14 &= b \end{aligned}$$

The equation of the line that is perpendicular to $y=\frac{2}{3}x-5$ is $y=-\frac{3}{2}x+14$.

If we write these lines in standard form, the equations would be $2x-3y=15$ and $3x+2y=28$, respectively.

Example B

Write the equation of the line that passes through (4, -7) and is perpendicular to $y=2$.

Solution: The line $y=2$ does not have an $x-$term, meaning it has no slope and a horizontal line. Therefore, to find the perpendicular line that passes through (4, -7), it would have to be a vertical line. Only need the $x-$coordinate. The perpendicular line would be $x=4$.

Example C

Write the equation of the line that passes through (6, -10) and is perpendicular to the line that passes through (4, -6) and (3, -4).

Solution: First, we need to find the slope of the line that our line will be perpendicular to. Use the points (4, -6) and (3, -4) to find the slope.

$$m = \frac{-4-(-6)}{3-4} = \frac{2}{-1} = -2$$

Therefore, the perpendicular slope is the opposite sign and the reciprocal of -2. That makes the slope $\frac{1}{2}$. Use the point (6, -10) to find the $y-$intercept.

$$\begin{aligned} -10 &= \frac{1}{2}(6)+b \\ -10 &= 3+b \\ -7 &= b \end{aligned}$$

The equation of the perpendicular line is $y = \frac{1}{2}x - 7$.

Guided Practice

1. Find the equation of the line that is perpendicular to $x-2y=8$ and passes through (4, -3).

2. Find the equation of the line that passes through (-8, 7) and is perpendicular to the line that passes through (6, -1) and (1, 3).

3. Are $x-4y=8$ and $2x+8y=-32$ parallel, perpendicular or neither?

Answers

1. First, we need to change this line from standard form to slope-intercept form.

$$\begin{aligned} x-2y &= 8 \\ -2y &= -x+8 \\ y &= \frac{1}{2}x-4 \end{aligned}$$

The perpendicular slope will be -2. Let's find the new $y-$intercept.

$$\begin{aligned} -3 &= -2(4)+b \\ -3 &= -8+b \\ 5 &= b \end{aligned}$$

The equation of the perpendicular line is $y=-2x+5$ or $2x+y=5$.

2. First, find the slope between (6, -1) and (1, 3).

$$m = \frac{-1-3}{6-1} = \frac{-4}{5} = -\frac{4}{5}$$

From this, the slope of the perpendicular line will be $\frac{5}{4}$. Now, use (-8, 7) to find the $y-$intercept.

$$\begin{aligned}7 &= \frac{5}{4}(-8)+b\\7 &= -10+b\\17 &= b\end{aligned}$$

The equation of the perpendicular line is $y=\frac{5}{4}x+17$.

3. To determine if the two lines are parallel or perpendicular, we need to change them both into slope-intercept form.

$$\begin{aligned}x-4y &= 8 & 2x+8y &= -32\\-4y &= -x+8 \quad \textit{and} & 8y &= -2x-32\\y &= \frac{1}{4}x-2 & y &= -\frac{1}{4}x-4\end{aligned}$$

Now, just look at the slopes. One is $\frac{1}{4}$ and the other is $-\frac{1}{4}$. They are not the same, so they are not parallel. To be perpendicular, the slopes need to be reciprocals, which they are not. Therefore, these two lines are not parallel or perpendicular.

Vocabulary

Perpendicular
When two lines intersect to form a right, or 90°, angle. The product of the slopes of two perpendicular lines is -1.

Problem Set

Find the equation of the line, given the following information. You may leave your answer in slope-intercept form.

1. Passes through (4, 7) and is perpendicular to $x-y=-5$.
2. Passes through (-6, -2) and is perpendicular to $y=4$.
3. Passes through (4, 5) and is perpendicular to $y=-\frac{1}{3}x-1$.
4. Passes through (1, -9) and is perpendicular to $x=8$.
5. Passes through (0, 6) and perpendicular to $x-4y=10$.
6. Passes through (-12, 4) and is perpendicular to $y=-3x+5$.
7. Passes through the $x-$intercept of $2x-3y=6$ and perpendicular to $x+6y=-3$.
8. Passes through (7, -8) and is perpendicular to $2x+5y=14$.
9. Passes through (1, 3) and is perpendicular to the line that passes through (-6, 2) and (-4, 6).
10. Passes through (3, -10) and is perpendicular to the line that passes through (-2, 2) and (-8, 1).
11. Passes through (-4, -1) and is perpendicular to the line that passes through (-15, 7) and (-3, 3).

Are the pairs of lines parallel, perpendicular or neither?

12. $4x+2y=5$ and $5x-10y=-20$
13. $9x+12y=8$ and $6x+8y=-1$
14. $5x-5y=20$ and $x+y=7$
15. $8x-4y=12$ and $4x-y=-15$

3.8 Graphing Lines

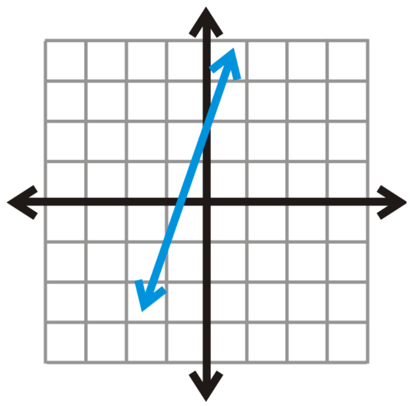
Objective

To be able to graph the equation of a line in slope-intercept or standard form.

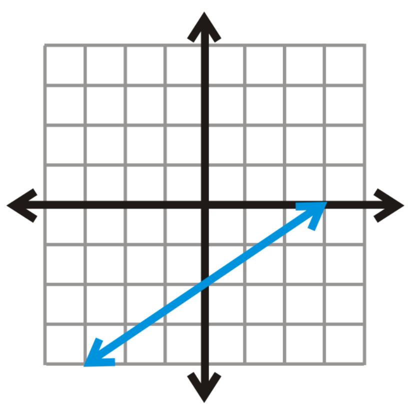
Review Queue

Find the equation of each line below. For the graphs, you may assume the $y-$intercepts are integers.

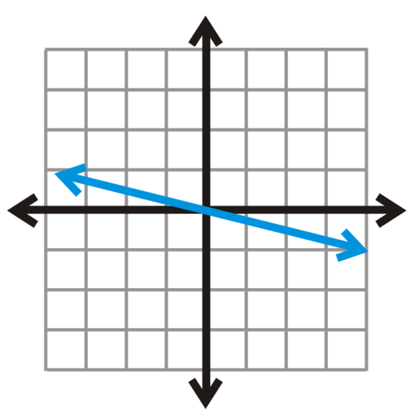
1.



2.



3.



4. What are the x and $y-$intercepts of:

a) $3x-5y=15$

b) $8x-5y=24$

Graph a Line in Slope-Intercept Form

Objective

To graph a line in slope-intercept form.

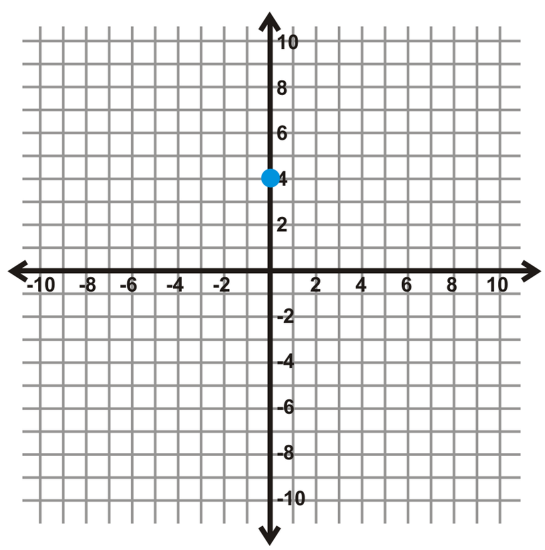
Guidance

From the previous lesson, we know that the equation of a line is $y=mx+b$, where m is the slope and b is the $y-$intercept. From these two pieces of information we can graph any line.

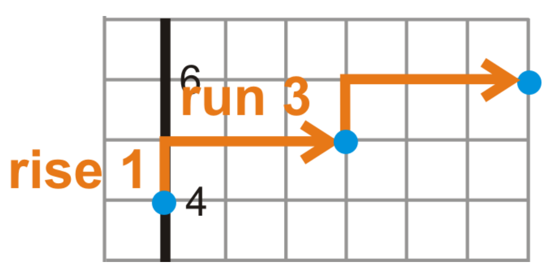
Example A

Graph $y=\frac{1}{3}x+4$ on the Cartesian plane.

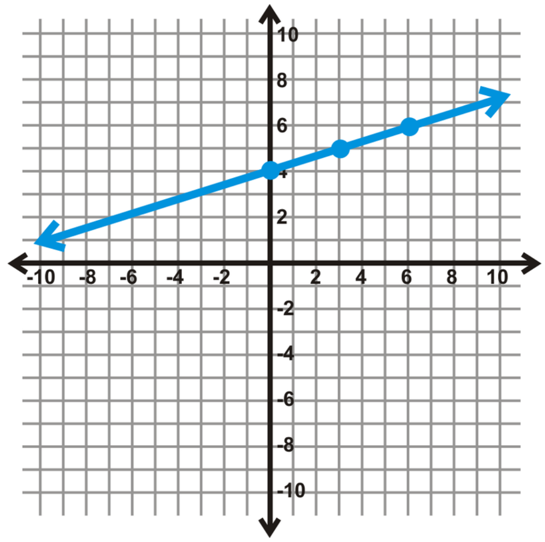
Solution: First, the Cartesian plane is the $x-y$ plane. Typically, when graphing lines, draw each axis from -10 to 10. To graph this line, you need to find the slope and $y-$intercept. By looking at the equation, $\frac{1}{3}$ is the slope and 4, or (0, 4), is the $y-$intercept. To start graphing this line, plot the $y-$intercept on the $y-$axis.



Now, we need to use the slope to find the next point on the line. Recall that the slope is also $\frac{rise}{run}$, so for $\frac{1}{3}$, we will rise 1 and run 3 from the $y-$intercept. Do this a couple of times to get at least three points.



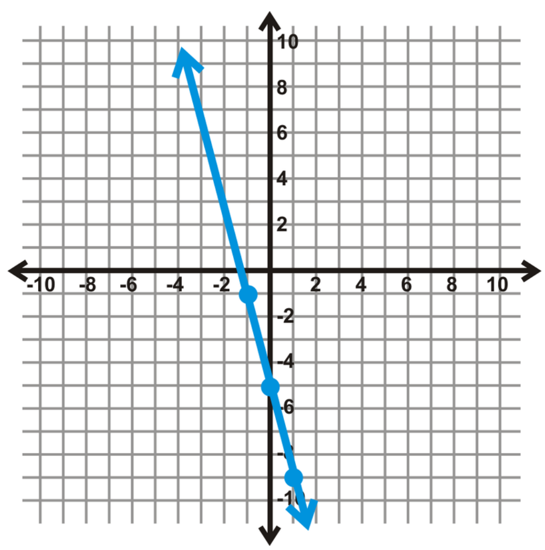
Now that we have three points, connect them to form the line $y=\frac{1}{3}x+4$.



Example B

Graph $y = -4x - 5$.

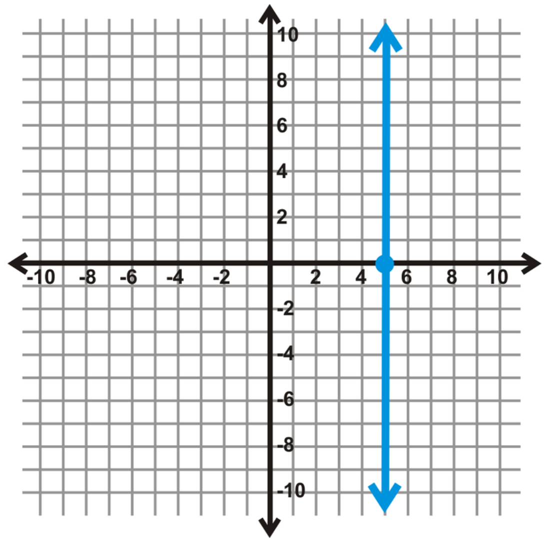
Solution: Now that the slope is negative, the vertical distance will "fall" instead of rise. Also, because the slope is a whole number, we need to put it over 1. Therefore, for a slope of -4, the line will fall 4 and run 1 OR rise 4 and run backward 1. Start at the $y-$intercept, and then use the slope to find a few more points.



Example C

Graph $x = 5$.

Solution: Any line in the form $x = a$ is a vertical line. To graph any vertical line, plot the value, in this case 5, on the $x-$axis. Then draw the vertical line.



To graph a horizontal line, $y = b$, it will be the same process, but plot the value given on the $y-$axis and draw a horizontal line.

Guided Practice

Graph the following lines.

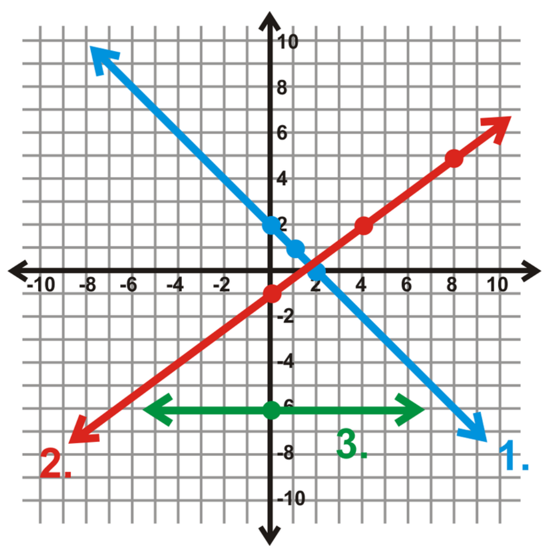
1. $y = -x + 2$

2. $y = \frac{3}{4}x - 1$

3. $y = -6$

Answers

All the answers are on the same grid below.



1. Plot (0, 2) and the slope is -1, which means you fall 1 and run 1.

2. Plot (0, -1) and then rise 3 and run 4 to the next point, (4, 2).

3. Plot -6 on the $y-$axis and draw a horizontal line.

Problem Set

Graph the following lines in the Cartesian plane.

1. $y = -2x - 3$
2. $y = x + 4$
3. $y = \frac{1}{3}x - 1$
4. $y = 9$
5. $y = -\frac{2}{5}x + 7$
6. $y = \frac{2}{4}x - 5$
7. $y = -5x - 2$
8. $y = -x$
9. $y = 4$
10. $x = -3$
11. $y = \frac{3}{2}x + 3$
12. $y = -\frac{1}{6}x - 8$
13. Graph $y = 4$ and $x = -6$ on the same set of axes. Where do they intersect?
14. If you were to make a general rule for the lines $y = b$ and $x = a$, where will they always intersect?
15. The cost per month, C (in dollars), of placing an ad on a website is $C = 0.25x + 50$, where x is the number of times someone clicks on your link. How much would it cost you if 500 people clicked on your link?

Graph a Line in Standard Form

Objective

To graph a line in standard form.

Guidance

When a line is in standard form, there are two different ways to graph it. The first is to change the equation to slope-intercept form and then graph as shown in the previous concept. The second is to use standard form to find the x and $y-$intercepts of the line and connect the two. Here are a few examples.

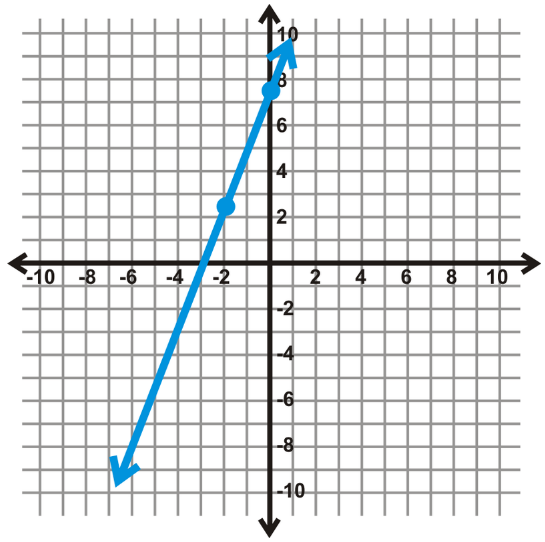
Example A

Graph $5x - 2y = -15$.

Solution: Let's use approach #1; change the equation to slope-intercept form.

$$\begin{aligned}5x - 2y &= -15 \\ -2y &= -5x - 15 \\ y &= \frac{5}{2}x + \frac{15}{2}\end{aligned}$$

The $y-$intercept is $\left(0, \frac{15}{2}\right)$. Change the improper fraction to a decimal and approximate it on the graph, (0, 7.5). Then use slope triangles. If you find yourself running out of room "rising 5" and "running 2," you could also "fall 5" and "run backwards 2" to find a point on the other side of the $y-$intercept.



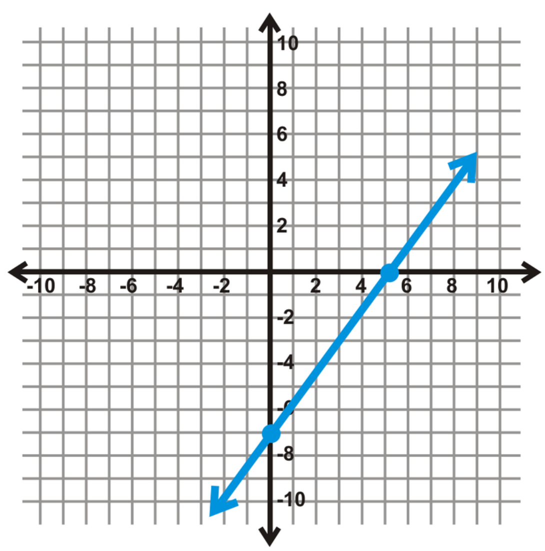
Example B

Graph $4x-3x=21$.

Solution: Let's use approach #2; find the x and $y-$intercepts (from *Standard Form of a Line* concept). Recall that the other coordinate will be zero at these points. Therefore, for the $x-$intercept, plug in zero for y and for the $y-$intercept, plug in zero for x.

$$\begin{aligned} 4x-3(0)&=21 \\ 4x&=21 \\ x&=\frac{21}{4} \ or \ 5.25 \end{aligned} \qquad\qquad \begin{aligned} 4(0)-3y&=21 \\ -3y&=21 \\ y&=-7 \end{aligned}$$

Now, plot each on their respective axes and draw a line.



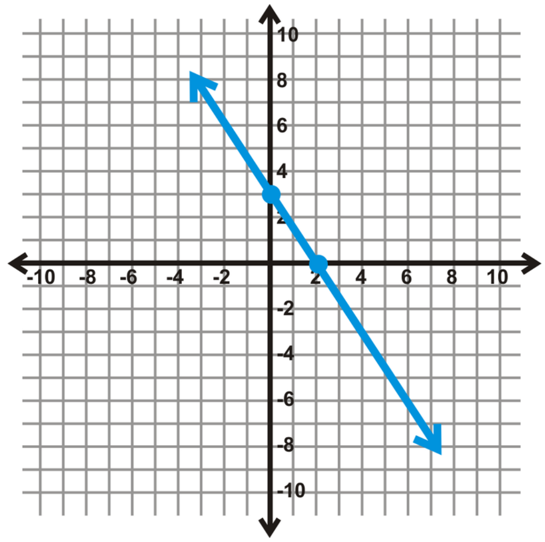
Guided Practice

1. Graph $4x+6y=18$ by changing it into slope-intercept form.

2. Graph $5x-3y=30$ by plotting the intercepts.

Answers

1. Change $4x+6y=18$ into slope-intercept form by solving for y, then graph.

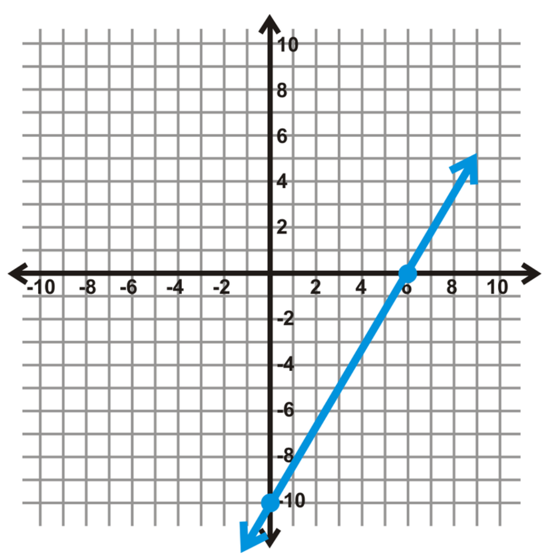


$$\begin{aligned}4x+6y&=18\\6y&=-4x+18\\y&=-\frac{2}{3}x+3\end{aligned}$$

2. Substitute in zero for x, followed by y and solve each equation.

$$\begin{aligned}5(0)-3y&=30\\-3y&=30\\y&=-10\end{aligned}\qquad\qquad\begin{aligned}5x-3(0)&=30\\5x&=30\\x&=6\end{aligned}$$

Now, plot each on their respective axes and draw a line.



Problem Set

Graph the following lines by changing the equation to slope-intercept form.

1. $-2x+y=5$
2. $3x+8y=16$
3. $4x-2y=10$
4. $6x+5y=-20$
5. $9x-6y=24$
6. $x+4y=-12$

Graph the following lines by finding the intercepts.

7. $2x+3y=12$
8. $-4x+5y=30$
9. $x-2y=8$
10. $7x+y=-7$
11. $6x+10y=15$
12. $4x-8y=-28$
13. **Writing** Which method do you think is easier? Why?
14. **Writing** Which method would you use to graph $x=-5$? Why?

3.9 Parallel and Perpendicular Lines in the Coordinate Plane

Learning Objectives

- Compute slope.
- Determine the equation of parallel and perpendicular lines to a given line.
- Graph parallel and perpendicular lines in slope-intercept and standard form.

Review Queue

Find the slope between the following points.

1. (-3, 5) and (2, -5)

2. (7, -1) and (-2, 2)

3. Is $x = 3$ horizontal or vertical? How do you know?

Graph the following lines on an $x - y$ plane.

4. $y = -2x + 3$

5. $y = \frac{1}{4}x - 2$

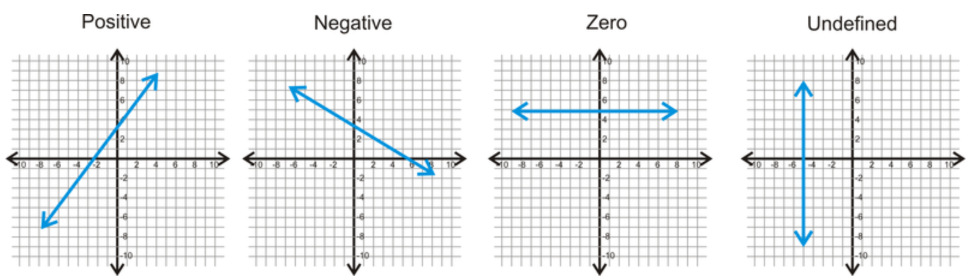
Know What? The picture to the right is the California Incline, a short piece of road that connects Highway 1 with the city of Santa Monica. The length of the road is 1532 feet and has an elevation of 177 feet. *You may assume that the base of this incline is sea level, or zero feet.* **Can you find the slope of the California Incline?**

HINT: You will need to use the Pythagorean Theorem, which has not been introduced in this class, but you may have seen it in a previous math class.

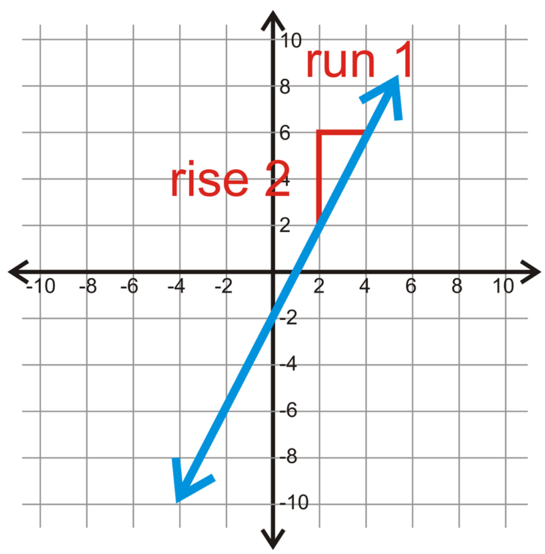
Slope in the Coordinate Plane

Recall from Algebra I, The slope of the line between two points (x_1, y_1) and (x_2, y_2) is $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

Different Types of Slope:



Example 1: What is the slope of the line through (2, 2) and (4, 6)?



Solution: Use the slope formula to determine the slope. Use (2, 2) as (x_1, y_1) and (4, 6) as (x_2, y_2).

$$m = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

Therefore, the slope of this line is 2.

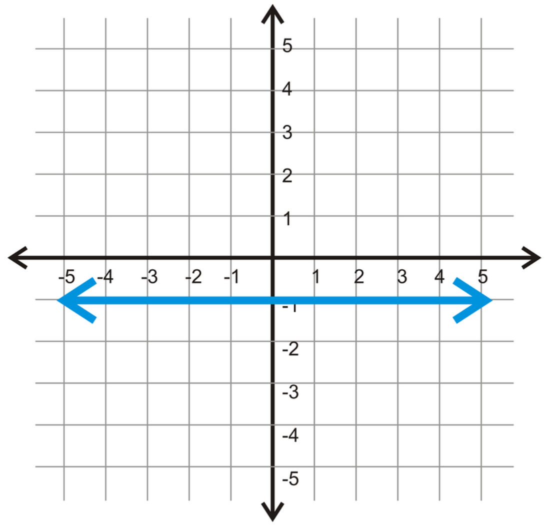
This slope is positive. Recall that slope can also be the "rise over run." In this case we "rise", or go up 2, and "run" in the positive direction 1.

Example 2: Find the slope between (-8, 3) and (2, -2).

Solution: $m = \frac{-2-3}{2-(-8)} = \frac{-5}{10} = -\frac{1}{2}$

This is a negative slope. Instead of "rising," the negative slope means that you would "fall," when finding points on the line.

Example 3: Find the slope between (-5, -1) and (3, -1).

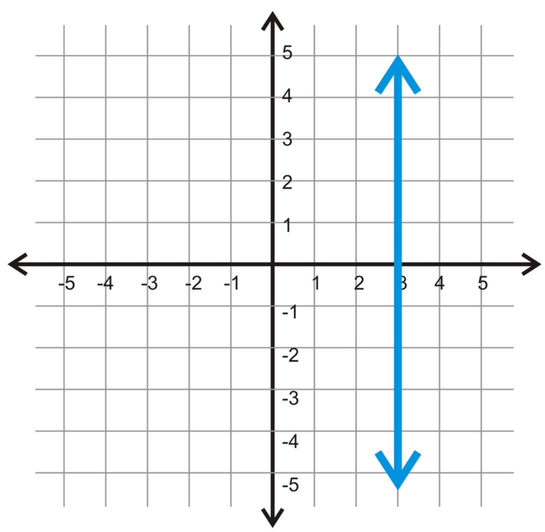


Solution:

$$m = \frac{-1-(-1)}{3-(-5)} = \frac{0}{8} = 0$$

Therefore, the slope of this line is 0, which means that it is a horizontal line. Horizontallines always pass through the $y-$axis. Notice that the $y-$coordinate for both points is -1. In fact, the $y-$coordinate for *any* point on this line is -1. This means that the horizontal line must cross $y = -1$.

Example 4: What is the slope of the line through (3, 2) and (3, 6)?



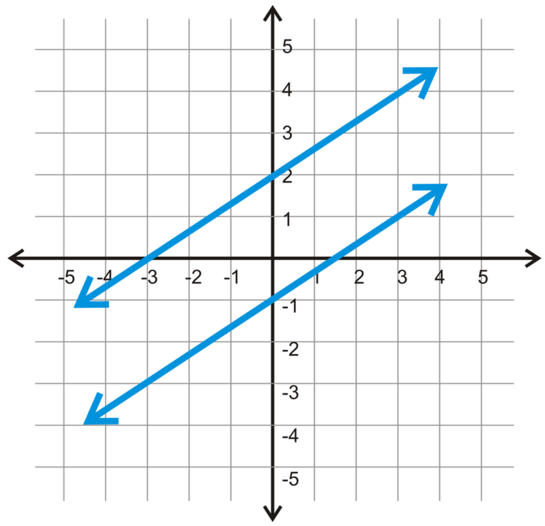
Solution:

$$m = \frac{6-2}{3-3} = \frac{4}{0} = undefined$$

Therefore, the slope of this line is undefined, which means that it is a *vertical* line. Verticallines always pass through the $x-$axis. Notice that the $x-$coordinate for both points is 3. In fact, the $x-$coordinate for *any* point on this line is 3. This means that the vertical line must cross $x = 3$.

Slopes of Parallel Lines

Recall from earlier in the chapter that the definition of parallel is two lines that never intersect. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, we see that the slopes of both are $\frac{2}{3}$.

This can be generalized to any pair of parallel lines in the coordinate plane.

Parallel lines have the same slope.

Example 5: Find the equation of the line that is parallel to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

Recall that the equation of a line in this form is called the slope-intercept form and is written as $y = mx + b$ where m is the slope and b is the $y-$intercept. Here, x and y represent any coordinate pair, (x, y) on the line.

Solution: We know that parallel lines have the same slope, so the line we are trying to find also has $m = -\frac{1}{3}$. Now, we need to find the $y-$intercept. 4 is the $y-$intercept of the given line, *not our new line*. We need to plug in 9 for x and -5 for y (this is our given coordinate pair that needs to be on the line) to solve for the *new* $y-$intercept (b).

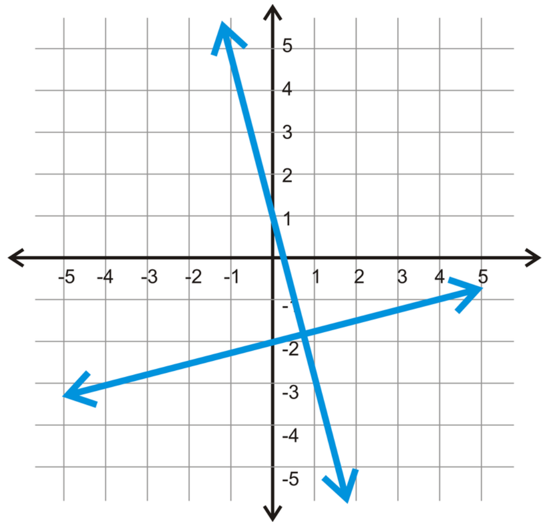
$$\begin{aligned} -5 &= -\frac{1}{3}(9) + b \\ -5 &= -3 + b \qquad \text{Therefore, the equation of line is } y = -\frac{1}{3}x - 2. \\ -2 &= b \end{aligned}$$

Reminder: the final equation contains the variables x and y to indicate that the line contains and infinite number of points or coordinate pairs that satisfy the equation.

Parallel lines always have the <u>same slope</u> and <u>different $y-$intercepts</u>.

Slopes of Perpendicular Lines

Recall from Chapter 1 that the definition of perpendicular is two lines that intersect at a 90°, or right, angle. In the coordinate plane, that would look like this:



If we take a closer look at these two lines, we see that the slope of one is -4 and the other is $\frac{1}{4}$.

This can be generalized to any pair of perpendicular lines in the coordinate plane.

The slopes of perpendicular lines are opposite signs and reciprocals of each other.

Example 6: Find the slope of the perpendicular lines to the lines below.

a) $y = 2x + 3$

b) $y = -\frac{2}{3}x - 5$

c) $y = x + 2$

Solution: We are only concerned with the slope for each of these.

a) $m = 2$, so $m_\perp$ is the reciprocal and negative, $m_\perp = -\frac{1}{2}$.

b) $m = -\frac{2}{3}$, take the reciprocal and make the slope positive, $m_\perp = \frac{3}{2}$.

c) Because there is no number in front of x, the slope is 1. The reciprocal of 1 is 1, so the only thing to do is make it negative, $m_\perp = -1$.

Example 7: Find the equation of the line that is perpendicular to $y = -\frac{1}{3}x + 4$ and passes through (9, -5).

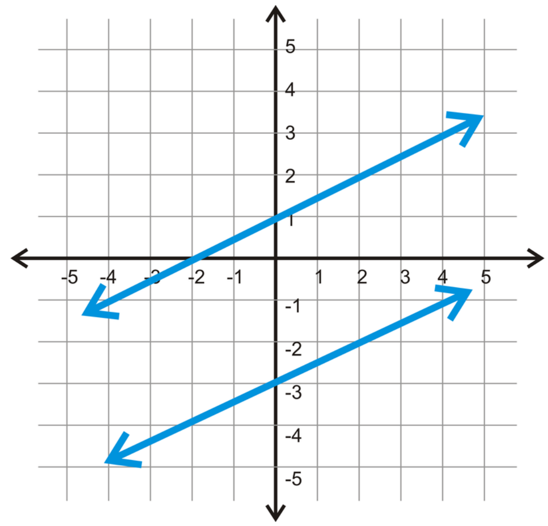
Solution: First, the slope is the reciprocal and opposite sign of $-\frac{1}{3}$. So, $m = 3$. Now, we need to find the $y-$intercept. 4 is the $y-$intercept of the given line, *not our new line*. We need to plug in 9 for x and -5 for y to solve for the *new* $y-$intercept (b).

$$\begin{aligned} -5 &= 3(9) + b \\ -5 &= 27 + b \\ -32 &= b \end{aligned}$$

Therefore, the equation of line is $y = 3x - 32$.

Graphing Parallel and Perpendicular Lines

Example 8: Find the equations of the lines below and determine if they are parallel, perpendicular or neither.



Solution: To find the equation of each line, start with the $y-$intercept. The top line has a $y-$intercept of 1. From there, determine the slope triangle, or the "rise over run." From the $y-$intercept, if you go up 1 and over 2, you hit the line again. Therefore, the slope of this line is $\frac{1}{2}$. The equation is $y = \frac{1}{2}x + 1$. For the second line, the $y-$intercept is -3. Again, start here to determine the slope and if you "rise" 1 and "run" 2, you run into the line again, making the slope $\frac{1}{2}$. The equation of this line is $y = \frac{1}{2}x - 3$. The lines are parallel because they have the same slope.

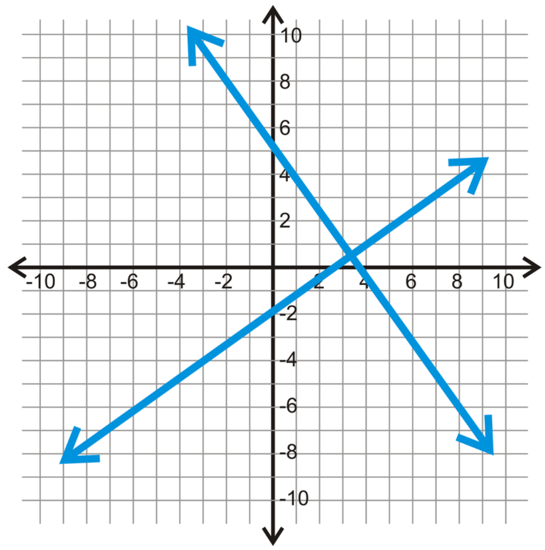
Example 9: Graph $3x - 4y = 8$ and $4x + 3y = 15$. Determine if they are parallel, perpendicular, or neither.

Solution: First, we have to change each equation into slope-intercept form. In other words, we need to solve each equation for y.

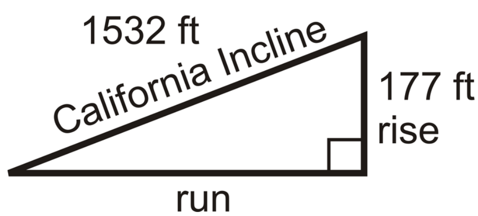
$$\begin{aligned} 3x - 4y &= 8 \\ -4y &= -3x + 8 \\ y &= \frac{3}{4}x - 2 \end{aligned} \qquad\qquad \begin{aligned} 4x + 3y &= 15 \\ 3y &= -4x + 15 \\ y &= -\frac{4}{3}x + 5 \end{aligned}$$

Now that the lines are in slope-intercept form (also called $y-$intercept form), we can tell they are perpendicular because the slopes are opposites signs and reciprocals.

To graph the two lines, plot the $y-$intercept on the $y-$axis. From there, use the slope to rise and then run. For the first line, you would plot -2 and then rise 3 and run 4, making the next point on the line (1, 4). For the second line, plot 5 and then fall (because the slop is negative) 4 and run 3, making the next point on the line (1, 3).



Know What? Revisited In order to find the slope, we need to first find the horizontal distance in the triangle to the right. This triangle represents the incline and the elevation. To find the horizontal distance, or the run, we need to use the Pythagorean Theorem, $a^2+b^2=c^2$, where c is the hypotenuse.



$$\begin{aligned}177^2+run^2&=1532^2\\31,329+run^2&=2,347,024\\run^2&=2,315,695\\run&\approx 1521.75\end{aligned}$$

The slope is then $\frac{177}{1521.75}$, which is roughly $\frac{3}{25}$.

Review Questions

Find the slope between the two given points.

1. (4, -1) and (-2, -3)
2. (-9, 5) and (-6, 2)
3. (7, 2) and (-7, -2)
4. (-6, 0) and (-1, -10)
5. (1, -2) and (3, 6)
6. (-4, 5) and (-4, -3)

Determine if each pair of lines are parallel, perpendicular, or neither. Then, graph each pair on the same set of axes.

7. $y = -2x + 3$ and $y = \frac{1}{2}x + 3$
8. $y = 4x - 2$ and $y = 4x + 5$
9. $y = -x + 5$ and $y = x + 1$
10. $y = -3x + 1$ and $y = 3x - 1$
11. $2x - 3y = 6$ and $3x + 2y = 6$
12. $5x + 2y = -4$ and $5x + 2y = 8$
13. $x - 3y = -3$ and $x + 3y = 9$
14. $x + y = 6$ and $4x + 4y = -16$

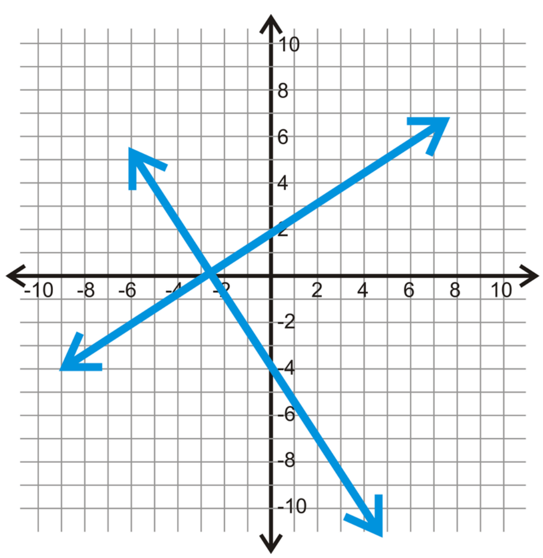
Determine the equation of the line that is ***parallel*** to the given line, through the given point.

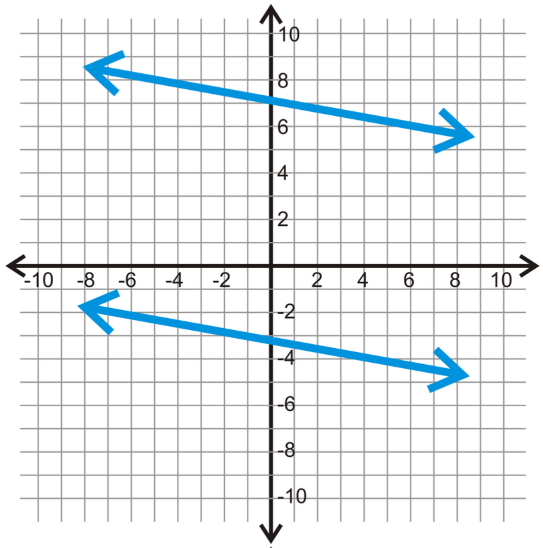
15. $y = -5x + 1;\ (-2,\ 3)$
16. $y = \frac{2}{3}x - 2;\ (9, 1)$
17. $x - 4y = 12;\ (-16,\ -2)$
18. $3x + 2y = 10;\ (8,\ -11)$
19. $2x - y = 15;\ (3,\ 7)$
20. $y = x - 5;\ (9,\ -1)$

Determine the equation of the line that is ***perpendicular*** to the given line, through the given point.

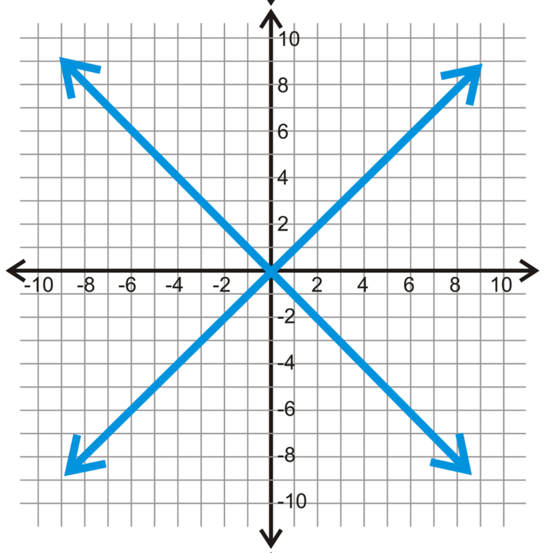
21. $y = x - 1;\ (-6,\ 2)$
22. $y = 3x + 4;\ (9,\ -7)$
23. $5x - 2y = 6;\ (5,\ 5)$
24. $y = 4;\ (-1,\ 3)$
25. $x = -3;\ (1,\ 8)$
26. $x - 3y = 11;\ (0,\ 13)$

Find the equation of the two lines in each graph below. Then, determine if the two lines are parallel, perpendicular or neither.

27. 



28.



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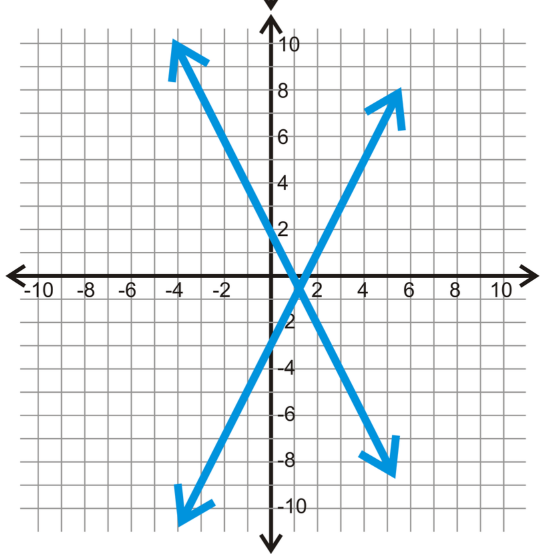
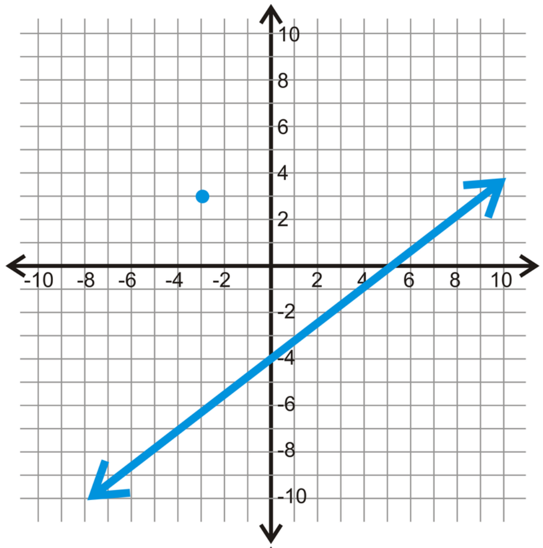


30.

For the line and point below, find:

a) A parallel line, through the given point.

b) A perpendicular line, through the given point.


10
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6
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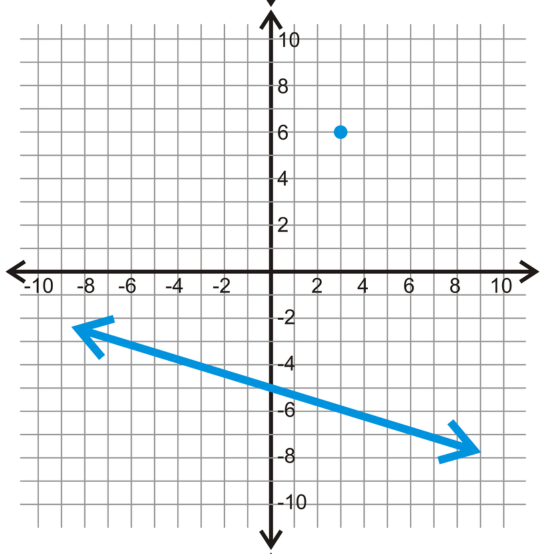
-10 -8 -6 -4 -2 2 4 6 8 10
-2
-4
-6
-8
-10

31.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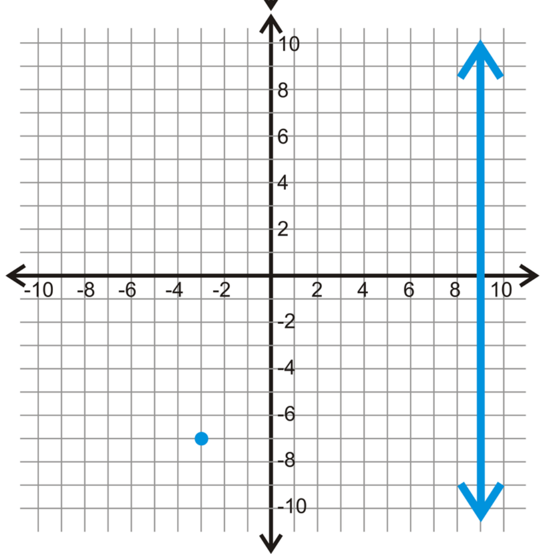
32.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-6
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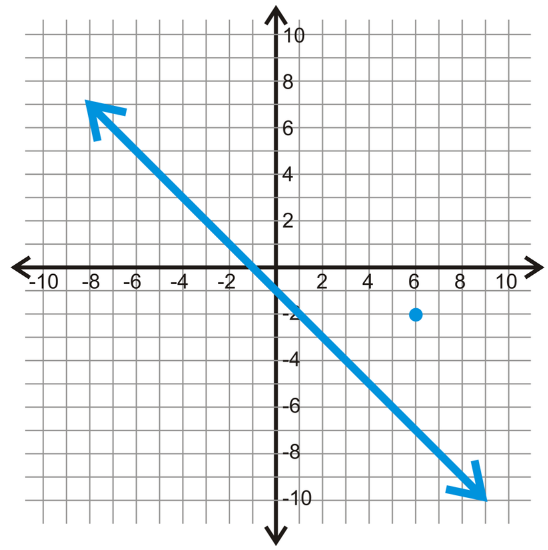
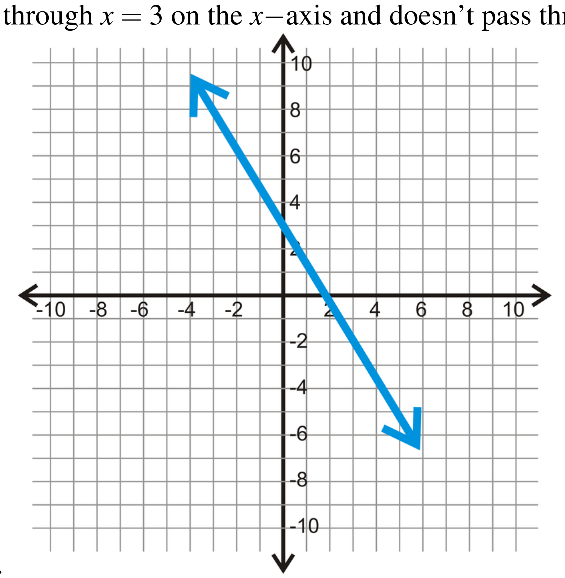
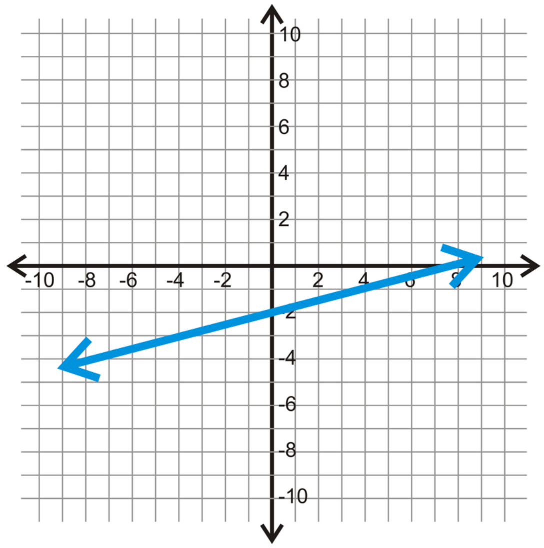
33.

34. 

Review Queue Answers

a. $m = \frac{-5-5}{2+3} = \frac{-10}{2} = -5$
b. $m = \frac{2+1}{-2-7} = \frac{3}{-9} = -\frac{1}{3}$
c. Vertical because it has to pass through $x = 3$ on the $x-$axis and doesn't pass through y at all.

d. 

e. 

3.10 Chapter 3 Review

Keywords and Theorems

Parallel
When two or more lines lie in the same plane and never intersect.

Skew Lines
Lines that are in different planes and never intersect.

Parallel Postulate
For a line and a point not on the line, there is exactly one line parallel to this line through the point.

Perpendicular Line Postulate
For a line and a point not on the line, there is exactly one line parallel to this line through the point.

Transversal
A line that intersects two distinct lines. These two lines may or may not be parallel.

Corresponding Angles
Two angles that are in the "same place" with respect to the transversal, but on different lines.

Alternate Interior Angles
Two angles that are on the interior of l and m, but on opposite sides of the transversal.

Alternate Exterior Angles
Two angles that are on the exterior of l and m, but on opposite sides of the transversal.

Same Side Interior Angles
Two angles that are on the same side of the transversal and on the interior of the two lines.

Corresponding Angles Postulate
If two parallel lines are cut by a transversal, then the corresponding angles are congruent.

Alternate Interior Angles Theorem
If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

Alternate Exterior Angles Theorem
If two parallel lines are cut by a transversal, then the alternate exterior angles are congruent.

Same Side Interior Angles Theorem
If two parallel lines are cut by a transversal, then the same side interior angles are supplementary.

Converse of Corresponding Angles Postulate
If corresponding angles are congruent when two lines are cut by a transversal, then the lines are parallel.

Converse of Alternate Interior Angles Theorem
If two lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel.

Converse of the Alternate Exterior Angles Theorem
If two lines are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel.

Converse of the Same Side Interior Angles Theorem
If two lines are cut by a transversal and the consecutive interior angles are supplementary, then the lines are parallel.

Parallel Lines Property
The Parallel Lines Property is a transitive property that can be applied to parallel lines.

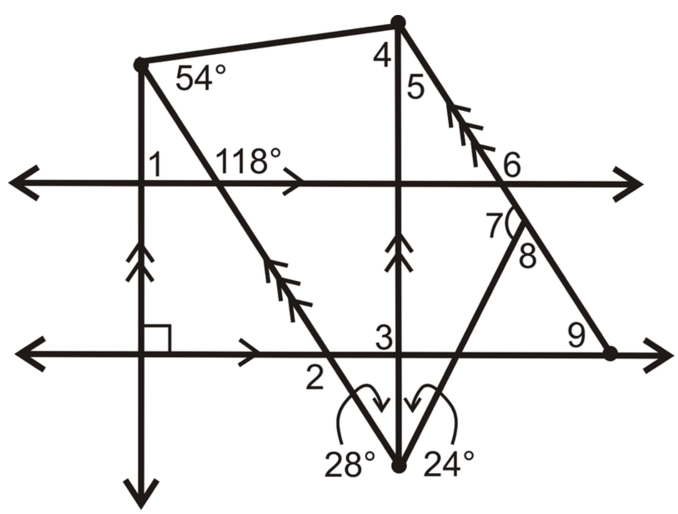
Theorem 3-1
If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

Theorem 3-2
If two lines are parallel and a third line is perpendicular to one of the parallel lines, it is also perpendicular to the other parallel line.

Distance Formula
$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Review

Find the value of each of the numbered angles below.



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9688 .

CHAPTER 4 Triangles and Congruence

Chapter Outline

In this chapter, you will learn all about triangles. First, we will learn about the properties of triangles and the angles within a triangle. Second, we will use that information to determine if two different triangles are congruent. After proving two triangles are congruent, we will use that information to prove other parts of the triangles are congruent as well as the properties of equilateral and isosceles triangles.

4.1 Triangle Sums

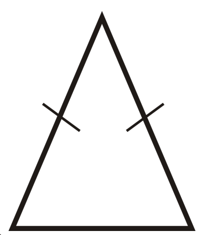
Learning Objectives

- Understand and apply the Triangle Sum Theorem.
- Identify interior and exterior angles in a triangle.
- Understand the Exterior Angle Theorem.

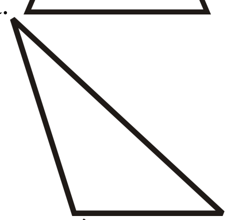
Review Queue

Classify the triangles below by their angles and sides.

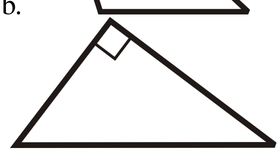
a.



b.



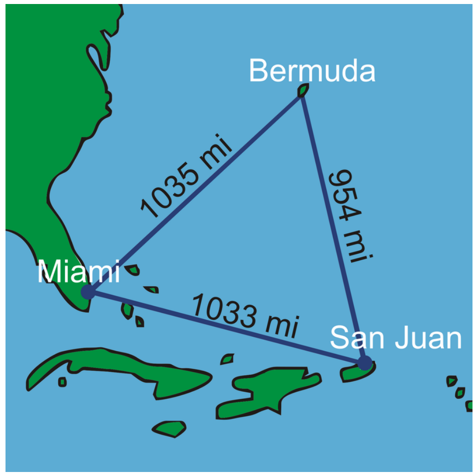
c.



d. How many degrees are in a straight angle? Draw and label a straight angle, $\angle ABC$.

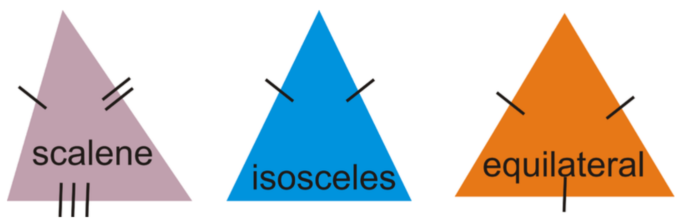
Know What? To the right is the Bermuda Triangle. You are probably familiar with the myth of this triangle; how several ships and planes passed through and mysteriously disappeared.

The measurements of the sides of the triangle are in the image. What type of triangle is this? Classify it by its sides and angles. Using a protractor, find the measure of each angle in the Bermuda Triangle. What do they add up to? Do you think the three angles in this image are the same as the three angles in the *actual* Bermuda triangle? Why or why not?



A Little Triangle Review

Recall that a triangle can be classified by its sides.



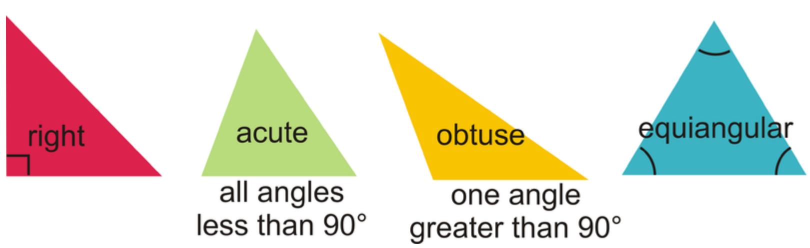
Scalene: All three sides are different lengths.

Isosceles: At least two sides are congruent.

Equilateral: All three sides are congruent.

By the definition, an equilateral triangle is also an isosceles triangle.

And, triangles can also be classified by their angles.



Right: One right angle.

Acute: All three angles are less than 90°.

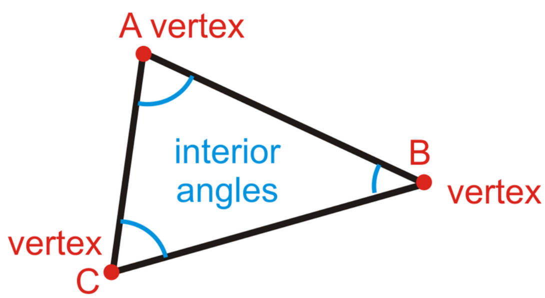
Obtuse: One angle is greater than 90°.

Equiangular: All three angles are congruent.

Triangle Sum Theorem

Interior Angles (in polygons): The angles inside of a closed figure with straight sides.

Vertex: The point where the sides of a polygon meet.



Triangles have three interior angles, three vertices and three sides.

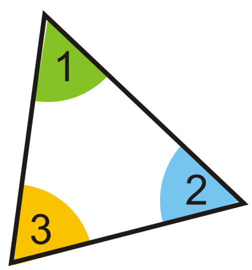
A triangle is labeled by its vertices with a $\triangle$. This triangle can be labeled $\triangle ABC, \triangle ACB, \triangle BCA, \triangle BAC, \triangle CBA$ or $\triangle CAB$. Order does not matter.

The angles in any polygon are measured in degrees. Each polygon has a different sum of degrees, depending on the number of angles in the polygon. How many degrees are in a triangle?

Investigation 4-1: Triangle Tear-Up

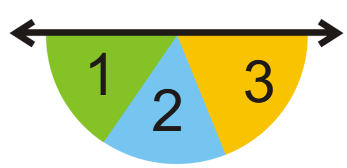
Tools Needed: paper, ruler, pencil, colored pencils

a. Draw a triangle on a piece of paper. Try to make all three angles different sizes. Color the three interior angles three different colors and label each one, $\angle 1, \angle 2$, and $\angle 3$.



b. Tear off the three colored angles, so you have three separate angles.

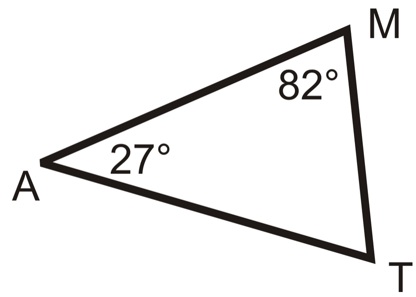
c. Attempt to line up the angles so their points all match up. What happens? What measure do the three angles add up to?



This investigation shows us that the sum of the angles in a triangle is 180° because the three angles fit together to form a straight line. Recall that a line is also a straight angle and all straight angles are 180°.

Triangle Sum Theorem: The interior angles of a triangle add up to 180°.

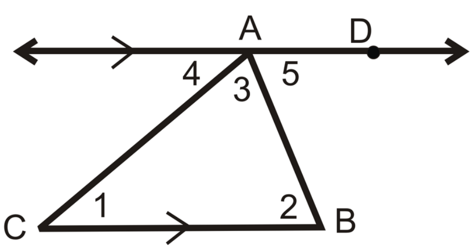
Example 1: What is the $m\angle T$?



Solution: From the Triangle Sum Theorem, we know that the three angles add up to 180°. Set up an equation to solve for $\angle T$.

$$\begin{aligned} m\angle M + m\angle A + m\angle T &= 180^\circ \\ 82^\circ + 27^\circ + m\angle T &= 180^\circ \\ 109^\circ + m\angle T &= 180^\circ \\ m\angle T &= 71^\circ \end{aligned}$$

Investigation 4-1 is one way to show that the angles in a triangle add up to 180°. However, it is not a two-column proof. Here we will prove the Triangle Sum Theorem.



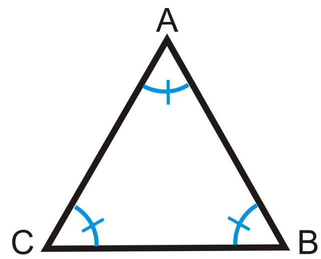
Given: $\triangle ABC$ with $\overleftrightarrow{AD} \parallel \overline{BC}$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

TABLE 4.1:

Statement	*Reason*
1. $\triangle ABC$ above with $\overleftrightarrow{AD} \parallel \overline{BC}$	Given
2. $\angle 1 \cong \angle 4, \angle 2 \cong \angle 5$	Alternate Interior Angles Theorem
3. $m\angle 1 = m\angle 4, m\angle 2 = m\angle 5$	$\cong$ angles have = measures
4. $m\angle 4 + m\angle CAD = 180^\circ$	Linear Pair Postulate
5. $m\angle 3 + m\angle 5 = m\angle CAD$	Angle Addition Postulate
6. $m\angle 4 + m\angle 3 + m\angle 5 = 180^\circ$	Substitution PoE
7. $m\angle 1 + m\angle 3 + m\angle 2 = 180^\circ$	Substitution PoE

Example 2: What is the measure of each angle in an equiangular triangle?



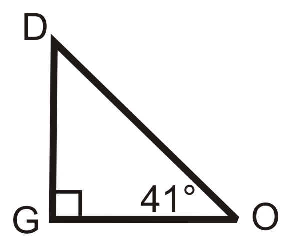
Solution: $\triangle ABC$ to the left is an example of an equiangular triangle, where all three angles are equal. Write an equation.

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= 180^\circ \\ m\angle A + m\angle A + m\angle A &= 180^\circ \\ 3m\angle A &= 180^\circ \\ m\angle A &= 60^\circ \end{aligned}$$

If $m\angle A = 60^\circ$, then $m\angle B = 60^\circ$ and $m\angle C = 60^\circ$.

Theorem 4-1: Each angle in an equiangular triangle measures 60°.

Example 3: Find the measure of the missing angle.



Solution: $m\angle O = 41^\circ$ and $m\angle G = 90^\circ$ because it is a right angle.

$$\begin{aligned} m\angle D + m\angle O + m\angle G &= 180^\circ \\ m\angle D + 41^\circ + 90^\circ &= 180^\circ \\ m\angle D + 41^\circ &= 90^\circ \\ m\angle D &= 49^\circ \end{aligned}$$

Notice that $m\angle D + m\angle O = 90^\circ$ because $\angle G$ is a right angle.

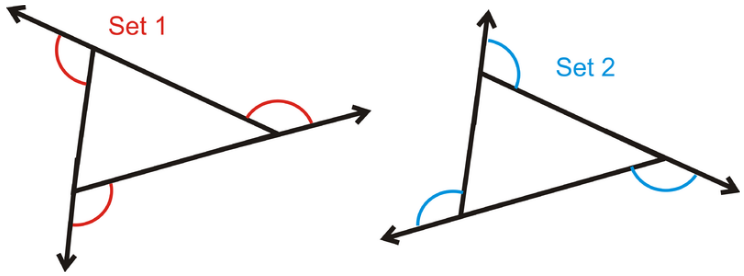
Theorem 4-2: The acute angles in a right triangle are always complementary.

Exterior Angles

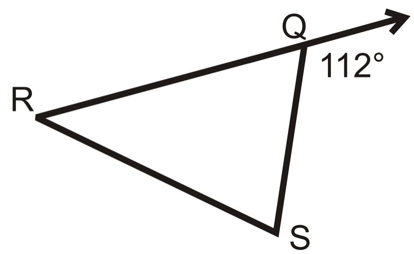
Exterior Angle: The angle formed by one side of a polygon and the extension of the adjacent side.

In all polygons, there are **two** sets of exterior angles, one going around the polygon clockwise and the other goes around the polygon counterclockwise.

By the definition, the interior angle and its adjacent exterior angle form a linear pair.



Example 4: Find the measure of $\angle RQS$.

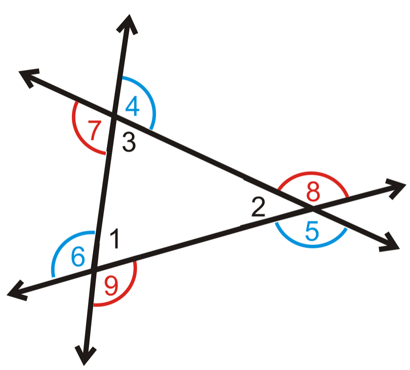


Solution: 112° is an exterior angle of $\triangle RQS$. Therefore, it is supplementary to $\angle RQS$ because they are a linear pair.

$$112^\circ + m\angle RQS = 180^\circ$$
$$m\angle RQS = 68^\circ$$

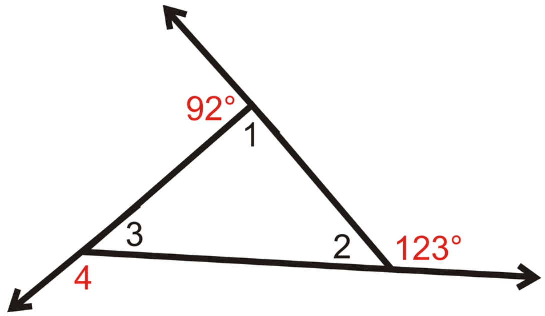
If we draw both sets of exterior angles on the same triangle, we have the following figure:

Notice, at each vertex, the exterior angles are also vertical angles, therefore they are congruent.



$$\angle 4 \cong \angle 7$$
$$\angle 5 \cong \angle 8$$
$$\angle 6 \cong \angle 9$$

Example 5: Find the measure of the numbered interior and exterior angles in the triangle.



Solution:

$m\angle 1 + 92^\circ = 180^\circ$ by the Linear Pair Postulate, so $m\angle 1 = 88^\circ$.

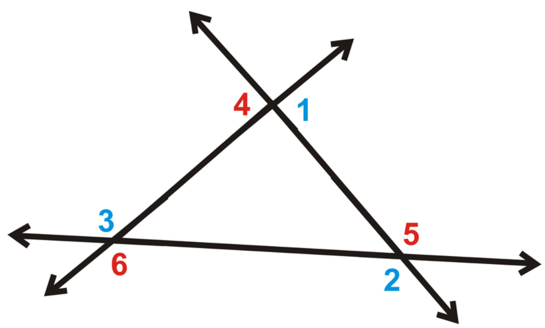
$m\angle 2 + 123^\circ = 180^\circ$ by the Linear Pair Postulate, so $m\angle 2 = 57^\circ$.

$m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$ by the Triangle Sum Theorem, so $88^\circ + 57^\circ + m\angle 3 = 180^\circ$ and $m\angle 3 = 35^\circ$.

$m\angle 3 + m\angle 4 = 180^\circ$ by the Linear Pair Postulate, so $m\angle 4 = 145^\circ$.

Looking at Example 5, the exterior angles are $92^\circ, 123^\circ$, and 145°. If we add these angles together, we get $92^\circ + 123^\circ + 145^\circ = 360^\circ$. This is always true for any set of exterior angles for any polygon.

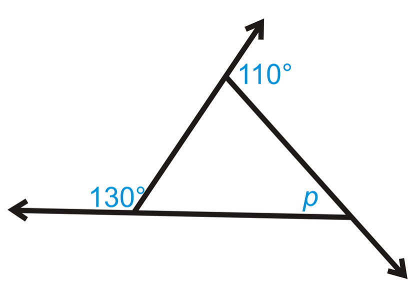
Exterior Angle Sum Theorem: Each set of exterior angles of a polygon add up to 360°.



$$m\angle 1 + m\angle 2 + m\angle 3 = 360^\circ$$
$$m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$$

We will prove this theorem for triangles in the review questions and will prove it for all polygons later in this text.

Example 6: What is the value of p in the triangle below?



Solution: First, we need to find the missing exterior angle, we will call it x. Set up an equation using the Exterior Angle Sum Theorem.

$$\begin{aligned}130^\circ + 110^\circ + x &= 360^\circ\\ x &= 360^\circ - 130^\circ - 110^\circ\\ x &= 120^\circ\end{aligned}$$

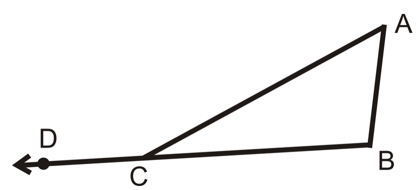
x and p are supplementary and add up to 180°.

$$\begin{aligned}x + p &= 180^\circ\\ 120^\circ + p &= 180^\circ\\ p &= 60^\circ\end{aligned}$$

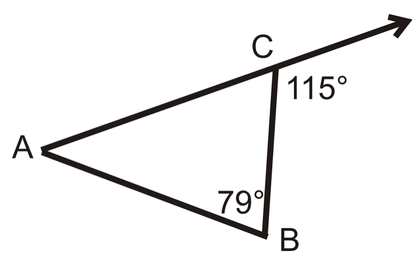
Exterior Angles Theorem

Remote Interior Angles: The two angles in a triangle that are not adjacent to the indicated exterior angle.

$\angle A$ and $\angle B$ are the remote interior angles for exterior angle $\angle ACD$.



Example 7: Find $m\angle A$.



Solution: First, find $m\angle ACB$. $m\angle ACB + 115^\circ = 180^\circ$ by the Linear Pair Postulate, so $m\angle ACB = 65^\circ$.

$m\angle A + 65^\circ + 79^\circ = 180^\circ$ by the Triangle Sum Theorem, so $m\angle A = 36^\circ$.

In Example 7, $m\angle A + m\angle B$ is $36^\circ + 79^\circ = 115^\circ$. This is the same as the exterior angle at C, 115°.

From this example, we can conclude the Exterior Angle Theorem.

Exterior Angle Theorem: The sum of the remote interior angles is equal to the non-adjacent exterior angle.

From the picture above, this means that $m\angle A + m\angle B = m\angle ACD$.

Here is the proof of the Exterior Angle Theorem. From the proof, you can see that this theorem is a combination of the Triangle Sum Theorem and the Linear Pair Postulate.

Given: $\triangle ABC$ with exterior angle $\angle ACD$

Prove: $m\angle A + m\angle B = m\angle ACD$

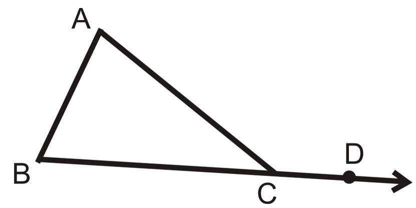
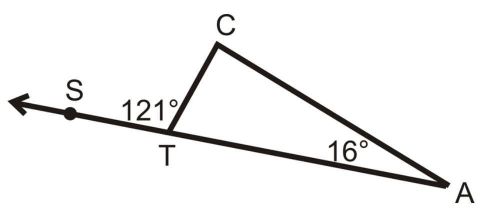


TABLE 4.2:

Statement	**Reason**
1. $\triangle ABC$ with exterior angle $\angle ACD$	Given
2. $m\angle A + m\angle B + m\angle ACB = 180^\circ$	Triangle Sum Theorem
3. $m\angle ACB + m\angle ACD = 180^\circ$	Linear Pair Postulate
4. $m\angle A + m\angle B + m\angle ACB = m\angle ACB + m\angle ACD$	Transitive PoE
5. $m\angle A + m\angle B = m\angle ACD$	Subtraction PoE

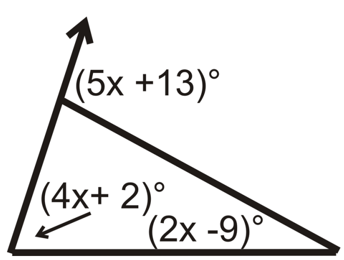
Example 8: Find $m\angle C$.



Solution: Using the Exterior Angle Theorem, $m\angle C + 16^\circ = 121^\circ$. Subtracting 16° from both sides, $m\angle C = 105^\circ$.

It is important to note that if you forget the Exterior Angle Theorem, you can do this problem just like we solved Example 7.

Example 9: ***Algebra Connection*** Find the value of x and the measure of each angle.



Solution: Set up an equation using the Exterior Angle Theorem.

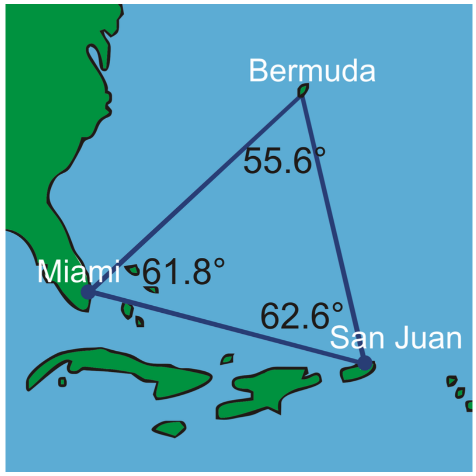
$$\underbrace{(4x+2)^\circ + (2x-9)^\circ}_{\text{interior angles}} = \underbrace{(5x+13)^\circ}_{\text{exterior angle}}$$

$$(6x-7)^\circ = (5x+13)^\circ$$
$$x = 20^\circ$$

Substituting 20° back in for x, the two interior angles are $(4(20)+2)^\circ = 82^\circ$ and $(2(20)-9)^\circ = 31^\circ$. The exterior angle is $(5(20)+13)^\circ = 113^\circ$. Double-checking our work, notice that $82^\circ + 31^\circ = 113^\circ$. If we had done the problem incorrectly, this check would not have worked.

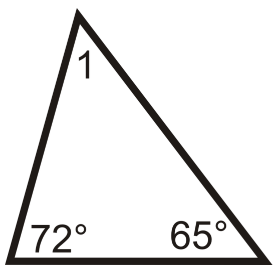
Know What? Revisited The Bermuda Triangle is an acute scalene triangle. The angle measures are in the picture to the right. Your measured angles should be within a degree or two of these measures. The angles should add up to 180°. However, because your measures are estimates using a protractor, they might not exactly add up.

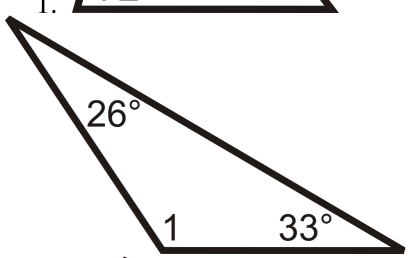
The angle measures in the picture are the actual measures, based off of the distances given, however, your measured angles might be off because the drawing is not to scale.



Review Questions

Determine $m\angle 1$.

1. 

2. 

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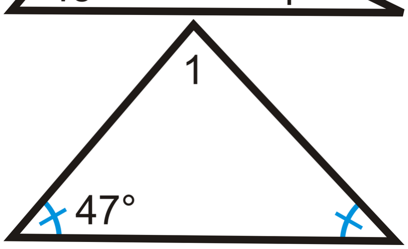
4. 

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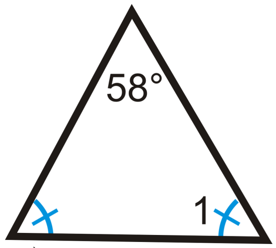

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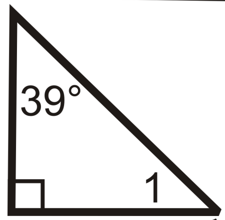

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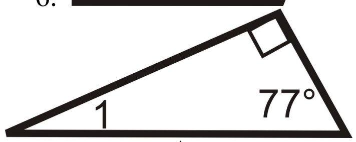

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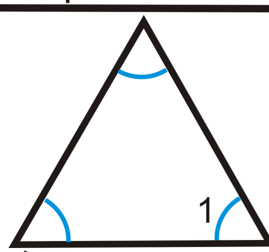

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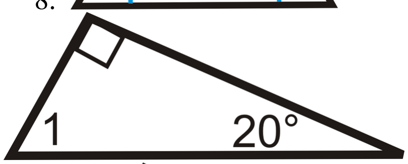

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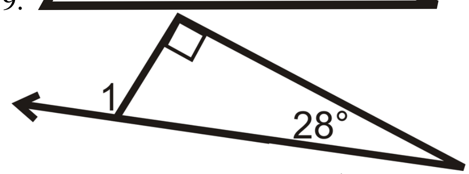

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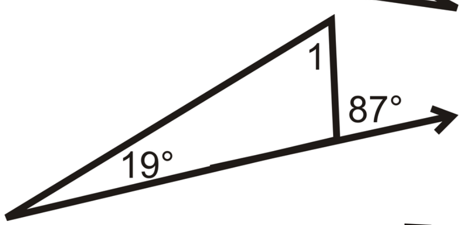

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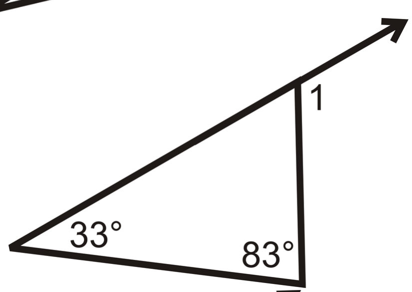

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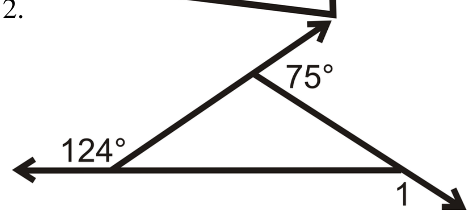
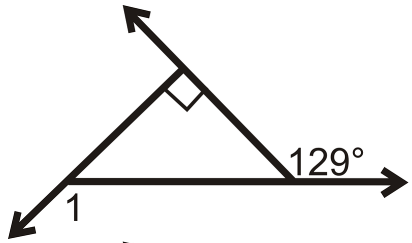
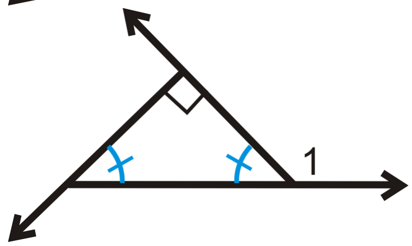
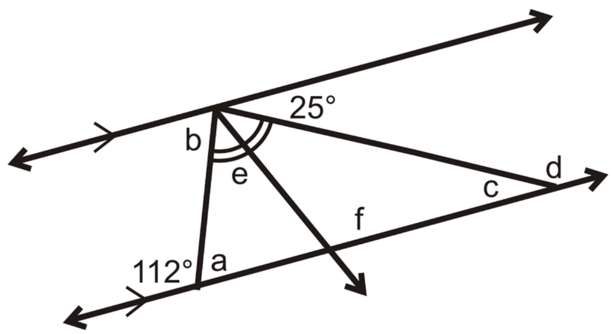


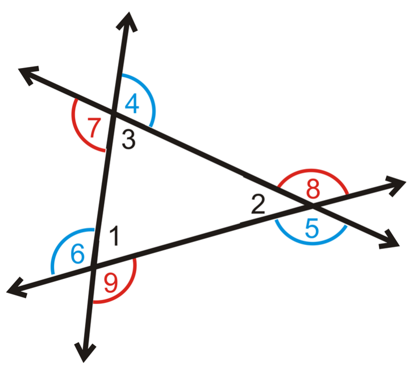
14. 

15. 

16. Find the lettered angles, $a-f$, in the picture to the right. Note that the two lines are parallel.



17. Fill in the blanks in the proof below. Given: The triangle to the right with interior angles and exterior angles. Prove: $m\angle 4+m\angle 5+m\angle 6=360^\circ$

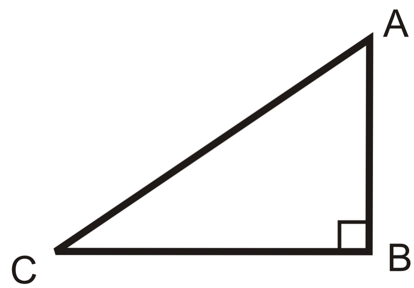


Only use the blue set of exterior angles for this proof.

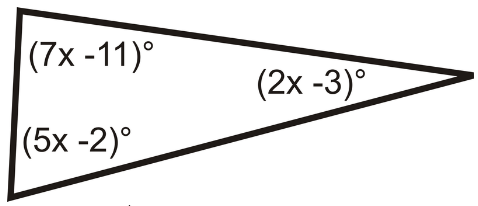
TABLE 4.3:

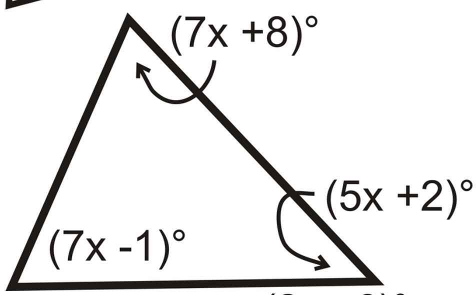
Statement	***Reason***
1. Triangle with interior and exterior angles.	Given
2. $m\angle 1+m\angle 2+m\angle 3=180^\circ$	
3. $\angle 3$ and $\angle 4$ are a linear pair, $\angle 2$ and $\angle 5$ are a linear pair, and $\angle 1$ and $\angle 6$ are a linear pair	
4.	Linear Pair Postulate (do all 3)
5. $m\angle 1+m\angle 6=180^\circ$ m $\angle 2+m\angle 5=180^\circ$ m $\angle 3+m\angle 4=180^\circ$	
6. $m\angle 1+m\angle 6+m\angle 2+m\angle 5+m\angle 3+m\angle 4=540^\circ$	
7. $m\angle 4+m\angle 5+m\angle 6=360^\circ$	

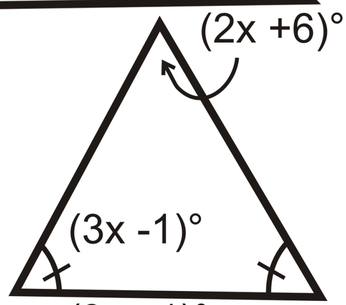
18. Write a two-column proof . Given: $\triangle ABC$ with right angle B. Prove: $\angle A$ and $\angle C$ are complementary.

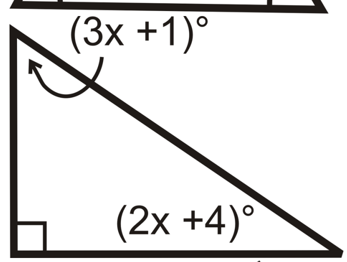


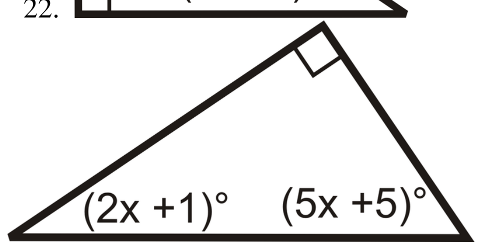
Algebra Connection Solve for x.

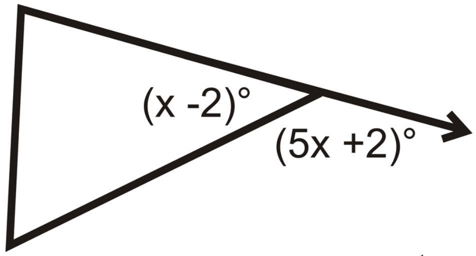
19. 

20. 

21. 

22. 

23. 



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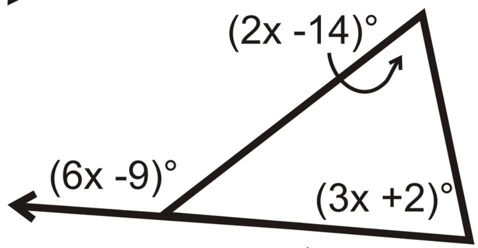

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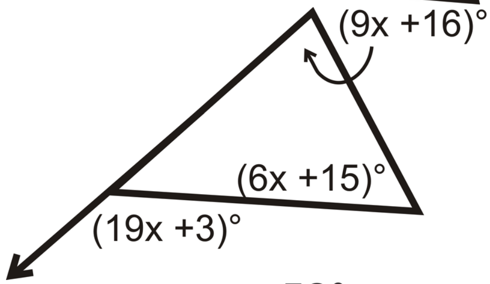

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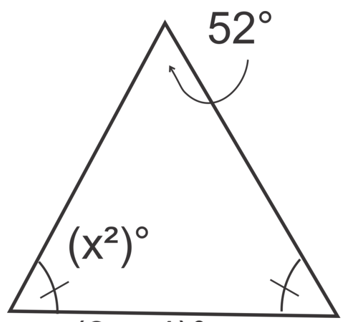

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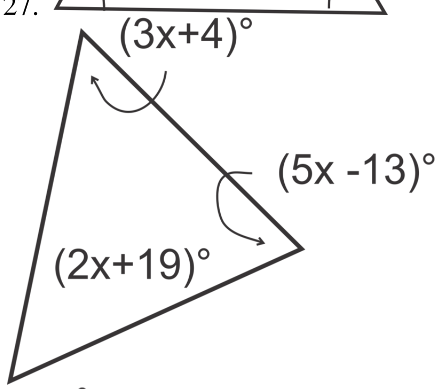

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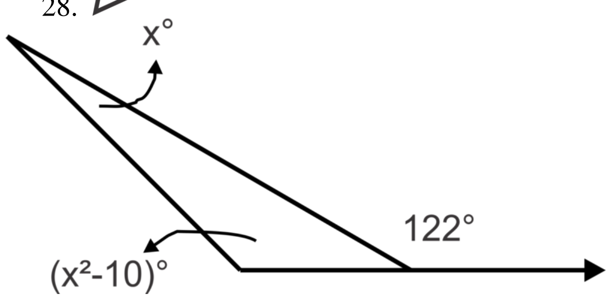

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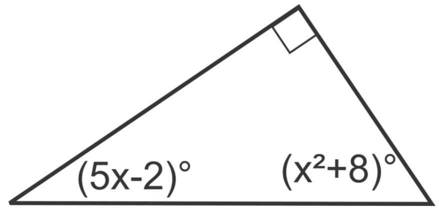
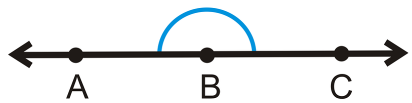
30. 

Review Queue Answers

a. acute isosceles
b. obtuse scalene
c. right scalene
d. 180°,



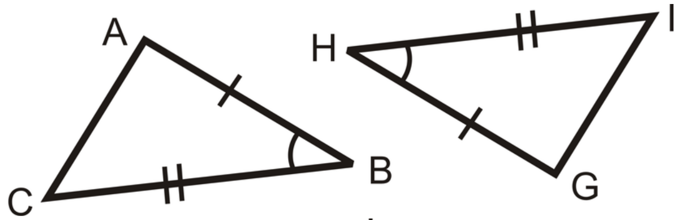
4.2 Congruent Figures

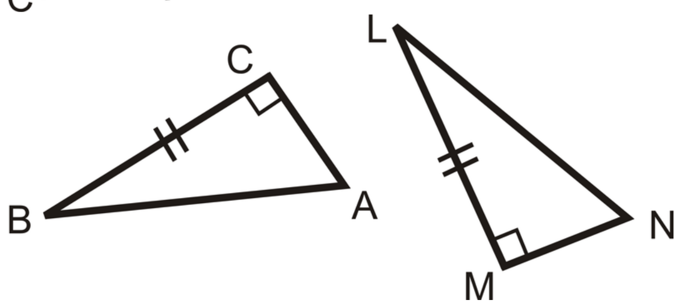
Learning Objectives

- Define congruent triangles and use congruence statements.
- Understand the Third Angle Theorem.
- Use properties of triangle congruence.

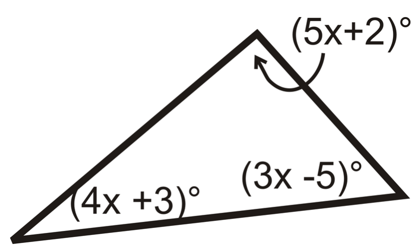
Review Queue

Which corresponding parts of each pair of triangles are congruent? Write all congruence statements for Questions 1 and 2.

a. 

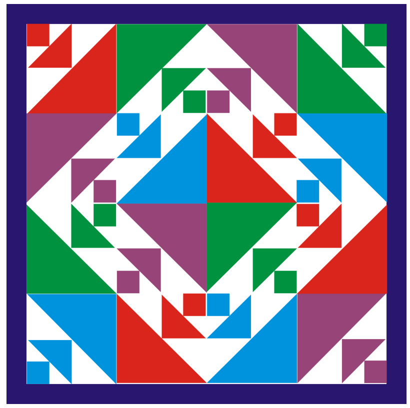
b. 

c. Determine the measure of x.



Know What? Quilt patterns are very geometrical. The pattern to the right is made up of several congruent figures. In order for these patterns to come together, the quilter rotates and flips each block (in this case, a large triangle, smaller triangle, and a smaller square) to get new patterns and arrangements.

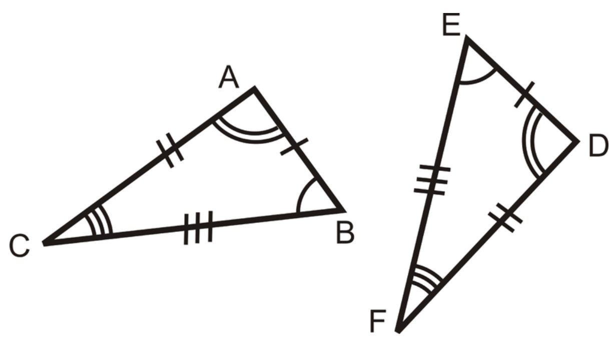
How many different sets of colored congruent triangles are there? How many triangles are in each set? How do you know these triangles are congruent?



Congruent Triangles

Recall that two figures are congruent if and only if they have exactly the same size and shape.

Congruent Triangles: Two triangles are congruent if the three corresponding angles and sides are congruent.

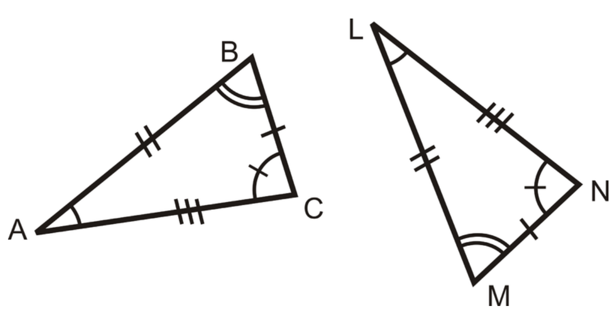


$\triangle ABC$ and $\triangle DEF$ are congruent because

$$\begin{array}{ccc} \overline{AB} \cong \overline{DE} & & \angle A \cong \angle D \\ \overline{BC} \cong \overline{EF} & \text{and} & \angle B \cong \angle E \\ \overline{AC} \cong \overline{DF} & & \angle C \cong \angle F \end{array}$$

When referring to corresponding congruent parts of triangles it is called **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent, or **CPCTC**.

Example 1: Are the two triangles below congruent?



Solution: To determine if the triangles are congruent, each pair of corresponding sides and angles must be congruent.

Start with the sides and match up sides with the same number of tic marks. Using the tic marks: $\overline{BC} \cong \overline{MN}, \overline{AB} \cong \overline{LM}, \overline{AC} \cong \overline{LN}$.

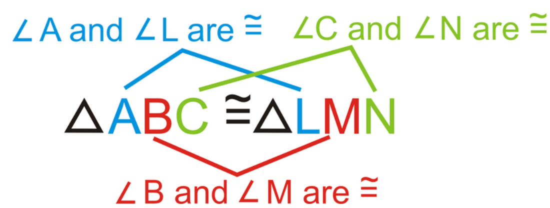
Next match the angles with the same markings; $\angle A \cong \angle L, \angle B \cong \angle M$, and $\angle C \cong \angle N$. Because all six parts are congruent, the two triangles are congruent.

We will learn, later in this chapter that it is impossible for two triangles to have all six parts be congruent and the triangles are not congruent, *when they are drawn to scale.*

Creating Congruence Statements

Looking at Example 1, we know that the two triangles are congruent because the three angles and three sides are congruent to the three angles and three sides in the other triangle.

When stating that two triangles are congruent, the order of the letters is very important. Corresponding parts must be written in the same order. Using Example 1, we would have:



Notice that the congruent sides also line up within the congruence statement.

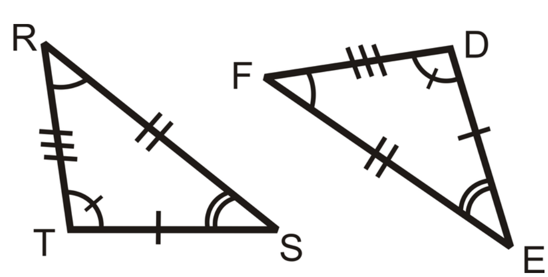
$$\overline{AB} \cong \overline{LM}, \overline{BC} \cong \overline{MN}, \overline{AC} \cong \overline{LN}$$

We can also write this congruence statement several other ways, as long as the congruent angles match up. For example, we can also write $\triangle ABC \cong \triangle LMN$ as:

$$\begin{array}{ll}
\triangle ACB \cong \triangle LNM & \triangle BCA \cong \triangle MNL \\
\triangle BAC \cong \triangle MLN & \triangle CBA \cong \triangle NML \\
\triangle CAB \cong \triangle NLM &
\end{array}$$

One congruence statement can always be written six ways. Any of the six ways above would be correct when stating that the two triangles in Example 1 are congruent.

Example 2: Write a congruence statement for the two triangles below.



Solution: To write the congruence statement, you need to line up the corresponding parts in the triangles: $\angle R \cong \angle F, \angle S \cong \angle E$, and $\angle T \cong \angle D$. Therefore, the triangles are $\triangle RST \cong \triangle FED$.

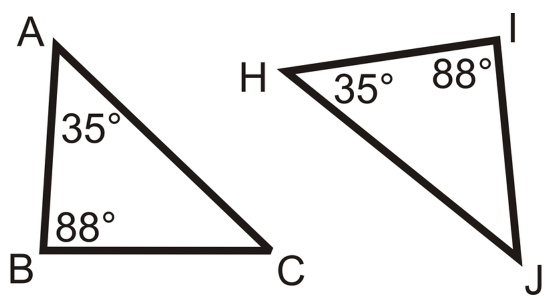
Example 3: If $\triangle CAT \cong \triangle DOG$, what else do you know?

Solution: From this congruence statement, we can conclude three pairs of angles and three pairs of sides are congruent.

$$\angle C \cong \angle D \qquad \angle A \cong \angle O \qquad \angle T \cong \angle G$$
$$\overline{CA} \cong \overline{DO} \qquad \overline{AT} \cong \overline{OG} \qquad \overline{CT} \cong \overline{DG}$$

The Third Angle Theorem

Example 4: Find $m\angle C$ and $m\angle J$.



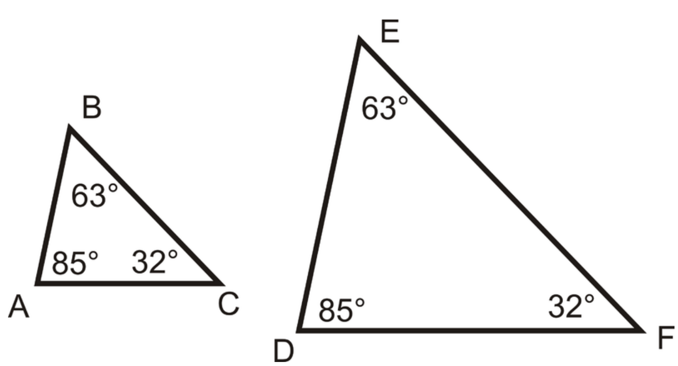
Solution: The sum of the angles in each triangle is 180°. So, for $\triangle ABC, 35^\circ + 88^\circ + m\angle C = 180^\circ$ and $m\angle C = 57^\circ$. For $\triangle HIJ$, $35^\circ + 88^\circ + m\angle J = 180^\circ$ and $m\angle J$ is also 57°.

Notice that we were given that $m\angle A = m\angle H$ and $m\angle B = m\angle I$ and we found out that $m\angle C = m\angle J$. This can be generalized into the Third Angle Theorem.

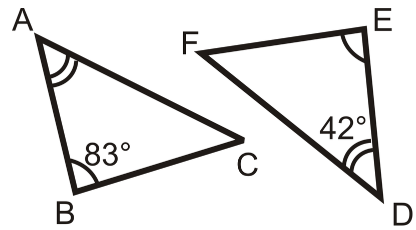
Third Angle Theorem: If two angles in one triangle are congruent to two angles in another triangle, then the third pair of angles must also congruent.

In other words, for triangles $\triangle ABC$ and $\triangle DEF, \angle A \cong \angle D$ and $\angle B \cong \angle E$, then $\angle C \cong \angle F$.

Notice that this theorem does not state that the triangles are congruent. That is because if two sets of angles are congruent, the sides could be different lengths. See the picture to the left.



Example 5: Determine the measure of the missing angles.



Solution: From the markings, we know that $\angle A \cong D$ and $\angle E \cong \angle B$. Therefore, the Third Angle Theorem tells us that $\angle C \cong \angle F$. So,

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= 180^\circ \\ m\angle D + m\angle B + m\angle C &= 180^\circ \\ 42^\circ + 83^\circ + m\angle C &= 180^\circ \\ m\angle C &= 55^\circ = m\angle F \end{aligned}$$

Congruence Properties

Recall the Properties of Congruence from Chapter 2. They will be very useful in the upcoming sections.

Reflexive Property of Congruence: Any shape is congruent to itself.

$\overline{AB} \cong \overline{AB}$ or $\triangle ABC \cong \triangle ABC$

Symmetric Property of Congruence: If two shapes are congruent, the statement can be written with either shape on either side of the $\cong$ sign.

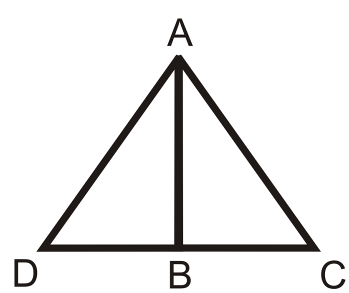
$\angle EFG \cong \angle XYZ$ and $\angle XYZ \cong \angle EFG$ or $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle ABC$

Transitive Property of Congruence: If two shapes are congruent and one of those is congruent to a third, the first and third shapes are also congruent.

$\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHI$, then $\triangle ABC \cong \triangle GHI$

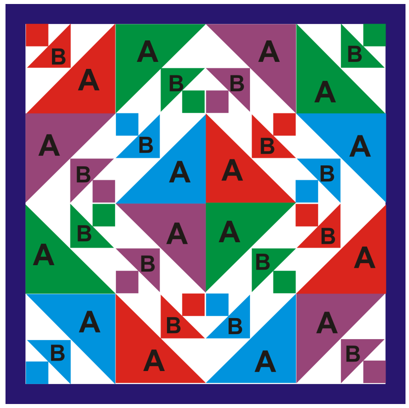
These three properties will be very important when you begin to prove that two triangles are congruent.

Example 6: In order to say that $\triangle ABD \cong \triangle ABC$, you must determine that the three corresponding angles and sides are congruent. Which pair of sides is congruent by the Reflexive Property?



Solution: The side $\overline{AB}$ is shared by both triangles. So, in a geometric proof, $\overline{AB} \cong \overline{AB}$ by the Reflexive Property of Congruence.

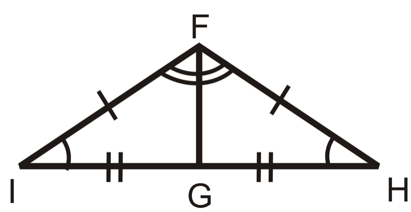
Know What? Revisited There are 16 "*A*" triangles and they are all congruent. There are 16 "*B*" triangles and they are all congruent. The quilt pattern is made from dividing up the square into smaller squares. The "*A*" triangles are all $\frac{1}{32}$ of the overall square and the "*B*" triangles are each $\frac{1}{128}$ of the large square. Both the "*A*" and "*B*" triangles are right triangles.



Review Questions

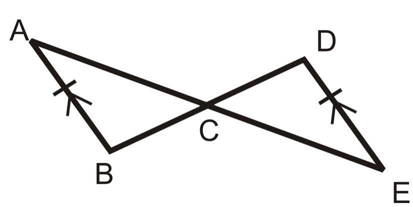
1. If $\triangle RAT \cong \triangle UGH$, what is also congruent?
2. If $\triangle BIG \cong \triangle TOP$, what is also congruent?

For questions 3-7, use the picture to the right.



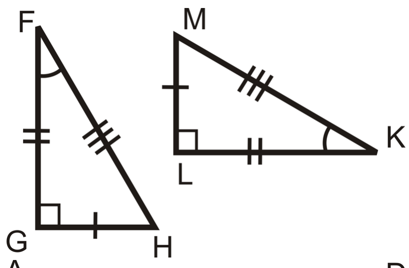
3. What theorem tells us that $\angle FGH \cong \angle FGI$?
4. What is $m\angle FGI$ and $m\angle FGH$? How do you know?
5. What property tells us that the third side of each triangle is congruent?
6. How does $\overline{FG}$ relate to $\angle IFH$?
7. Write the congruence statement for these two triangles.

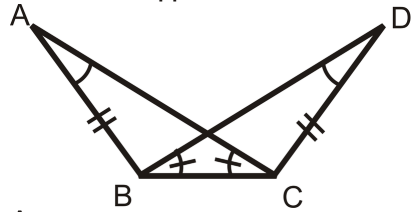
For questions 8-12, use the picture to the right.

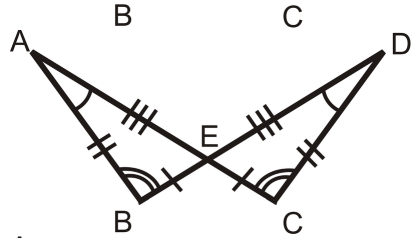


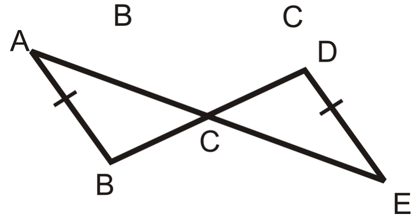
8. If $\overline{AB} \parallel \overline{DE}$, what angles are congruent? How do you know?
9. Why is $\angle ACB \cong \angle ECD$? It is not the same reason as #8.
10. Are the two triangles congruent with the information you currently have? Why or why not?
11. If you are told that C is the midpoint of $\overline{AE}$ and $\overline{BD}$, what segments are congruent?
12. Write a congruence statement for the two triangles.

For questions 13-16, determine if the triangles are congruent. If they are, write the congruence statement.

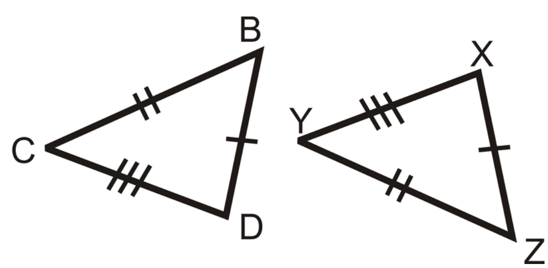
13. 

14. 

15. 

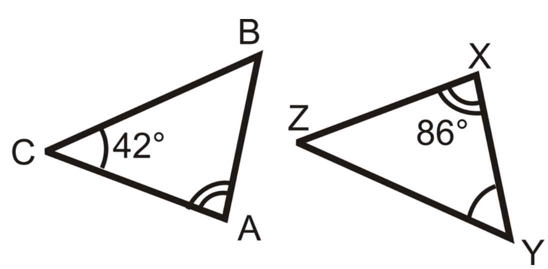
16. 

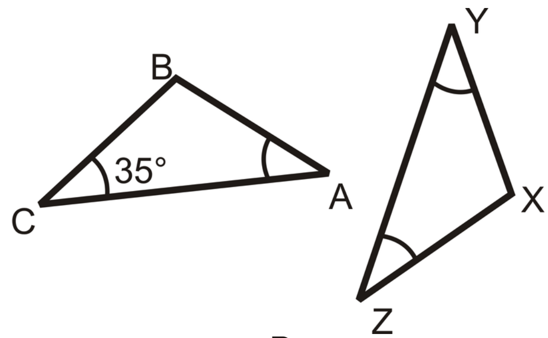
17. Suppose the two triangles to the right are congruent. Write a congruence statement for these triangles.



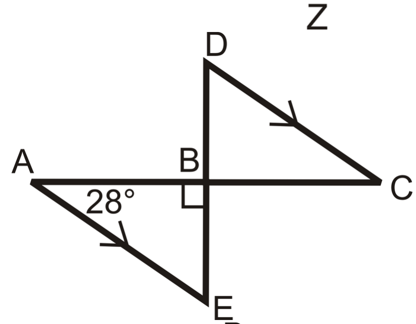
18. Explain how we know that if the two triangles are congruent, then $\angle B \cong \angle Z$.

For questions 19-22, determine the measure of all the angles in the each triangle.

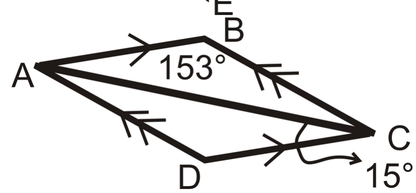
19. 



20.



21.



22.

23. Fill in the blanks in the Third Angle Theorem proof below. Given: $\angle A \cong \angle D, \angle B \cong \angle E$Prove: $\angle C \cong \angle F$

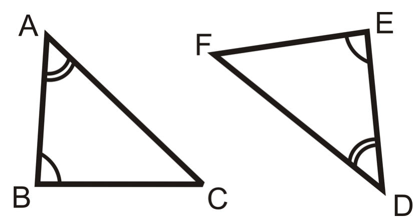


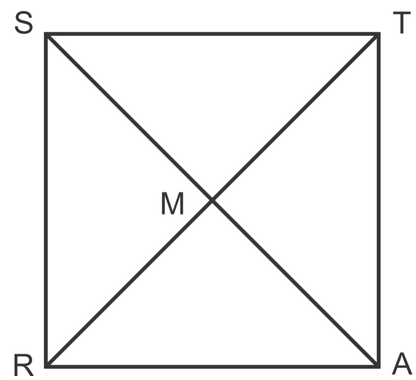
TABLE 4.4:

Statement	**Reason**
1. $A \cong \angle D, \angle B \cong \angle E$	
2.	$\cong$ angles have = measures
3. $m\angle A + m\angle B + m\angle C = 180^\circ$	
m $\angle D + m\angle E + m\angle F = 180^\circ$	
4.	Substitution PoE
5.	Substitution PoE
6. $m\angle C = m\angle F$	
7. $\angle C \cong \angle F$	

For each of the following questions, determine if the Reflexive, Symmetric or Transitive Properties of Congruence is used.

24. $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$
25. $\overline{AB} \cong \overline{AB}$
26. $\triangle XYZ \cong \triangle LMN$ and $\triangle LMN \cong \triangle XYZ$
27. $\triangle ABC \cong \triangle BAC$
28. What type of triangle is $\triangle ABC$ in #27? How do you know?

Use the following diagram for questions 29 and 30.



29. Mark the diagram with the following information. $\overline{ST} \parallel \overline{RA}; \overline{SR} \parallel \overline{TA}; \overline{ST} \perp \overline{TA}$ and $\overline{SR}; \overline{SA}$ and $\overline{RT}$ are perpendicularly bisect each other.
30. Using the given information and your markings, name all of the congruent triangles in the diagram.

Review Queue Answers

1. $\angle B \cong \angle H, \overline{AB} \cong \overline{GH}, \overline{BC} \cong \overline{HI}$

2. $\angle C \cong \angle M, \overline{BC} \cong \overline{LM}$

3. The angles add up to 180°

$$\begin{aligned}(5x+2)^\circ + (4x+3)^\circ + (3x-5)^\circ &= 180^\circ \\ 12x &= 180^\circ \\ x &= 15^\circ\end{aligned}$$

4.3 Triangle Congruence using SSS and SAS

Learning Objectives

- Use the distance formula to analyze triangles on the $x - y$ plane.
- Apply the SSS Postulate to prove two triangles are congruent.
- Apply the SAS Postulate to prove two triangles are congruent.

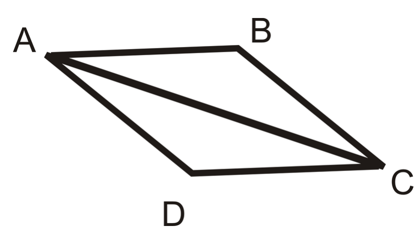
Review Queue

a. Determine the distance between the two points.

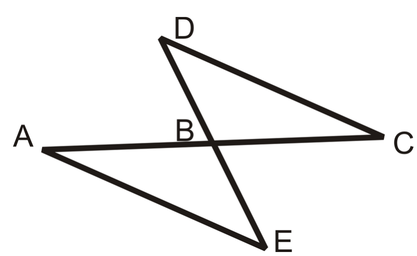
   a. (-1, 5) and (4, 12)
   b. (-6, -15) and (-3, 8)

b. Are the two triangles congruent? Explain why or why not.

   a. $\overline{AB} \parallel \overline{CD}, \overline{AD} \parallel \overline{BC}$
      $\overline{AB} \cong \overline{CD}, \overline{AD} \cong \overline{BC}$

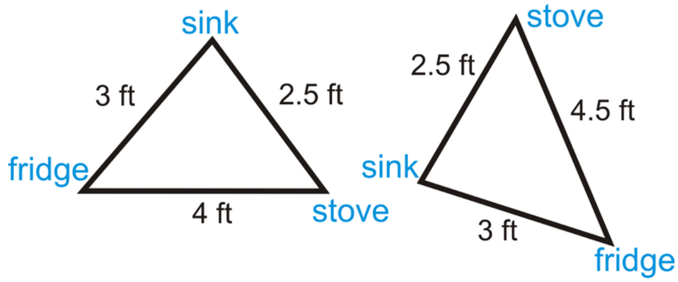


   b. B is the midpoint of $\overline{AC}$ and $\overline{DE}$



c. At this point in time, how many angles and sides do we have to know are congruent in order to say that two triangles are congruent?

Know What? The "ideal" measurements in a kitchen from the sink, refrigerator and oven are as close to an equilateral triangle as possible. Your parents are remodeling theirs to be as close to this as possible and the measurements are in the picture at the left, below. Your neighbor's kitchen has the measurements on the right. Are the two triangles congruent? Why or why not?



SSS Postulate of Triangle Congruence

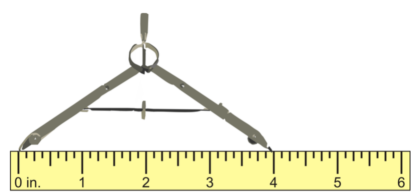
Consider the question: If I have three lengths, 3 in, 4 in, and 5 in, can I construct more than one triangle with these measurements? In other words, can I construct two different triangles with these same three lengths?

Investigation 4-2: Constructing a Triangle Given Three Sides

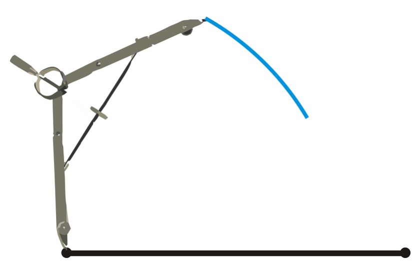
Tools Needed: compass, pencil, ruler, and paper

a. Draw the longest side (5 in) horizontally, halfway down the page. *The drawings in this investigation are to scale.*

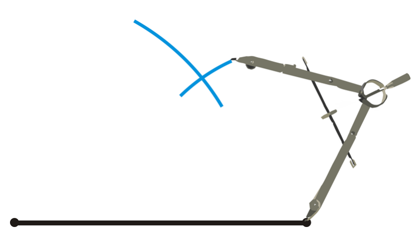
b. Take the compass and, using the ruler, widen the compass to measure 4 in, the next side.



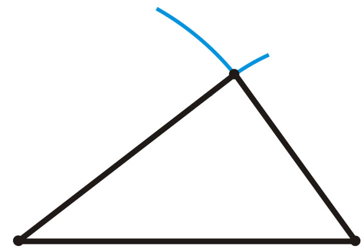
c. Using the measurement from Step 2, place the pointer of the compass on the left endpoint of the side drawn in Step 1. Draw an arc mark above the line segment.



d. Repeat Step 2 with the last measurement, 3 in. Then, place the pointer of the compass on the right endpoint of the side drawn in Step 1. Draw an arc mark above the line segment. Make sure it intersects the arc mark drawn in Step 3.



e. Draw lines from each endpoint to the arc intersections. These lines will be the other two sides of the triangle.



Can you draw another triangle, with these measurements that looks different? The answer is NO. ***Only one triangle can be created from any given three lengths.***

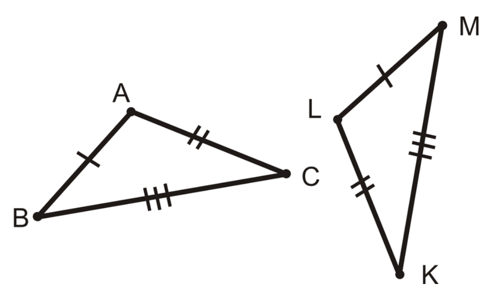
An animation of this investigation can be found at: http://www.mathsisfun.com/geometry/construct-ruler-compass-1.html

Side-Side-Side (SSS) Triangle Congruence Postulate: If three sides in one triangle are congruent to three sides in another triangle, then the triangles are congruent.

Now, we only need to show that all three sides in a triangle are congruent to the three sides in another triangle. This is a postulate so we accept it as true without proof.

Think of the SSS Postulate as a shortcut. You no longer have to show 3 sets of angles are congruent and 3 sets of sides are congruent in order to say that the two triangles are congruent.

Example 1: Write a triangle congruence statement based on the diagram below:

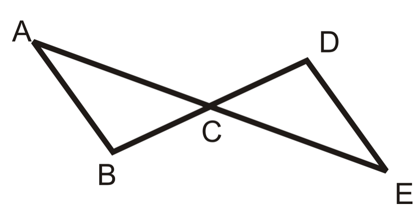


Solution: From the tic marks, we know $\overline{AB} \cong \overline{LM}, \overline{AC} \cong \overline{LK}, \overline{BC} \cong \overline{MK}$. Using the SSS Postulate we know the two triangles are congruent. Lining up the corresponding sides, we have $\triangle ABC \cong \triangle LMK$.

Don't forget ORDER MATTERS when writing triangle congruence statements. Here, we lined up the sides with one tic mark, then the sides with two tic marks, and finally the sides with three tic marks.

Example 2: Write a two-column proof to show that the two triangles are congruent.

Given: $\overline{AB} \cong \overline{DE}$



C is the midpoint of $\overline{AE}$ and $\overline{DB}$.

Prove: $\triangle ACB \cong \triangle ECD$

Solution:

TABLE 4.5:

Statement	***Reason***
1. $\overline{AB} \cong \overline{DE}$	Given
C is the midpoint of $\overline{AE}$ and $\overline{DB}$	
2. $\overline{AC} \cong \overline{CE}, \overline{BC} \cong \overline{CD}$	Definition of a midpoint
3. $\triangle ACB \cong \triangle ECD$	SSS Postulate

Make sure that you clearly state the three sets of congruent sides BEFORE stating that the triangles are congruent.

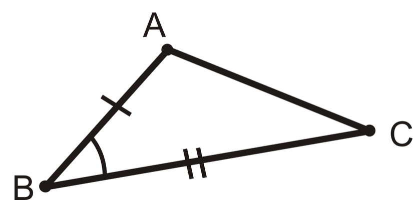
Prove Move: Feel free to mark the picture with the information you are given as well as information that you can infer (vertical angles, information from parallel lines, midpoints, angle bisectors, right angles).

SAS Triangle Congruence Postulate

First, it is important to note that SAS refers to Side-Angle-Side. The placement of the word Angle is important because it indicates that the angle that you are given is between the two sides.

Included Angle: When an angle is between two given sides of a triangle (or polygon).

In the picture to the left, the markings indicate that $\overline{AB}$ and $\overline{BC}$ are the given sides, so $\angle B$ would be the included angle.

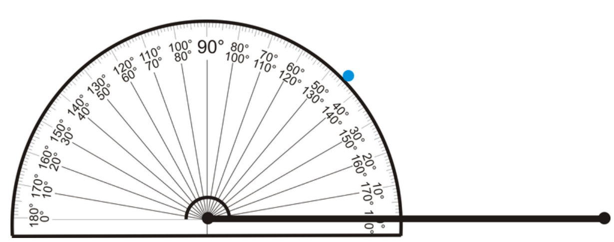


Consider the question: If I have two sides of length 2 in and 5 in and the angle between them is 45°, can I construct only one triangle?

Investigation 4-3: Constructing a Triangle Given Two Sides and Included Angle Tools Needed: protractor, pencil, ruler, and paper

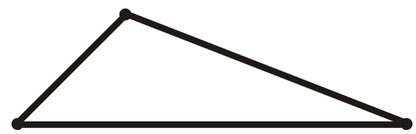
a. Draw the longest side (5 in) horizontally, halfway down the page. *The drawings in this investigation are to scale.*

b. At the left endpoint of your line segment, use the protractor to measure a 45° angle. Mark this measurement.



c. Connect your mark from Step 2 with the left endpoint. Make your line 2 in long, the length of the second side.

d. Connect the two endpoints by drawing the third side.

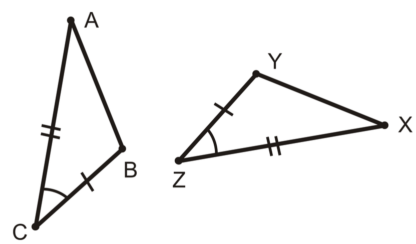


Can you draw another triangle, with these measurements that looks different? The answer is NO. ***Only one triangle can be created from any given two lengths and the INCLUDED angle.***

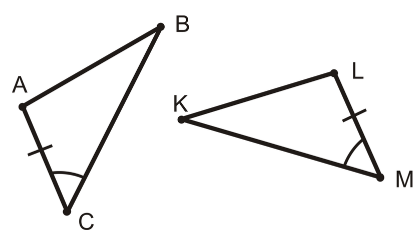
Side-Angle-Side (SAS) Triangle Congruence Postulate: If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.

The markings in the picture are enough to say that $\triangle ABC \cong \triangle XYZ$.

So, in addition to SSS congruence, we now have SAS. Both of these postulates can be used to say that two triangles are congruent. When doing proofs, you might be able to use either SSS or SAS to prove that two triangles are congruent. There is no set way to complete a proof, so when faced with the choice to use SSS or SAS, it does not matter. Either would be correct.



Example 3: What additional piece of information would you need to prove that these two triangles are congruent using the SAS Postulate?



a) $\angle ABC \cong \angle LKM$

b) $\overline{AB} \cong \overline{LK}$

c) $\overline{BC} \cong \overline{KM}$

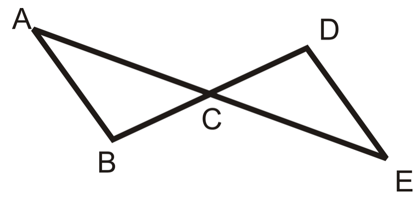
d) $\angle BAC \cong \angle KLM$

Solution: For the SAS Postulate, you need two sides and the included angle in both triangles. So, you need the side on the other side of the angle. In $\triangle ABC$, that is $\overline{BC}$ and in $\triangle LKM$ that is $\overline{KM}$. The correct answer is c.

Example 4: Write a two-column proof to show that the two triangles are congruent.

Given: C is the midpoint of $\overline{AE}$ and $\overline{DB}$

Prove: $\triangle ACB \cong \triangle ECD$



Solution:

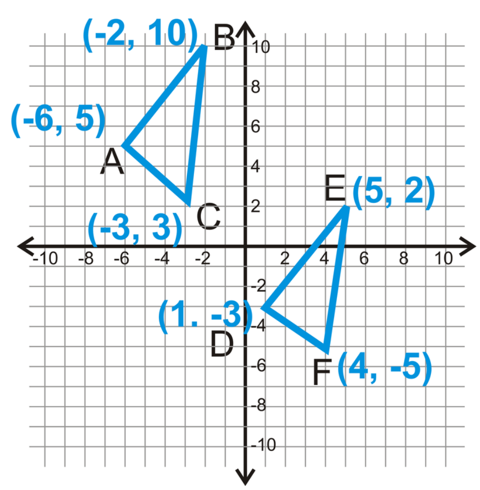
TABLE 4.6:

Statement	***Reason***
1. C is the midpoint of $\overline{AE}$ and $\overline{DB}$	Given
2. $\overline{AC} \cong \overline{CE}, \overline{BC} \cong \overline{CD}$	Definition of a midpoint
3. $\angle ACB \cong \angle DCE$	Vertical Angles Postulate
4. $\triangle ACB \cong \triangle ECD$	SAS Postulate

In Example 4, we could have only proven the two triangles congruent by SAS. If we were given that $\overline{AB} \cong \overline{DE}$, then we could have also proven the two triangles congruent by SSS.

SSS in the Coordinate Plane

In the coordinate plane, the easiest way to show two triangles are congruent is to find the lengths of the 3 sides in each triangle. Finding the measure of an angle in the coordinate plane can be a little tricky, so we will avoid it in this text. Therefore, you will only need to apply SSS in the coordinate plane. To find the lengths of the sides, you will need to use the distance formula, $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.



Example 5: Find the distances of all the line segments from both triangles to see if the two triangles are congruent.

Solution: Begin with $\triangle ABC$ and its sides.

$$\begin{aligned}
AB &= \sqrt{(-6-(-2))^2+(5-10)^2} \\
&= \sqrt{(-4)^2+(-5)^2} \\
&= \sqrt{16+25} \\
&= \sqrt{41}
\end{aligned}$$

$$\begin{aligned}
BC &= \sqrt{(-2-(-3))^2+(10-3)^2} \\
&= \sqrt{(1)^2+(7)^2} \\
&= \sqrt{1+49} \\
&= \sqrt{50} = 5\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
AC &= \sqrt{(-6-(-3))^2+(5-3)^2} \\
&= \sqrt{(-3)^2+(2)^2} \\
&= \sqrt{9+4} \\
&= \sqrt{13}
\end{aligned}$$

Now, find the distances of all the sides in $\triangle DEF$.

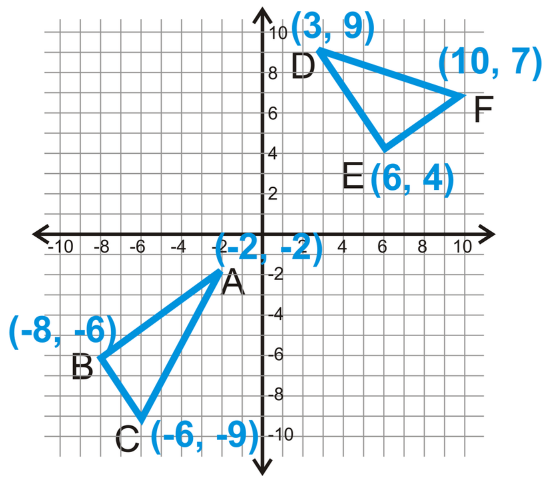
$$\begin{aligned}
DE &= \sqrt{(1-5)^2+(-3-2)^2} \\
&= \sqrt{(-4)^2+(-5)^2} \\
&= \sqrt{16+25} \\
&= \sqrt{41}
\end{aligned}$$

$$\begin{aligned}
EF &= \sqrt{(5-4)^2+(2-(-5))^2} \\
&= \sqrt{(1)^2+(7)^2} \\
&= \sqrt{1+49} \\
&= \sqrt{50} = 5\sqrt{2}
\end{aligned}$$

$$\begin{aligned}
DF &= \sqrt{(1-4)^2+(-3-(-5))^2} \\
&= \sqrt{(-3)^2+(2)^2} \\
&= \sqrt{9+4} \\
&= \sqrt{13}
\end{aligned}$$

We see that $AB = DE, BC = EF$, and $AC = DF$. Recall that if two lengths are equal, then they are also congruent. Therefore, $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$. Because the corresponding sides are congruent, we can say that $\triangle ABC \cong \triangle DEF$ by SSS.

Example 6: Determine if the two triangles are congruent.



Solution: Use the distance formula to find all the lengths. Start with $\triangle ABC$.

$$\begin{aligned}AB &= \sqrt{(-2-(-8))^2+(-2-(-6))^2}\\ &= \sqrt{(6)^2+(4)^2}\\ &= \sqrt{36+16}\\ &= \sqrt{52} = 2\sqrt{13}\end{aligned}$$

$$\begin{aligned}BC &= \sqrt{(-8-(-6))^2+(-6-(-9))^2}\\ &= \sqrt{(-2)^2+(3)^2}\\ &= \sqrt{4+9}\\ &= \sqrt{13}\end{aligned}$$

$$\begin{aligned}AC &= \sqrt{(-2-(-6))^2+(-2-(-9))^2}\\ &= \sqrt{(-4)^2+(7)^2}\\ &= \sqrt{16+49}\\ &= \sqrt{65}\end{aligned}$$

Now find the sides of $\triangle DEF$.

$$\begin{aligned} DE &= \sqrt{(3-6)^2+(9-4)^2} \\ &= \sqrt{(-3)^2+(5)^2} \\ &= \sqrt{9+25} \\ &= \sqrt{34} \end{aligned}$$

$$\begin{aligned} EF &= \sqrt{(6-10)^2+(4-7)^2} \\ &= \sqrt{(-4)^2+(-3)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25}=5 \end{aligned}$$

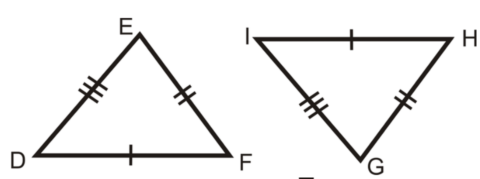
$$\begin{aligned} DF &= \sqrt{(3-10)^2+(9-7)^2} \\ &= \sqrt{(-7)^2+(2)^2} \\ &= \sqrt{49+4} \\ &= \sqrt{53} \end{aligned}$$

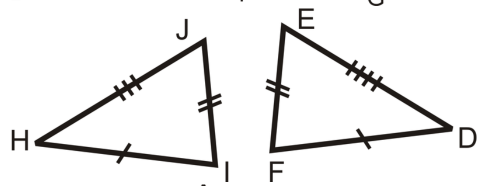
No sides have equal measures, so the triangles are not congruent.

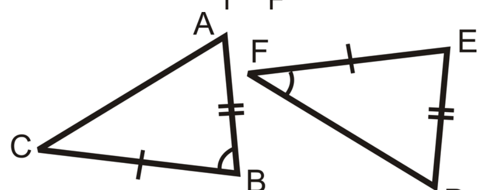
Know What? Revisited From what we have learned in this section, the two triangles are not congruent because the distance from the fridge to the stove in your house is 4 feet and in your neighbor's it is 4.5 ft. The SSS Postulate tells us that all three sides have to be congruent.

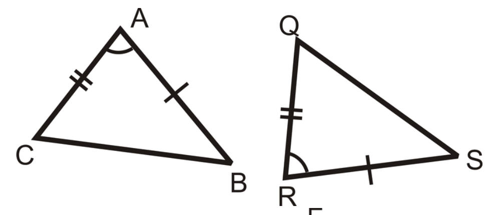
Review Questions

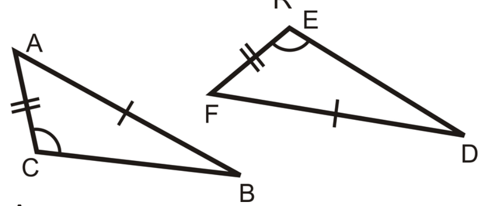
Are the pairs of triangles congruent? If so, write the congruence statement.

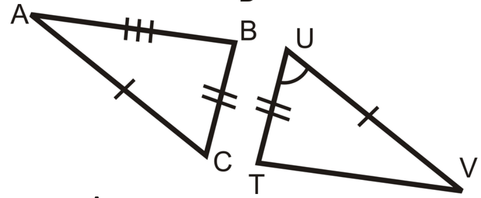
1. 

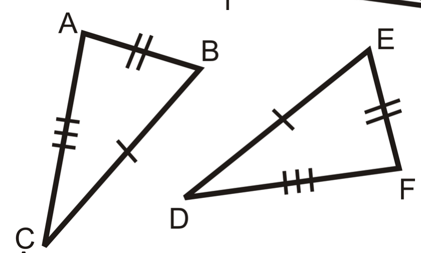
2. 

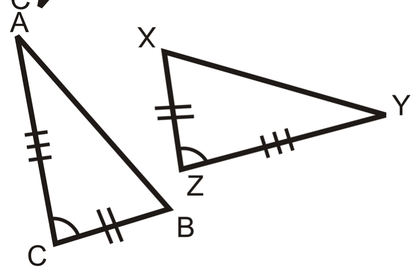
3. 

4. 

5. 

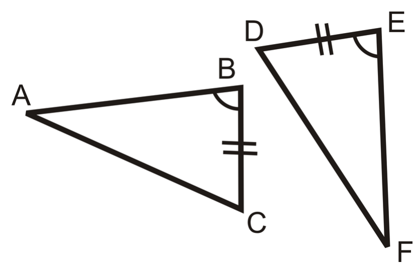
6. 

7. 

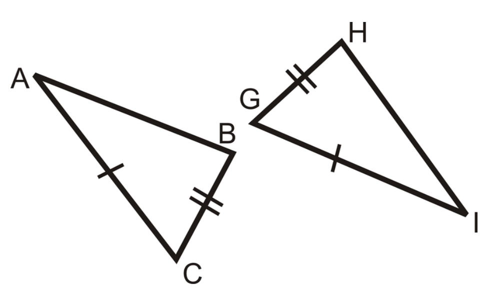
8. 

State the additional piece of information needed to show that each pair of triangles are congruent.

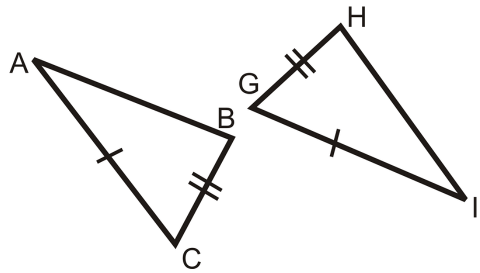
9. Use SAS



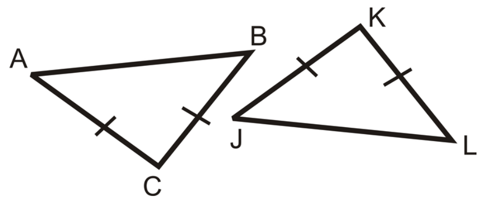
10. Use SSS



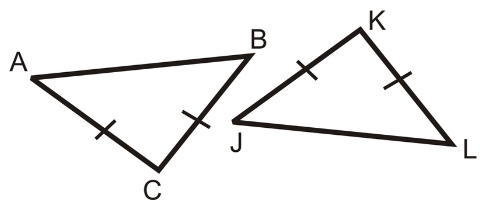
11. Use SAS



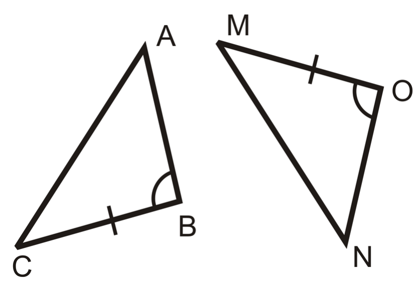
12. Use SAS



13. Use SSS



14. Use SAS



Fill in the blanks in the proofs below.

15. <u>Given</u>: $\overline{AB} \cong \overline{DC}, \overline{BE} \cong \overline{CE}$<u>Prove</u>: $\triangle ABE \cong \triangle ACE$

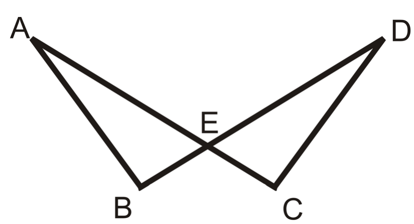


TABLE 4.7:

Statement	***Reason***
1.	1.
2. $\angle AEB \cong \angle DEC$	2.
3. $\triangle ABE \cong \triangle ACE$	3.

16. Given: $\overline{AB} \cong \overline{DC}, \overline{AC} \cong \overline{DB}$Prove: $\triangle ABC \cong \triangle DCB$

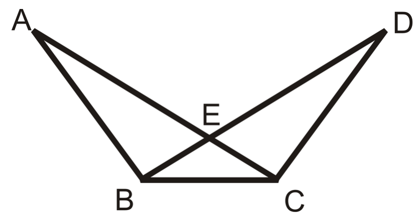


TABLE 4.8:

Statement	Reason
1.	1.
2.	2. Reflexive PoC
3. $\triangle ABC \cong \triangle DCB$	3.

17. Given: B is a midpoint of $\overline{DC}\overline{AB} \perp \overline{DC}$Prove: $\triangle ABD \cong \triangle ABC$

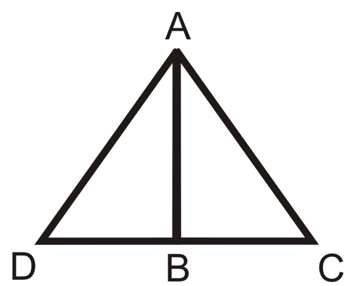
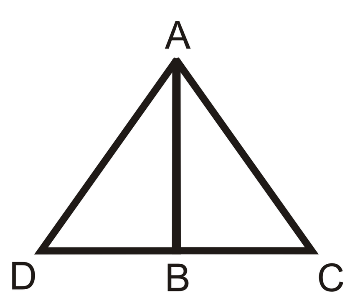


TABLE 4.9:

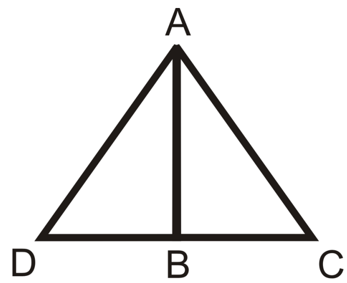
Statement	Reason
1. B is a midpoint of $\overline{DC}, \overline{AB} \perp \overline{DC}$	1.
2.	2. Definition of a midpoint
3. $\angle ABD$ and $\angle ABC$ are right angles	3.
4.	4. All right angles are $\cong$
5.	5.
6. $\triangle ABD \cong \triangle ABC$	6.

Write a two-column proof for the given information below.

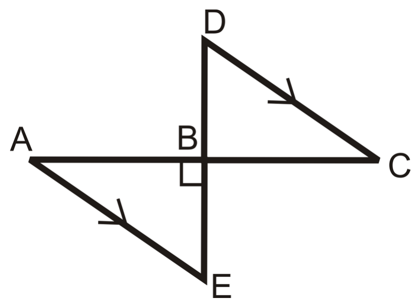
18. Given: $\overline{AB}$ is an angle bisector of $\angle DAC\overline{AD} \cong \overline{AC}$Prove: $\triangle ABD \cong \triangle ABC$



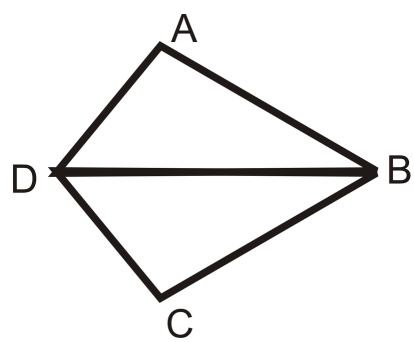
19. Given: B is the midpoint of $\overline{DC}\overline{AD} \cong \overline{AC}$Prove: $\triangle ABD \cong \triangle ABC$



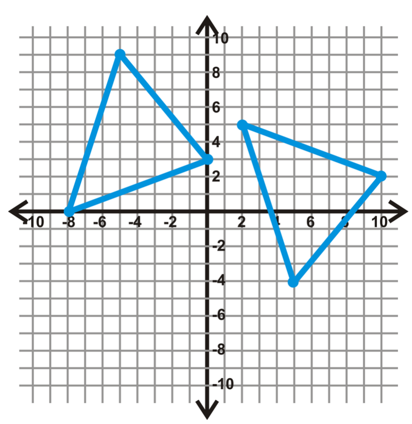
20. Given: B is the midpoint of $\overline{DE}$ and $\overline{AC}\angle ABE$ is a right angle Prove: $\triangle ABE \cong \triangle CBD$

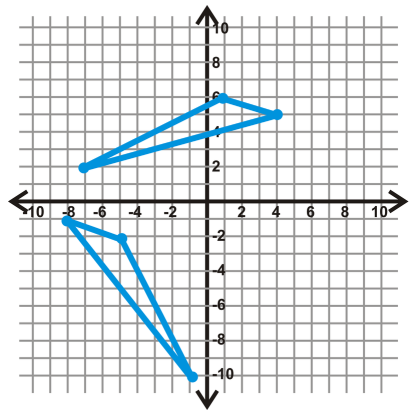


21. Given: $\overline{DB}$ is the angle bisector of $\angle ADC\overline{AD} \cong \overline{DC}$Prove: $\triangle ABD \cong \triangle CBD$



Determine if the two triangles are congruent, using the distance formula. Leave all of your answers in simplest radical form (simplify all radicals, no decimals).

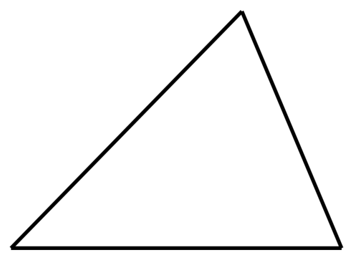
22. 

23. 

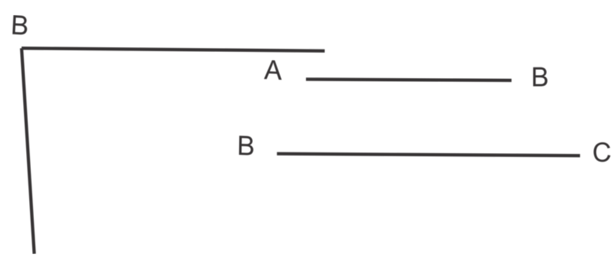
24. $\triangle ABC: A(-1,5), B(-4,2), C(2,-2)$ and $\triangle DEF: D(7,-5), E(4,2), F(8,-9)$
25. $\triangle ABC: A(-8,-3), B(-2,-4), C(-5,-9)$ and $\triangle DEF: D(-7,2), E(-1,3), F(-4,8)$

Constructions

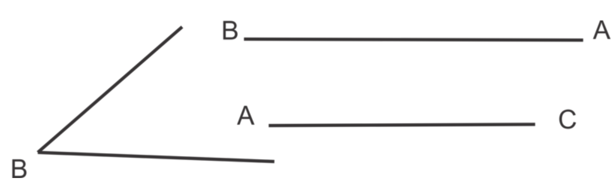
26. Construct a triangle with sides of length 5cm, 3cm, 2cm.
27. Copy the triangle below using a straightedge and compass.



28. Use the two sides and the given angle to construct $\triangle ABC$.



29. Use the two sides and the given angle to construct $\triangle ABC$.



30. Was the information given in problem 29 in SAS order? If not, your triangle may not be the only triangle that you can construct using the given information. Construct the second possible triangle.

Review Queue Answers

a. a. $\sqrt{74}$

b. $\sqrt{538}$

a. Yes, $\triangle CAD \cong \triangle ACB$ because $\angle CAD \cong \angle ACB$ and $\angle BAC \cong \angle ACD$ by Alternate Interior Angles. $\overline{AC} \cong \overline{AC}$ by the Reflexive PoC and $\angle ADC \cong \angle ABC$ by the 3^{rd} Angle Theorem.
b. At this point in time, we do not have enough information to show that the two triangles are congruent. We know that $\overline{AB} \cong \overline{BC}$ and $\overline{DB} \cong \overline{BE}$ from the definition of a midpoint. By vertical angles, we know that $\angle DBC \cong \angle ABE$. This is only two sides and one pair of angles; not enough info, yet.

b. We need to know three pairs of congruent sides and two pairs of congruent angles. From this, we can assume the third pair of angles are congruent from the 3^{rd} Angle Theorem.

4.4 Triangle Congruence Using ASA, AAS, and HL

Learning Objectives

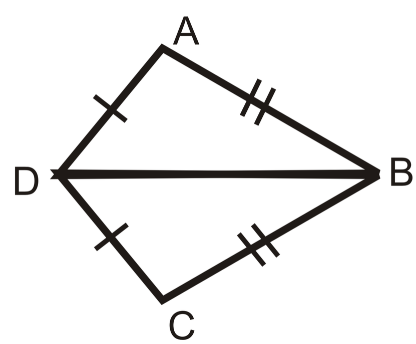
- Use the ASA Congruence Postulate, AAS Congruence Theorem, and the HL Congruence Theorem.
- Complete two-column proofs using SSS, SAS, ASA, AAS, and HL.

Review Queue

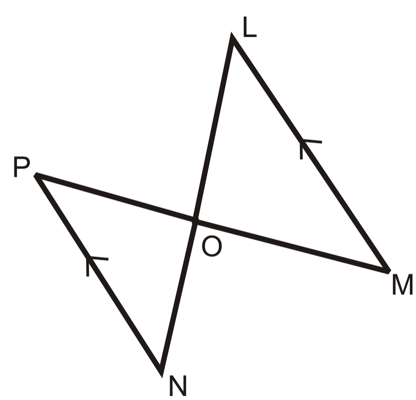
1. Write a two-column proof.

Given: $\overline{AD} \cong \overline{DC}, \overline{AB} \cong \overline{CB}$

Prove: $\triangle DAB \cong \triangle DCB$



2. Is $\triangle PON \cong \triangle MOL$? Why or why not?

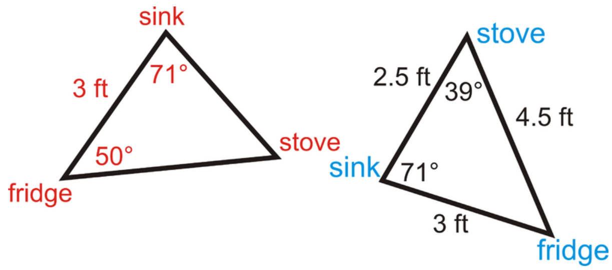


3. If $\triangle DEF \cong \triangle PQR$, can it be assumed that:

a) $\angle F \cong \angle R$? Why or why not?

b) $\overline{EF} \cong \overline{PR}$? Why or why not?

Know What? Your parents changed their minds at the last second about their kitchen layout. Now, they have decided they to have the distance between the sink and the fridge be 3 ft, the angle at the sink 71° and the angle at the fridge is 50°. You used your protractor to measure the angle at the stove and sink at your neighbor's house. Are the kitchen triangles congruent now?



ASA Congruence

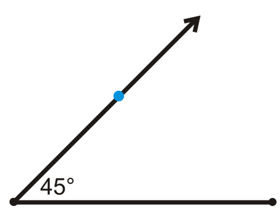
Like SAS, ASA refers to Angle-Side-Angle. The placement of the word Side is important because it indicates that the side that you are given is between the two angles.

Consider the question: If I have two angles that are 45° and 60° and the side between them is 5 in, can I construct only one triangle? We will investigate it here.

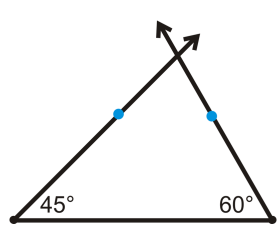
Investigation 4-4: Constructing a Triangle Given Two Angles and Included Side Tools Needed: protractor, pencil, ruler, and paper

a. Draw the side (5 in) horizontally, halfway down the page. *The drawings in this investigation are to scale.*

b. At the left endpoint of your line segment, use the protractor to measure the 45° angle. Mark this measurement and draw a ray from the left endpoint through the 45° mark.



c. At the right endpoint of your line segment, use the protractor to measure the 60° angle. Mark this measurement and draw a ray from the left endpoint through the 60° mark. Extend this ray so that it crosses through the ray from Step 2.

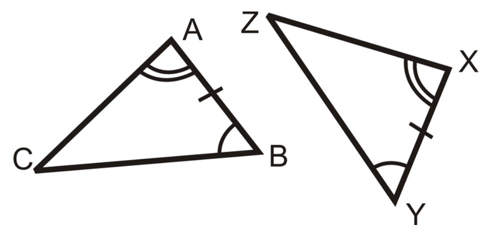


d. Erase the extra parts of the rays from Steps 2 and 3 to leave only the triangle.

Can you draw another triangle, with these measurements that looks different? The answer is NO. ***Only one triangle can be created from any given two angle measures and the INCLUDED side.***

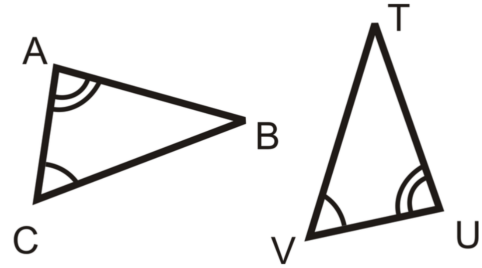
Angle-Side-Angle (ASA) Congruence Postulate: If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent.

The markings in the picture are enough to say $\triangle ABC \cong \triangle XYZ$.



Now, in addition to SSS and SAS, you can use ASA to prove that two triangles are congruent.

Example 1: What information would you need to prove that these two triangles are congruent using the ASA Postulate?



a) $\overline{AB} \cong \overline{UT}$

b) $\overline{AC} \cong \overline{UV}$

c) $\overline{BC} \cong \overline{TV}$

d) $\angle B \cong \angle T$

Solution: For ASA, we need the side between the two given angles, which is $\overline{AC}$ and $\overline{UV}$. The answer is b.

Example 2: Write a 2-column proof.

Given: $\angle C \cong \angle E, \overline{AC} \cong \overline{AE}$

Prove: $\triangle ACF \cong \triangle AEB$

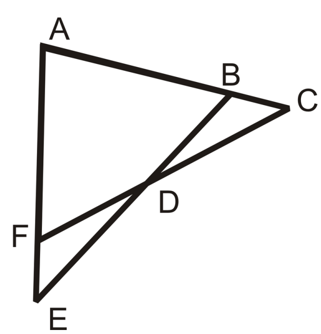
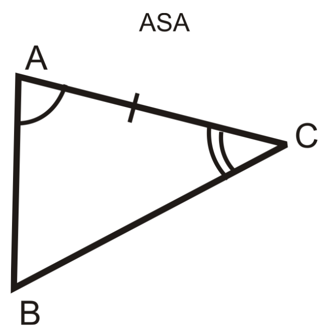


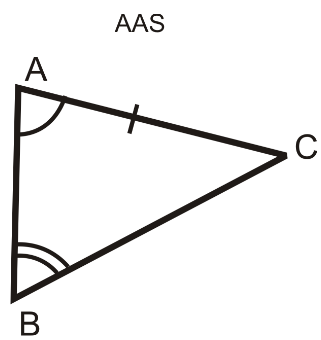
TABLE 4.10:

Statement	*Reason*
1. $\angle C \cong \angle E, \overline{AC} \cong \overline{AE}$	Given
2. $\angle A \cong \angle A$	Reflexive PoC
3. $\triangle ACF \cong \triangle AEB$	ASA

AAS Congruence

A variation on ASA is AAS, which is Angle-Angle-Side. Recall that for ASA you need two angles and the side between them. But, if you know two pairs of angles are congruent, then the third pair will also be congruent by the 3^{rd} Angle Theorem. Therefore, you can prove a triangle is congruent whenever you have any two angles and a side.





Be careful to note the placement of the side for ASA and AAS. As shown in the pictures above, the side is ***between*** the two angles for ASA and it is not for AAS.

Angle-Angle-Side (AAS or SAA) Congruence Theorem: If two angles and a non-included side in one triangle are congruent to two corresponding angles and a non-included side in another triangle, then the triangles are congruent.

Proof of AAS Theorem:

Given: $\angle A \cong \angle Y, \angle B \cong \angle Z, \overline{AC} \cong \overline{XY}$

Prove: $\triangle ABC \cong \triangle YZX$

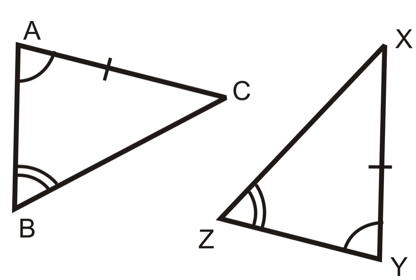


TABLE 4.11:

Statement	*Reason*
1. $\angle A \cong \angle Y, \angle B \cong \angle Z, \overline{AC} \cong \overline{XY}$	Given
2. $\angle C \cong \angle X$	3^{rd} Angle Theorem
3. $\triangle ABC \cong \triangle YZX$	ASA

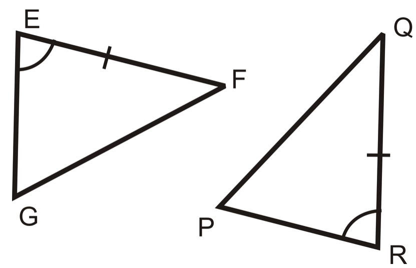
By proving $\triangle ABC \cong \triangle YZX$ with ASA, we have also shown that the AAS Theorem is valid. You can now use this theorem to show that two triangles are congruent.

Example 3: What information do you need to prove that these two triangles are congruent using:

a) ASA?

b) AAS?

c) SAS?



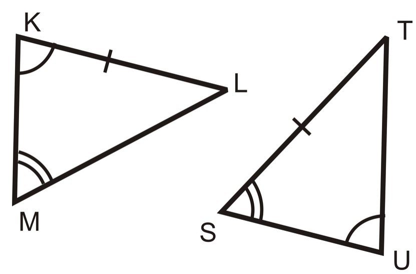
Solution:

a) For ASA, we need the angles on the other side of $\overline{EF}$ and $\overline{QR}$. Therefore, we would need $\angle F \cong \angle Q$.

b) For AAS, we would need the angle on the other side of $\angle E$ and $\angle R$. $\angle G \cong \angle P$.

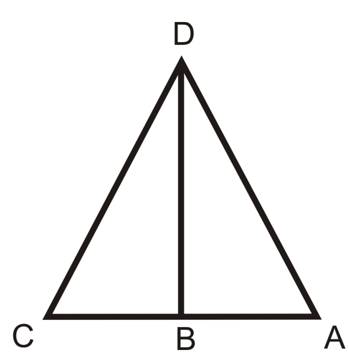
c) For SAS, we would need the side on the other ***side*** of $\angle E$ and $\angle R$. So, we would need $\overline{EG} \cong \overline{RP}$.

Example 4: Can you prove that the following triangles are congruent? Why or why not?



Solution: Even though $\overline{KL} \cong \overline{ST}$, they are not corresponding. Look at the angles around $\overline{KL}, \angle K$ and $\angle L$. $\angle K$ has **one** arc and $\angle L$ is unmarked. The angles around $\overline{ST}$ are $\angle S$ and $\angle T$. $\angle S$ has **two** arcs and $\angle T$ is unmarked. In order to use AAS, $\angle S$ needs to be congruent to $\angle K$. They are not congruent because the arcs marks are different. Therefore, we cannot conclude that these two triangles are congruent.

Example 5: Write a 2-column proof.



Given: $\overline{BD}$ is an angle bisector of $\angle CDA, \angle C \cong \angle A$

Prove: $\triangle CBD \cong \angle ABD$

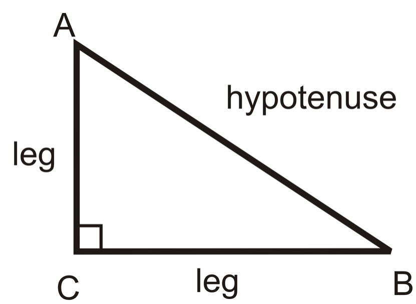
Solution:

TABLE 4.12:

Statement	Reason
1. $\overline{BD}$ is an angle bisector of $\angle CDA$, $\angle C \cong \angle A$	Given
2. $\angle CDB \cong \angle ADB$	Definition of an Angle Bisector
3. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
3. $\triangle CBD \cong \triangle ABD$	AAS

Hypotenuse-Leg Congruence Theorem

So far, the congruence postulates we have learned will work on any triangle. The last congruence theorem can only be used on right triangles. Recall that a right triangle has exactly one right angle. The two sides adjacent to the right angle are called legs and the side opposite the right angle is called the hypotenuse.



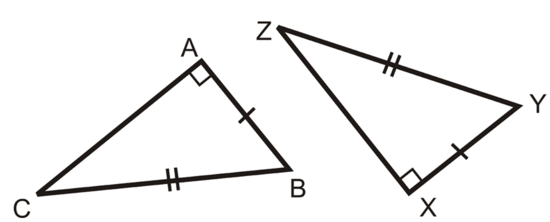
You may or may not know the Pythagorean Theorem (which will be covered in more depth later in this text). It says, for any ***right*** triangle, this equation is true:

$(leg)^2 + (leg)^2 = (hypotenuse)^2$. What this means is that if you are given two sides of a right triangle, you can always find the third.

Therefore, if you know that two sides of a ***right*** triangle are congruent to two sides of another ***right*** triangle, you can conclude that third sides are also congruent.

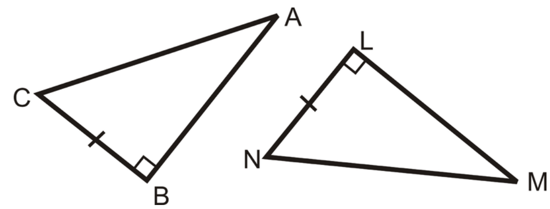
HL Congruence Theorem: If the hypotenuse and leg in one right triangle are congruent to the hypotenuse and leg in another right triangle, then the two triangles are congruent.

The markings in the picture are enough to say $\triangle ABC \cong \triangle XYZ$.



Notice that this theorem is only used with a hypotenuse and a leg. If you know that the two legs of a right triangle are congruent to two legs of another triangle, the two triangles would be congruent by SAS, because the right angle would be between them. We will not prove this theorem here because we have not proven the Pythagorean Theorem yet.

Example 6: What information would you need to prove that these two triangles are congruent using the: a) HL Theorem? b) SAS Theorem?



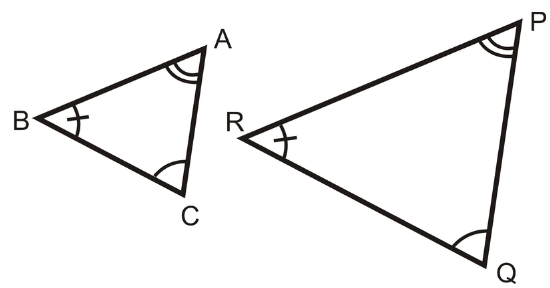
Solution:

a) For HL, you need the hypotenuses to be congruent. So, $\overline{AC} \cong \overline{MN}$.

b) To use SAS, we would need the other legs to be congruent. So, $\overline{AB} \cong \overline{ML}$.

AAA and SSA Relationships

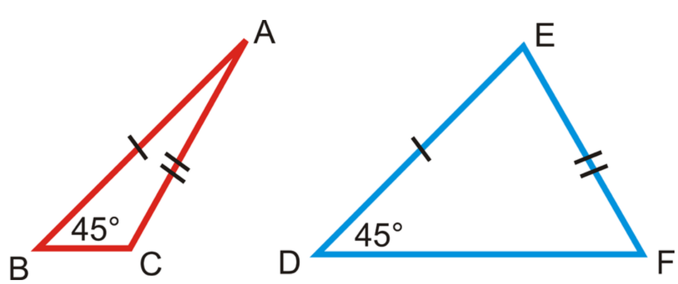
There are two other side-angle relationships that we have not discussed: AAA and SSA.



AAA implied that all the angles are congruent, however, that does not mean the triangles are congruent.

As you can see, $\triangle ABC$ and $\triangle PRQ$ are not congruent, even though all the angles are. These triangles are similar, a topic that will be discussed later in this text.

SSA relationships do not prove congruence either. In review problems 29 and 30 of the last section you illustrated an example of how SSA could produce two different triangles. $\triangle ABC$ and $\triangle DEF$ below are another example of SSA.



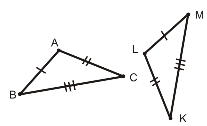
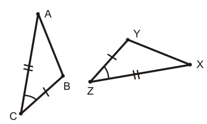
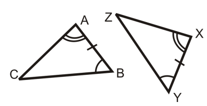
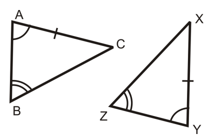
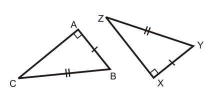
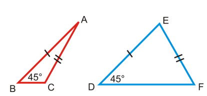
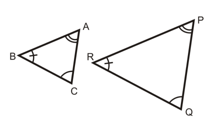
$\angle B$ and $\angle D$ are ***not*** the included angles between the congruent sides, so we cannot prove that these two triangles are congruent. Notice, that two different triangles can be drawn even though $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{EF}$, and $m\angle B = m\angle D$.

You might have also noticed that SSA could also be written ASS. This is true, however, in this text we will write SSA.

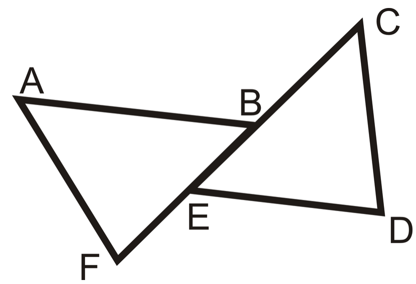
Triangle Congruence Recap

To recap, here is a table of all of the possible side-angle relationships and if you can use them to determine congruence or not.

TABLE 4.13:

Side-Angle Relationship	*Picture*	*Determine Congruence?*
SSS		**Yes** $\triangle ABC \cong \triangle LKM$
SAS		**Yes** $\triangle ABC \cong \triangle XYZ$
ASA		**Yes** $\triangle ABC \cong \triangle XYZ$
AAS (or SAA)		**Yes** $\triangle ABC \cong \triangle YZX$
HL		**Yes, Right Triangles Only** $\triangle ABC \cong \triangle XYZ$
SSA (or ASS)		**NO**
AAA		**NO**

Example 7: Write a 2-column proof.



Given: $\overline{AB} \parallel \overline{ED}, \angle C \cong \angle F, \overline{AB} \cong \overline{ED}$

Prove: $\overline{AF} \cong \overline{CD}$

Solution:

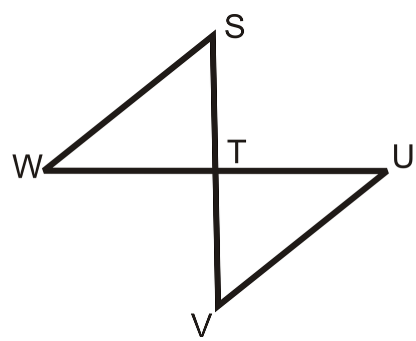
TABLE 4.14:

Statement	*Reason*
1. $\overline{AB} \parallel \overline{ED}, \angle C \cong \angle F, \overline{AB} \cong \overline{ED}$	Given
2. $\angle ABE \cong \angle DEB$	Alternate Interior Angles Theorem
3. $\triangle ABF \cong \triangle DEC$	ASA
4. $\overline{AF} \cong \overline{CD}$	CPCTC

Example 8: Write a 2-column proof.

Given: T is the midpoint of $\overline{WU}$ and $\overline{SV}$

Prove: $\overline{WS} \parallel \overline{VU}$



Solution:

TABLE 4.15:

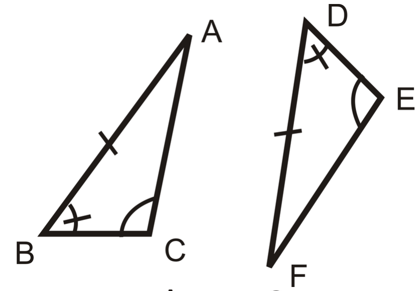
Statement	*Reason*
1. T is the midpoint of $\overline{WU}$ and $\overline{SV}$	Given
2. $\overline{WT} \cong \overline{TU}, \overline{ST} \cong \overline{TV}$	Definition of a midpoint
3. $\angle STW \cong \angle UTV$	Vertical Angle Theorem
4. $\triangle STW \cong \triangle VTU$	SAS
5. $\angle S \cong \angle V$	CPCTC
6. $\overline{WS} \parallel \overline{VU}$	Converse of the Alternate Interior Angles Theorem

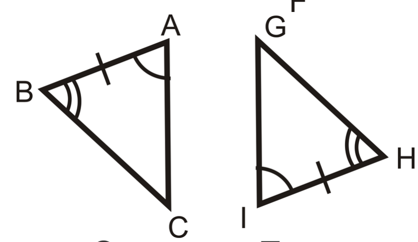
Prove Move: At the beginning of this chapter we introduced CPCTC. Now, it can be used in a proof once two triangles are proved congruent. It is used to prove the parts of congruent triangles are congruent in order to prove that sides are parallel (like in Example 8), midpoints, or angle bisectors. You will do proofs like these in the review questions.

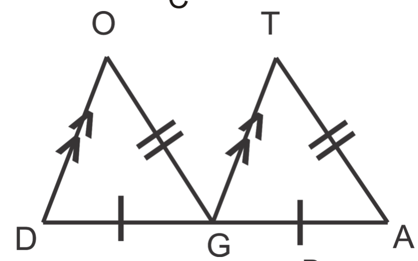
Know What? Revisited Even though we do not know all of the angle measures in the two triangles, we can find the missing angles by using the Third Angle Theorem. In your parents' kitchen, the missing angle is 39°. The missing angle in your neighbor's kitchen is 50°. From this, we can conclude that the two kitchens are now congruent, either by ASA or AAS.

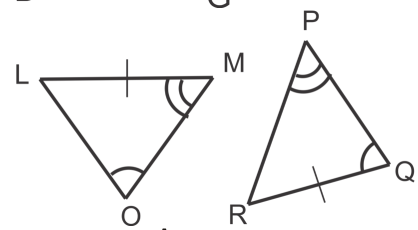
Review Questions

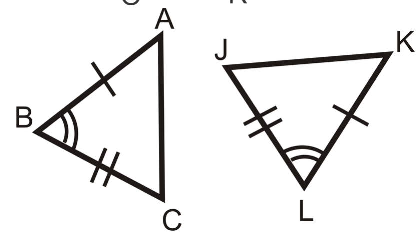
For questions 1-10, determine if the triangles are congruent. If they are, write the congruence statement and which congruence postulate or theorem you used.

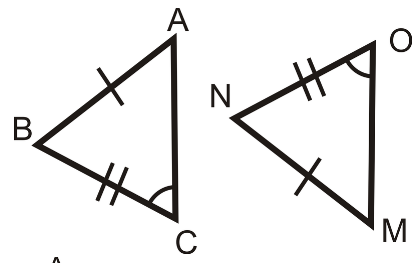
1. 

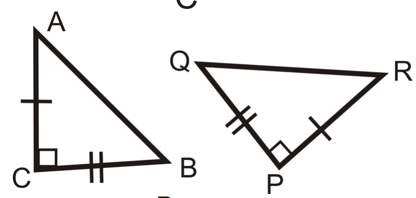
2. 

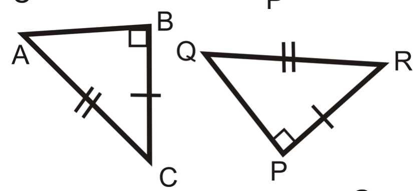
3. 

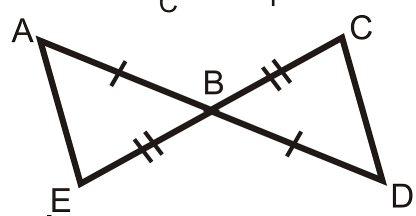
4. 

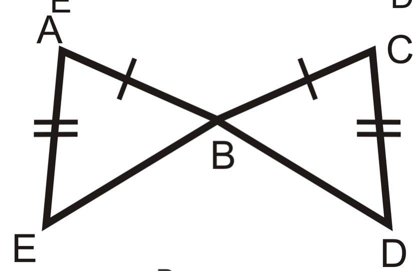
5. 

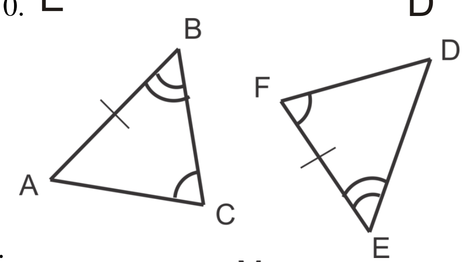
6. 

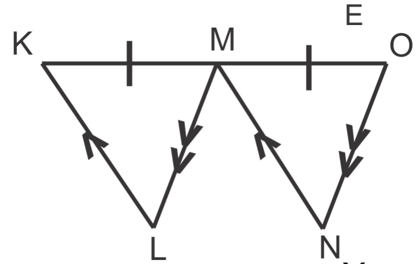
7. 

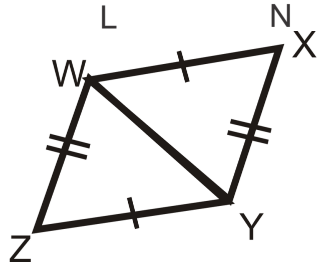
8. 

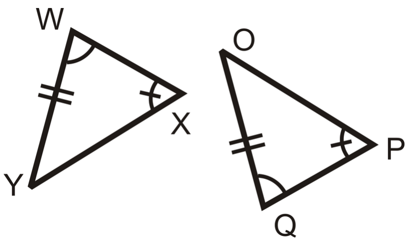
9. 

10. 

11. 

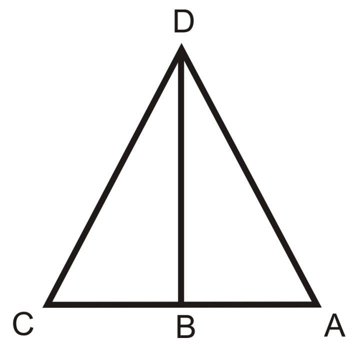
12. 

13. 



14.

For questions 11-15, use the picture to the right and the given information below.

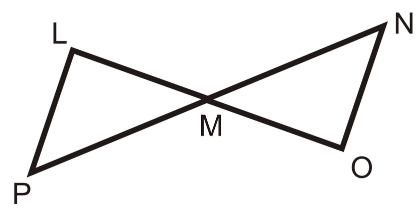


Given: $\overline{DB} \perp \overline{AC}, \overline{DB}$ is the angle bisector of $\angle CDA$

15. From $\overline{DB} \perp \overline{AC}$, which angles are congruent and why?
16. Because $\overline{DB}$ is the angle bisector of $\angle CDA$, what two angles are congruent?
17. From looking at the picture, what additional piece of information are you given? Is this enough to prove the two triangles are congruent?
18. Write a 2-column proof to prove $\triangle CDB \cong \triangle ADB$.
19. What would be your reason for $\angle C \cong \angle A$?

For questions 16-20, use the picture to the right and the given information.

Given: $\overline{LP} \parallel \overline{NO}, \overline{LP} \cong \overline{NO}$



20. From $\overline{LP} \parallel \overline{NO}$, which angles are congruent and why?
21. From looking at the picture, what additional piece of information can you conclude?
22. Write a 2-column proof to prove $\triangle LMP \cong \triangle OMN$.
23. What would be your reason for $\overline{LM} \cong \overline{MO}$?
24. Fill in the blanks for the proof below. Use the given and the picture from above. Prove: M is the midpoint of $\overline{PN}$

TABLE 4.16:

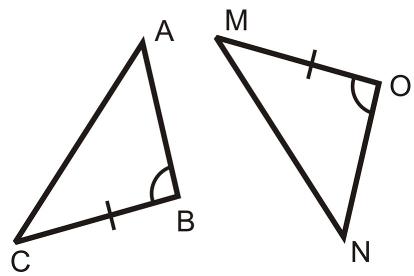
Statement	***Reason***
1. $\overline{LP} \parallel \overline{NO}, \overline{LP} \cong \overline{NO}$	Given

TABLE 4.16: (continued)

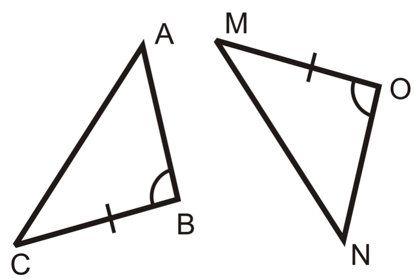
Statement	*Reason*
2.	Alternate Interior Angles
3.	ASA
4. $\overline{LM} \cong \overline{MO}$	
5. M is the midpoint of $\overline{PN}$	

Determine the additional piece of information needed to show the two triangles are congruent by the given postulate.

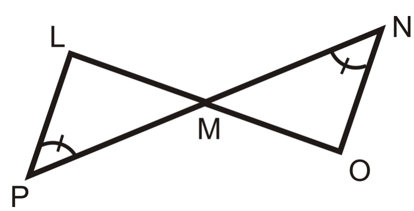
25. AAS



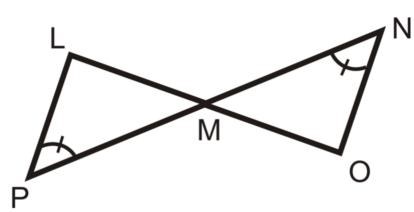
26. ASA



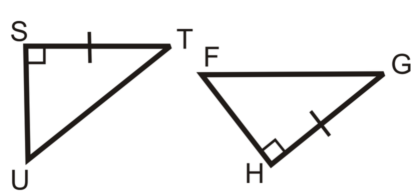
27. ASA



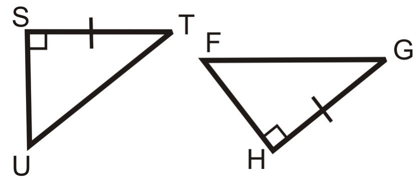
28. AAS



29. HL

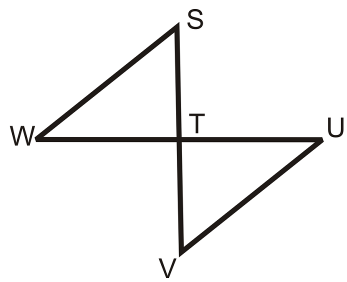


30. SAS

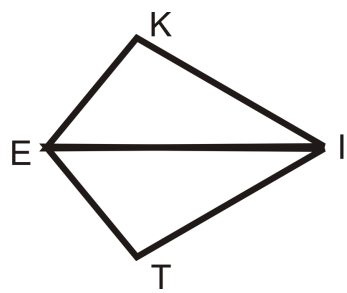


Write a 2-column proof.

31. Given: $\overline{SV} \perp \overline{WU} T$ is the midpoint of $\overline{SV}$ and $\overline{WU}$Prove: $\overline{WS} \cong \overline{UV}$



32. Given: $\angle K \cong \angle T$, $\overline{EI}$ is the angle bisector of $\angle KET$Prove: $\overline{EI}$ is the angle bisector of $\angle KIT$



Review Queue Answers

1.

TABLE 4.17:

Statement	*Reason*
1. $\overline{AD} \cong \overline{DC}$, $\overline{AB} \cong \overline{CB}$	Given
2. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
3. $\triangle DAB \cong \triangle DCB$	SSS

2. No, only the angles are congruent, you need at least one side to prove the triangles are congruent.

3. (a) Yes, CPCTC

(b) No, these sides do not line up in the congruence statement.

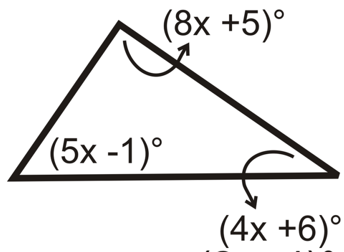
4.5 Isosceles and Equilateral Triangles

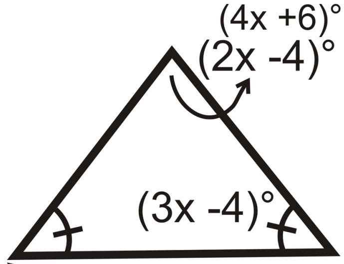
Learning Objectives

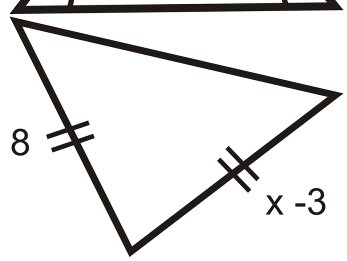
- Understand the properties of isosceles and equilateral triangles.
- Use the Base Angles Theorem and its converse.
- Prove an equilateral triangle is also equiangular.

Review Queue

Find the value of x.

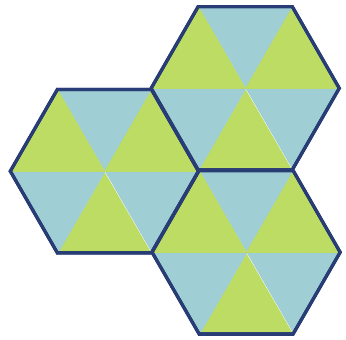
a.


b.


c.


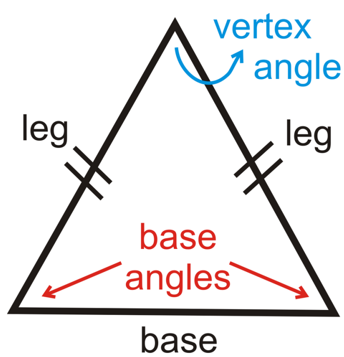
d. If a triangle is equiangular, what is the measure of each angle?

Know What? Your parents now want to redo the bathroom. To the right is the tile they would like to place in the shower. The blue and green triangles are all equilateral. What type of polygon is dark blue outlined figure? Can you determine how many degrees are in each of these figures? Can you determine how many degrees are around a point? HINT: For a "point" you can use a point where the six triangles meet.



Isosceles Triangle Properties

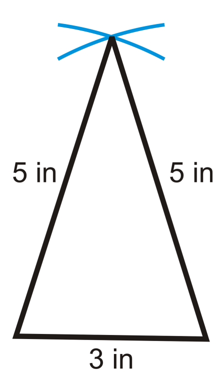
An isosceles triangle is a triangle that has ***at least*** two congruent sides. The congruent sides of the isosceles triangle are called the ***legs***. The other side is called the **base** and the angles between the base and the congruent sides are called **base angles**. The angle made by the two legs of the isosceles triangle is called the **vertex angle.**



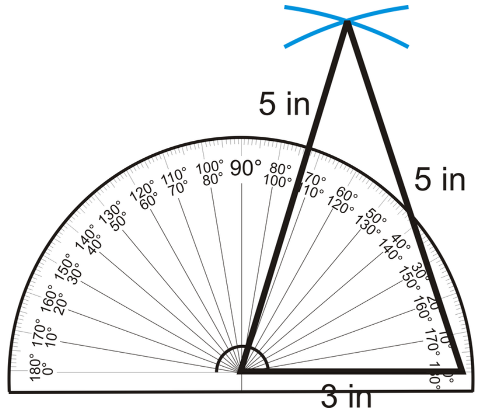
Investigation 4-5: Isosceles Triangle Construction

Tools Needed: pencil, paper, compass, ruler, protractor

a. Refer back to Investigation 4-2. Using your compass and ruler, draw an isosceles triangle with sides of 3 in, 5 in and 5 in. Draw the 3 in side (the base) horizontally 6 inches from the top of the page.



b. Now that you have an isosceles triangle, use your protractor to measure the base angles and the vertex angle.



The base angles should each be 72.5° *and the vertex angle should be* 35°.

We can generalize this investigation into the Base Angles Theorem.

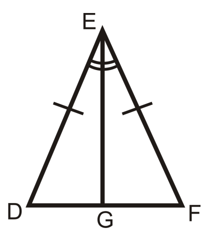
Base Angles Theorem: The base angles of an isosceles triangle are congruent.

To prove the Base Angles Theorem, we will construct the angle bisector (Investigation 1-5) through the vertex angle of an isosceles triangle.

Given: Isosceles triangle $\triangle DEF$ with $\overline{DE} \cong \overline{EF}$

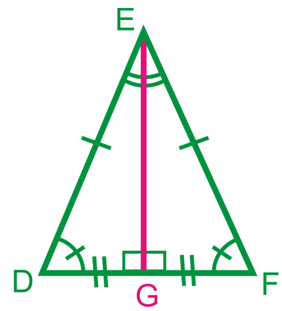
Prove: $\angle D \cong \angle F$

TABLE 4.18:

Statement	Reason
1. Isosceles triangle $\triangle DEF$ with $\overline{DE} \cong \overline{EF}$	Given
2. Construct angle bisector $\overline{EG}$ for $\angle E$ 	Every angle has one angle bisector
3. $\angle DEG \cong \angle FEG$	Definition of an angle bisector
4. $\overline{EG} \cong \overline{EG}$	Reflexive PoC
5. $\triangle DEG \cong \triangle FEG$	SAS
6. $\angle D \cong \angle F$	CPCTC

By constructing the angle bisector, $\overline{EG}$, we designed two congruent triangles and then used CPCTC to show that the base angles are congruent. Now that we have proven the Base Angles Theorem, you do not have to construct the angle bisector every time. It can now be assumed that base angles of any isosceles triangle are always equal.

Let's further analyze the picture from step 2 of our proof.



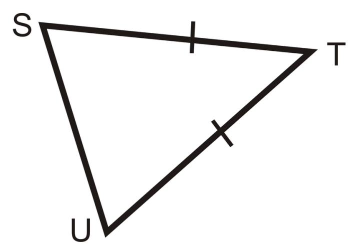
Because $\triangle DEG \cong \triangle FEG$, we know that $\angle EGD \cong \angle EGF$ by CPCTC. Thes two angles are also a linear pair, so they are congruent supplements, or 90° each. Therefore, $\overline{EG} \perp \overline{DF}$.

Additionally, $\overline{DG} \cong \overline{GF}$ by CPCTC, so G is the midpoint of $\overline{DF}$. This means that $\overline{EG}$ is the **perpendicular bisector** of $\overline{DF}$, in addition to being the angle bisector of $\angle DEF$.

Isosceles Triangle Theorem: The angle bisector of the vertex angle in an isosceles triangle is also the perpendicular bisector to the base.

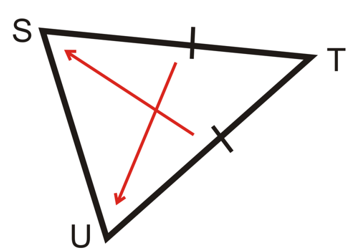
This is ONLY true for the vertex angle. We will prove this theorem in the review questions for this section.

Example 1: Which two angles are congruent?



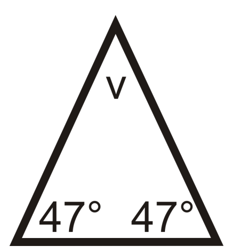
Solution: This is an isosceles triangle. The congruent angles, are opposite the congruent sides.

From the arrows we see that $\angle S \cong \angle U$.



Example 2: If an isosceles triangle has base angles with measures of 47°, what is the measure of the vertex angle?

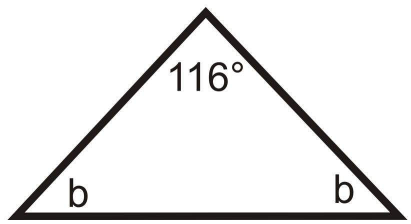
Solution: Draw a picture and set up an equation to solve for the vertex angle, v.



$$
\begin{aligned}
47^\circ + 47^\circ + v &= 180^\circ \\
v &= 180^\circ - 47^\circ - 47^\circ \\
v &= 86^\circ
\end{aligned}
$$

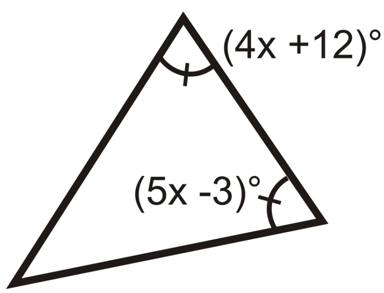
Example 3: If an isosceles triangle has a vertex angle with a measure of 116°, what is the measure of each base angle?

Solution: Draw a picture and set up and equation to solve for the base angles, b. Recall that the base angles are equal.



$$
\begin{aligned}
116^\circ + b + b &= 180^\circ \\
2b &= 64^\circ \\
b &= 32^\circ
\end{aligned}
$$

Example 4: ***Algebra Connection*** Find the value of x and the measure of each angle.



Solution: Set the angles equal to each other and solve for x.

$$
\begin{aligned}
(4x + 12)^\circ &= (5x - 3)^\circ \\
15^\circ &= x
\end{aligned}
$$

If $x = 15^\circ$, then the base angles are $4(15^\circ) + 12^\circ$, or 72°. The vertex angle is $180^\circ - 72^\circ - 72^\circ = 36^\circ$.

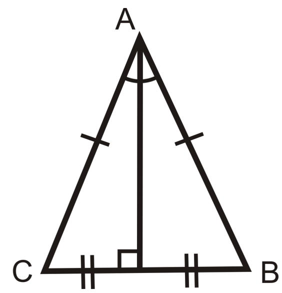
The converses of the Base Angles Theorem and the Isosceles Triangle Theorem are both true.

Base Angles Theorem Converse: If two angles in a triangle are congruent, then the opposite sides are also congruent.

So, for a triangle $\triangle ABC$, if $\angle A \cong \angle B$, then $\overline{CB} \cong \overline{CA}$. $\angle C$ would be the vertex angle.

Isosceles Triangle Theorem Converse: The perpendicular bisector of the base of an isosceles triangle is also the angle bisector of the vertex angle.

In other words, if $\triangle ABC$ is isosceles, $\overline{AD} \perp \overline{CB}$ and $\overline{CD} \cong \overline{DB}$, then $\angle CAD \cong \angle BAD$.



Equilateral Triangles

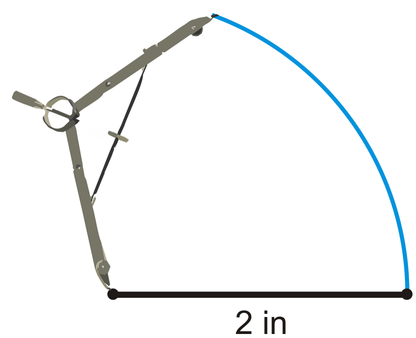
By definition, all sides in an equilateral triangle have exactly the same length. Therefore, ***every equilateral triangle is also an isosceles triangle.***

Investigation 4-6: Constructing an Equilateral Triangle

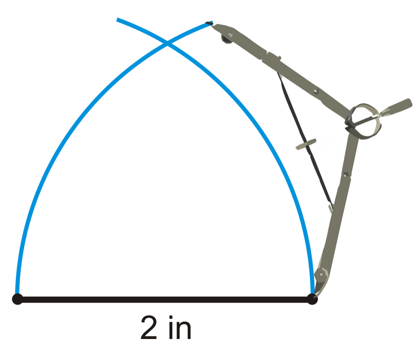
Tools Needed: pencil, paper, compass, ruler, protractor

1. Because all the sides of an equilateral triangle are equal, pick a length to be all the sides of the triangle. Measure this length and draw it horizontally on your paper.

2. Put the pointer of your compass on the left endpoint of the line you drew in Step 1. Open the compass to be the same width as this line. Make an arc above the line.

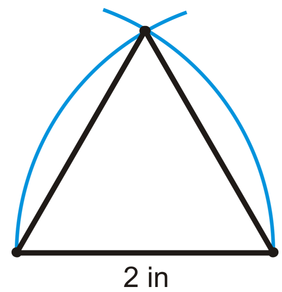


3. Repeat Step 2 on the right endpoint.



4. Connect each endpoint with the arc intersections to make the equilateral triangle.

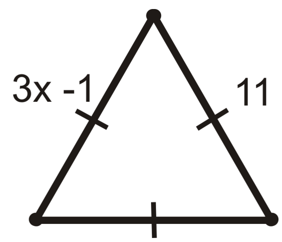
Use the protractor to measure each angle of your constructed equilateral triangle. What do you notice?



From the Base Angles Theorem, the angles opposite congruent sides in an isosceles triangle are congruent. So, if all three sides of the triangle are congruent, then all of the angles are congruent or 60° each.

Equilateral Triangles Theorem: All equilateral triangles are also equiangular. Also, all equiangular triangles are also equilateral.

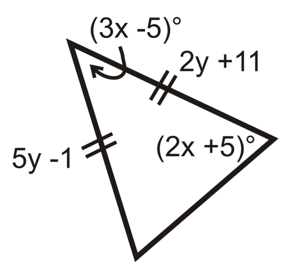
Example 5: ***Algebra Connection*** Find the value of x.



Solution: Because this is an equilateral triangle $3x - 1 = 11$. Now, we have an equation, solve for x.

$$\begin{aligned}3x - 1 &= 11\\ 3x &= 12\\ x &= 4\end{aligned}$$

Example 6: ***Algebra Connection*** Find the values of x and y.



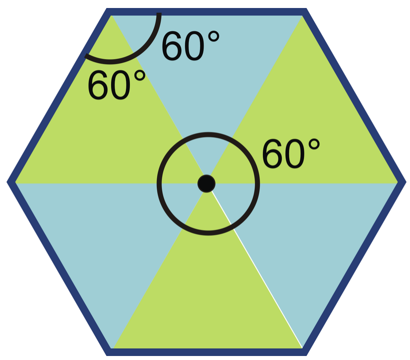
Solution: Let's start with y. Both sides are equal, so set the two expressions equal to each other and solve for y.

$$\begin{aligned}5y - 1 &= 2y + 11\\ 3y &= 12\\ y &= 4\end{aligned}$$

For x, we need to use two $(2x+5)^\circ$ expressions because this is an isosceles triangle and that is the base angle measurement. Set all the angles equal to 180° and solve.

$$\begin{aligned}(2x+5)^\circ+(2x+5)^\circ+(3x-5)^\circ &= 180^\circ\\ (7x+5)^\circ &= 180^\circ\\ 7x &= 175^\circ\\ x &= 25^\circ\end{aligned}$$

Know What? Revisited Let's focus on one tile. First, these triangles are all equilateral, so this is an equilateral hexagon (6 sided polygon). Second, we now know that every equilateral triangle is also equiangular, so every triangle within this tile has 360° angles. This makes our equilateral hexagon also equiangular, with each angle measuring 120°. Because there are 6 angles, the sum of the angles in a hexagon are 6.120° or 720°. Finally, the point in the center of this tile, has 660° angles around it. That means there are 360° around a point.



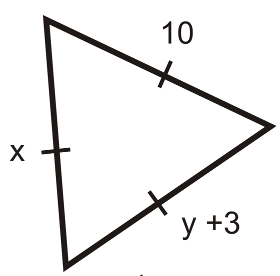
Review Questions

Constructions For questions 1-5, use your compass and ruler to:

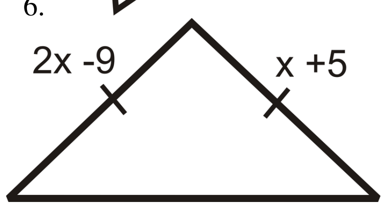
1. Draw an isosceles triangle with sides 3.5 in, 3.5 in, and 6 in.
2. Draw an isosceles triangle that has a vertex angle of 100° and legs with length of 4 cm. (you will also need your protractor for this one)
3. Draw an equilateral triangle with sides of length 7 cm.
4. Using what you know about constructing an equilateral triangle, construct (without your protractor) a 60° angle.
5. Draw an isosceles right triangle. What is the measure of the base angles?

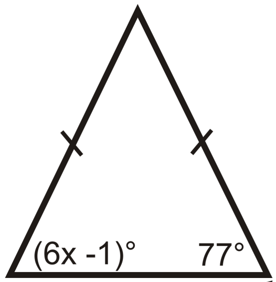
For questions 6-17, find the measure of x and/or y.

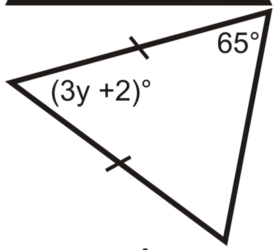
6.

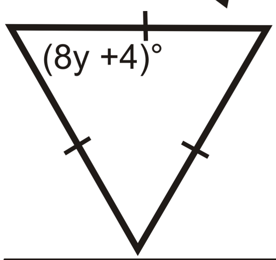


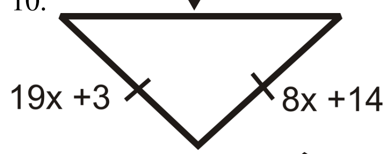
7.

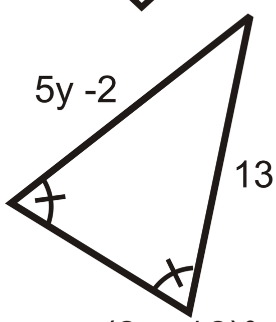


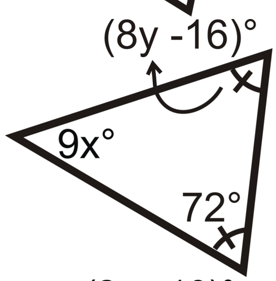
8. 

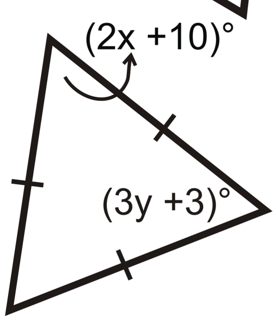
9. 

10. 

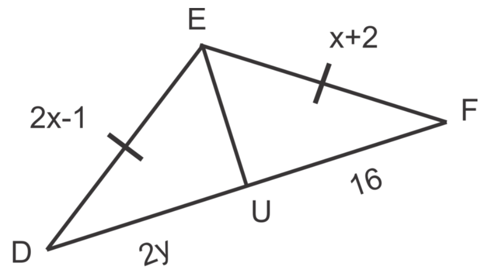
11. 

12. 

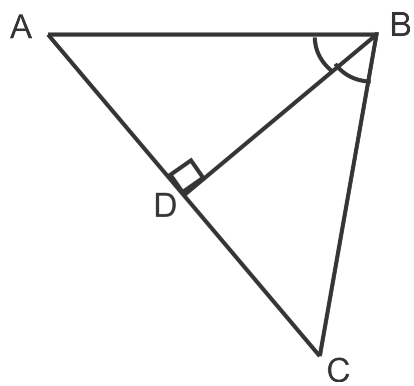
13. 

14. 

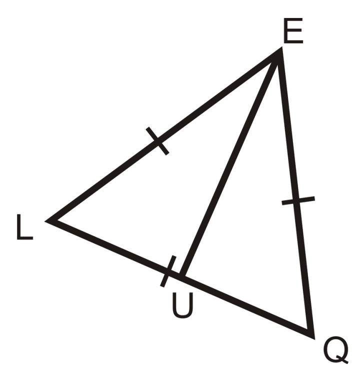
15. $\angle DEF$ in triangle $\triangle DEF$ is bisected by $\overline{EU}$. Find x and y.



16. Is $\triangle ABC$ isosceles? Explain your reasoning.



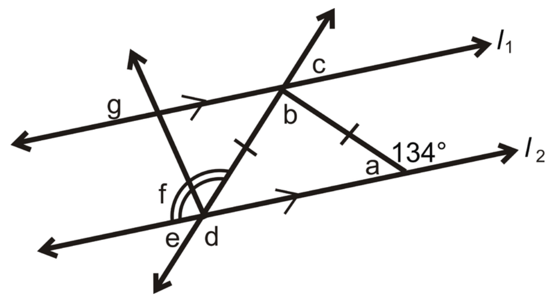
17. $\triangle EQG$ is an equilateral triangle. If $\overline{EU}$ bisects $\angle LEQ$, find:



a. $m\angle EUL$
b. $m\angle UEL$
c. $m\angle ELQ$
d. If $EQ = 4$, find LU.

Determine if the following statements are ALWAYS, SOMETIMES, or NEVER true. Explain your reasoning.

18. Base angles of an isosceles triangle are congruent.
19. Base angles of an isosceles triangle are complementary.
20. Base angles of an isosceles triangle can be equal to the vertex angle.
21. Base angles of an isosceles triangle can be right angles.
22. Base angles of an isosceles triangle are acute.
23. In the diagram below, $l_1 \parallel l_2$. Find all of the lettered angles.



Fill in the blanks in the proofs below.

24. Given: Isosceles $\triangle CIS$, with base angles $\angle C$ and $\angle S\overline{IO}$ is the angle bisector of $\angle CIS$Prove: $\overline{IO}$ is the perpendicular bisector of $\overline{CS}$

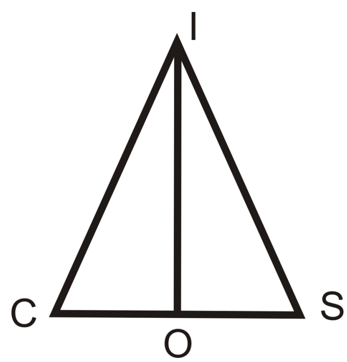
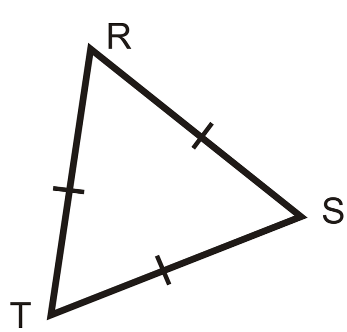


TABLE 4.19:

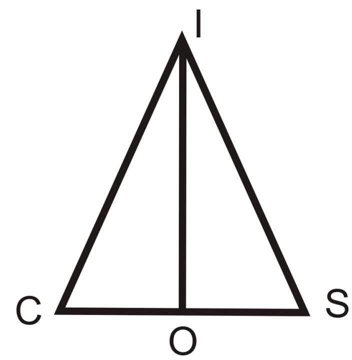
Statement	***Reason***
1.	Given
2.	Base Angles Theorem
3. $\angle CIO \cong \angle SIO$	
4.	Reflexive PoC
5. $\triangle CIO \cong \triangle SIO$	
6. $\overline{CO} \cong \overline{OS}$	
7.	CPCTC
8. $\angle IOC$ and $\angle IOS$ are supplementary	
9.	Congruent Supplements Theorem
10. $\overline{IO}$ is the perpendicular bisector of $\overline{CS}$	

Write a 2-column proof.

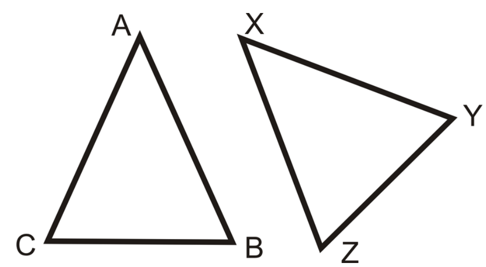
25. Given: Equilateral $\triangle RST$ with $\overline{RT} \cong \overline{ST} \cong \overline{RS}$Prove: $\triangle RST$ is equiangular



26. Given: Isosceles $\triangle ICS$ with $\angle C$ and $\angle S\overline{IO}$ is the perpendicular bisector of $\overline{CS}$Prove: $\overline{IO}$ is the angle bisector of $\angle CIS$



27. Given: Isosceles $\triangle ABC$ with base angles $\angle B$ and $\angle C$ Isosceles $\triangle XYZ$ with base angles $\angle Y$ and $\angle Z\angle C \cong \angle Z, \overline{BC} \cong \overline{YZ}$Prove: $\triangle ABC \cong \triangle XYZ$



Constructions

28. Using the construction of an equilateral triangle (investigation 4-6), construct a 30° angle. *Hint: recall how to bisect an angle from investigation 1-4.*
29. Use the construction of a 60° angle to construct a 120° angle.
30. Is there another way to construction a 120° angle? Describe the method.
31. Describe how you could construct a 45° angle (there is more than one possible way).

Review Queue Answers

a. $$\begin{aligned}(5x-1)^\circ+(8x+5)^\circ+(4x+6)^\circ &= 180^\circ\\ 17x+10 &= 180^\circ\\ 17x &= 170^\circ\\ x &= 10^\circ\end{aligned}$$

b. $$\begin{aligned}(2x-4)^\circ+(3x-4)^\circ+(3x-4)^\circ &= 180^\circ\\ 8x-12 &= 180^\circ\\ 8x &= 192^\circ\\ x &= 24^\circ\end{aligned}$$

c. $$\begin{aligned}x-3 &= 8\\ x &= 5\end{aligned}$$

d. Each angle is $\frac{180^\circ}{3}$, or 60°

4.6 Chapter 4 Review

Definitions, Postulates, and Theorems

Interior Angles
The angles inside of a closed figure with straight sides.

Vertex
The point where the sides of a polygon meet.

Triangle Sum Theorem
The interior angles of a triangle add up to 180°.

Exterior Angle
The angle formed by one side of a polygon and the extension of the adjacent side.

Exterior Angle Sum Theorem
Each set of exterior angles of a polygon add up to 360°.

Remote Interior Angles
The two angles in a triangle that are not adjacent to the indicated exterior angle.

Exterior Angle Theorem
The sum of the remote interior angles is equal to the non-adjacent exterior angle.

Congruent Triangles
Two triangles are congruent if the three corresponding angles and sides are congruent.

Third Angle Theorem
If two angles in one triangle are congruent to two angles in another triangle, then the third pair of angles must also congruent.

Reflexive Property of Congruence
Any shape is congruent to itself.

Symmetric Property of Congruence
If two shapes are congruent, the statement can be written with either shape on either side of the $\cong$ sign.

Transitive Property of Congruence
If two shapes are congruent and one of those is congruent to a third, the first and third shapes are also congruent.

Side-Side-Side (SSS) Triangle Congruence Postulate
If three sides in one triangle are congruent to three sides in another triangle, then the triangles are congruent.

Included Angle
When an angle is between two given sides of a triangle (or polygon).

Side-Angle-Side (SAS) Triangle Congruence Postulate
If two sides and the included angle in one triangle are congruent to two sides and the included angle in another triangle, then the two triangles are congruent.

Angle-Side-Angle (ASA) Congruence Postulate
If two angles and the included side in one triangle are congruent to two angles and the included side in another triangle, then the two triangles are congruent.

Angle-Angle-Side (AAS or SAA) Congruence Theorem
If two angles and a non-included side in one triangle are congruent to two corresponding angles and a non-included side in another triangle, then the triangles are congruent.

HL Congruence Theorem
If the hypotenuse and leg in one right triangle are congruent to the hypotenuse and leg in another right triangle, then the two triangles are congruent.

Base Angles Theorem
The base angles of an isosceles triangle are congruent.

Isosceles Triangle Theorem
The angle bisector of the vertex angle in an isosceles triangle is also the perpendicular bisector to the base.

Base Angles Theorem Converse
If two angles in a triangle are congruent, then the opposite sides are also congruent.

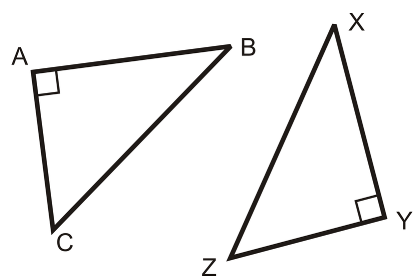
Isosceles Triangle Theorem Converse
The perpendicular bisector of the base of an isosceles triangle is also the angle bisector of the vertex angle.

Equilateral Triangles Theorem
all sides in an equilateral triangle have exactly the same length.

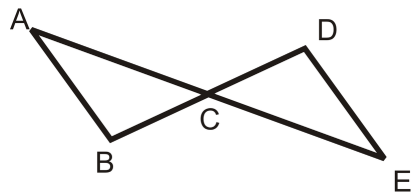
Review

For each pair of triangles, write what needs to be congruent in order for the triangles to be congruent. Then, write the congruence statement for the triangles.

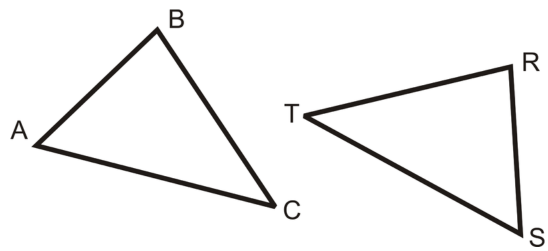
1. HL



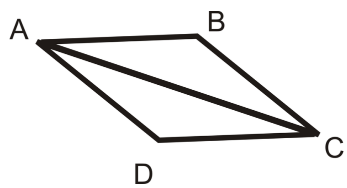
2. ASA



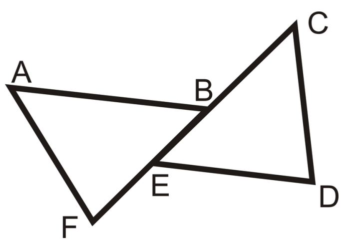
3. AAS



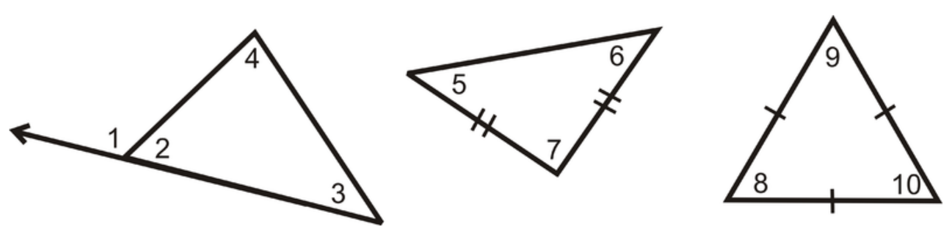
4. SSS



5. SAS



Using the pictures below, determine which theorem, postulate or definition that supports each statement below.



6. $m\angle 1 + m\angle 2 = 180^\circ$
7. $\angle 5 \cong \angle 6$
8. $m\angle 1 = m\angle 4 + m\angle 3$
9. $m\angle 8 = 60^\circ$
10. $m\angle 5 + m\angle 6 + m\angle 7 = 180^\circ$
11. $\angle 8 \cong \angle 9 \cong \angle 10$
12. If $m\angle 7 = 90^\circ$, then $m\angle 5 = m\angle 6 = 45^\circ$

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9689 .

CHAPTER 5 Relationships with Triangles

Chapter Outline

This chapter introduces different segments within triangles and how they relate to each other. We will explore the properties of midsegments, perpendicular bisectors, angle bisectors, medians, and altitudes. Next, we will look at the relationship of the sides of a triangle, how they relate to each other and how the sides of one triangle can compare to another.

5.1 Midsegments of a Triangle

Learning Objectives

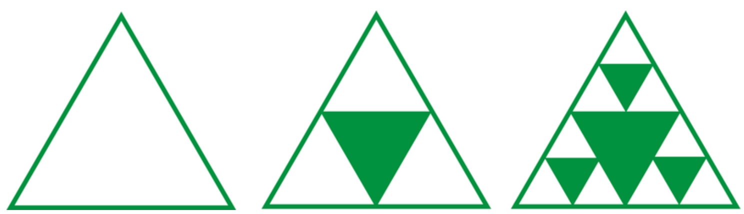
- Identify the midsegments of a triangle.
- Use the Midsegment Theorem to solve problems involving side lengths, midsegments, and algebra.

Review Queue

Find the midpoint between the given points.

a. (-4, 1) and (6, 7)
b. (5, -3) and (11, 5)
c. (0, -2) and (-4, 14)
d. Find the equation of the line between (-2, -3) and (-1, 1).
e. Find the equation of the line that is parallel to the line from #4 through (2, -7).

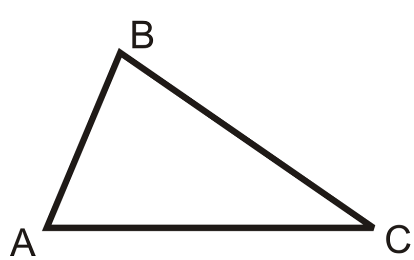
Know What? A fractal is a repeated design using the same shape (or shapes) of different sizes. Below, is an example of the first few steps of a fractal. What does the next figure look like? How many triangles are in each figure (green and white triangles)? Is there a pattern?



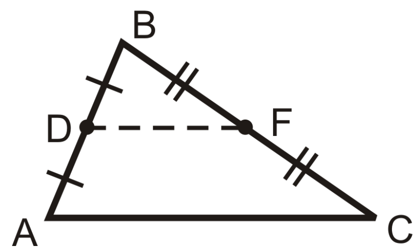
Defining Midsegment

Midsegment: A line segment that connects two midpoints of adjacent sides of a triangle.

Example 1: Draw the midsegment $\overline{DF}$ between $\overline{AB}$ and $\overline{BC}$. Use appropriate tic marks.

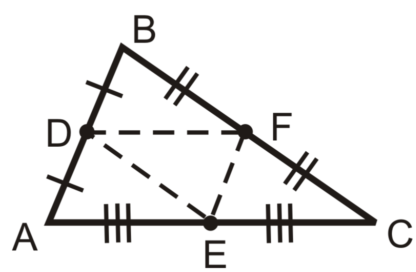


Solution: Find the midpoints of $\overline{AB}$ and $\overline{BC}$ using your ruler. Label these points D and F. Connect them to create the midsegment.



Don't forget to put the tic marks, indicating that D and F are midpoints, $\overline{AD} \cong \overline{DB}$ and $\overline{BF} \cong \overline{FC}$.

Example 2: Find the midpoint of $\overline{AC}$ from $\triangle ABC$. Label it E and find the other two midsegments of the triangle.



Solution:

For every triangle there are three midsegments.

Let's transfer what we know about midpoints in the coordinate plane to midsegments in the coordinate plane. We will need to use the midpoint formula, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 3: The vertices of $\triangle LMN$ are $L(4,5), M(-2,-7)$ and $N(-8,3)$. Find the midpoints of all three sides, label them O, P and Q. Then, graph the triangle, it's midpoints and draw in the midsegments.

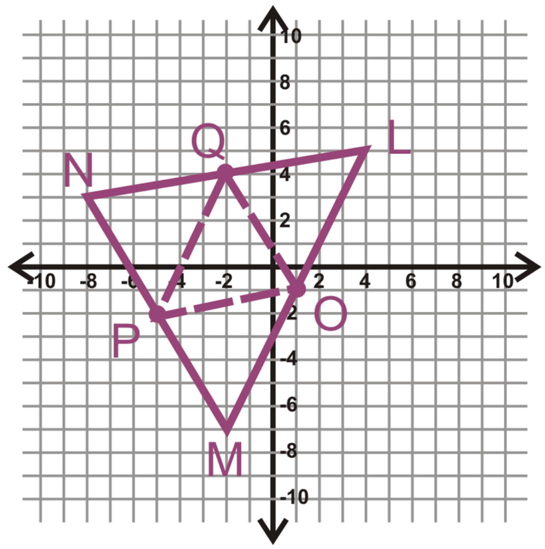
Solution: Use the midpoint formula 3 times to find all the midpoints.

L and $M = \left(\frac{4+(-2)}{2}, \frac{5+(-7)}{2}\right) = (1,-1)$, point O

L and $N = \left(\frac{4+(-8)}{2}, \frac{5+3}{2}\right) = (-2,4)$, point Q

M and $N = \left(\frac{-2+(-8)}{2}, \frac{-7+3}{2}\right) = (-5,-2)$, point P

The graph would look like the graph to the right. We will use this graph to explore the properties of midsegments.



Example 4: Find the slopes of $\overline{NM}$ and $\overline{QO}$.

Solution: The slope of $\overline{NM}$ is $\frac{-7-3}{-2-(-8)} = \frac{-10}{6} = -\frac{5}{3}$.

The slope of $\overline{QO}$ is $\frac{-1-4}{1-(-2)} = -\frac{5}{3}$.

From this we can conclude that $\overline{NM} \parallel \overline{QO}$. If we were to find the slopes of the other sides and midsegments, we would find $\overline{LM} \parallel \overline{QP}$ and $\overline{NL} \parallel \overline{PO}$. ***This is a property of all midsegments.***

Example 5: Find NM and QO.

Solution: Now, we need to find the lengths of $\overline{NM}$ and $\overline{QO}$. Use the distance formula.

$$NM = \sqrt{(-7-3)^2 + (-2-(-8))^2} = \sqrt{(-10)^2 + 6^2} = \sqrt{100+36} = \sqrt{136} \approx 11.66$$
$$QO = \sqrt{(1-(-2))^2 + (-1-4)^2} = \sqrt{3^2 + (-5)^2} = \sqrt{9+25} = \sqrt{34} \approx 5.83$$

From this we can conclude that QO is **half** of NM. If we were to find the lengths of the other sides and midsegments, we would find that OP is **half** of NL and QP is **half** of LM. ***This is a property of all midsegments.***

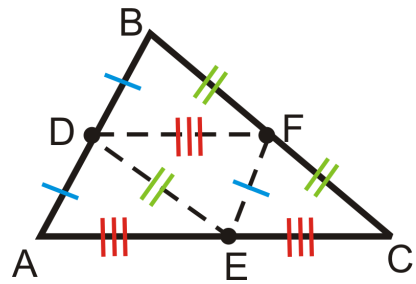
The Midsegment Theorem

The conclusions drawn in Examples 4 and 5 can be generalized into the Midsegment Theorem.

Midsegment Theorem: The midsegment of a triangle is half the length of the side it is parallel to.

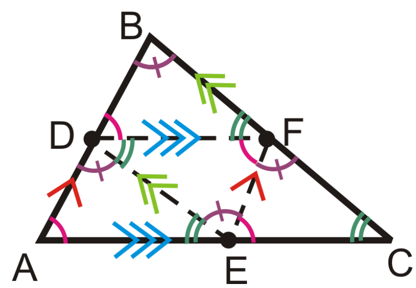
Example 6: Mark everything you have learned from the Midsegment Theorem on $\triangle ABC$ above.

Solution: Let's draw two different triangles, one for the congruent sides, and one for the parallel lines.



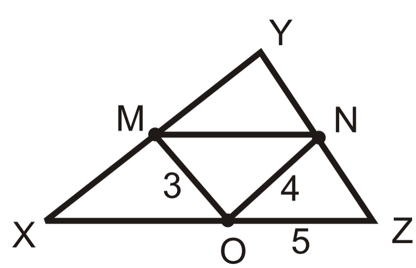
Because the midsegments are half the length of the sides they are parallel to, they are congruent to half of each of those sides (as marked). Also, this means that all four of the triangles in $\triangle ABC$, created by the midsegments are congruent by SSS.

As for the parallel midsegments and sides, several congruent angles are formed. In the picture to the right, the pink and teal angles are congruent because they are corresponding or alternate interior angles. Then, the purple angles are congruent by the 3^{rd} Angle Theorem.



To play with the properties of midsegments, go to http://www.mathopenref.com/trianglemidsegment.html .

Example 7: M, N, and O are the midpoints of the sides of the triangle.



Find

a) MN

b) XY

c) The perimeter of $\triangle XYZ$

Solution: Use the Midsegment Theorem.

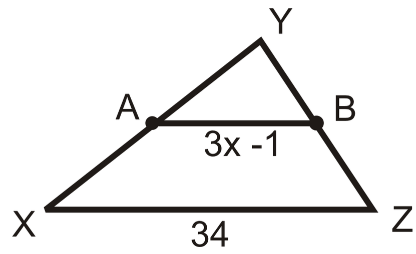
a) $MN = OZ = 5$

b) $XY = 2(ON) = 2 \cdot 4 = 8$

c) The perimeter is the sum of the three sides of $\triangle XYZ$.

$$XY + YZ + XZ = 2 \cdot 4 + 2 \cdot 3 + 2 \cdot 5 = 8 + 6 + 10 = 24$$

Example 8: ***Algebra Connection*** Find the value of x and AB.



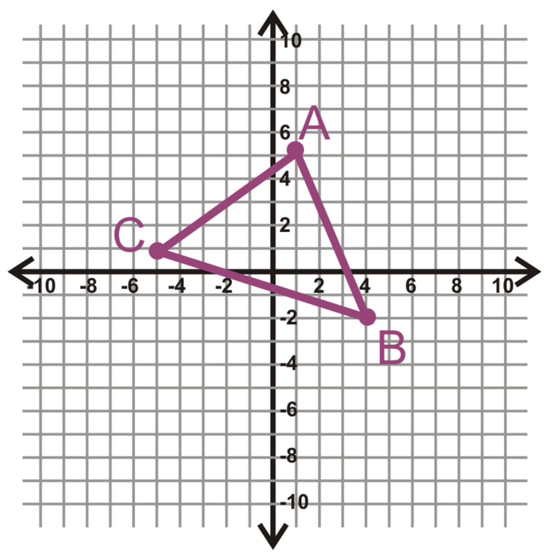
Solution: First, AB is half of 34, or 17. To find x, set $3x-1$ equal to 17.

$$\begin{aligned}3x-1&=17\\3x&=18\\x&=6\end{aligned}$$

Let's go back to the coordinate plane.

Example 9: If the midpoints of the sides of a triangle are $A(1,5), B(4,-2)$, and $C(-5,1)$, find the vertices of the triangle.

Solution: The easiest way to solve this problem is to graph the midpoints and then apply what we know from the Midpoint Theorem.



Now that the points are plotted, find the slopes between all three.

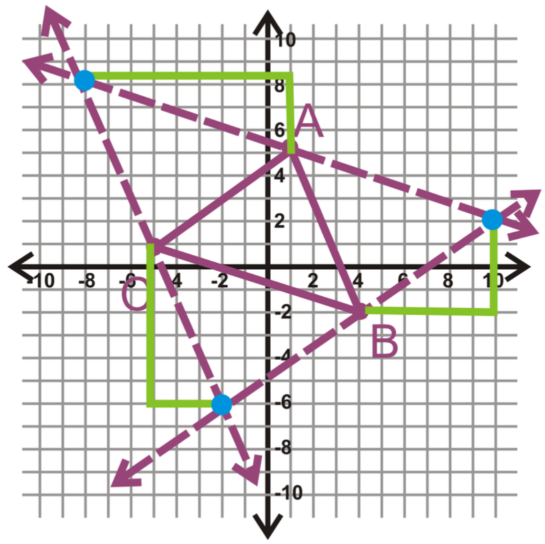
slope $AB = \frac{5+2}{1-4} = -\frac{7}{3}$

slope $BC = \frac{-2-1}{4+5} = \frac{-3}{9} = -\frac{1}{3}$

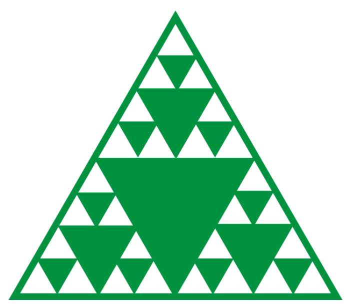
slope $AC = \frac{5-1}{1+5} = \frac{4}{6} = \frac{2}{3}$

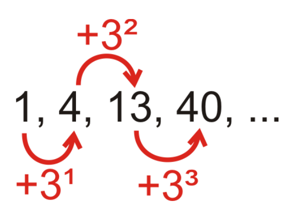
Using the slope between two of the points and the third, plot the slope triangle on either side of the third point and extend the line. Repeat this process for all three midpoints. For example, use the slope of AB with point C.

The green lines in the graph to the left represent the slope triangles of each midsegment. The three dotted lines represent the sides of the triangle. Where they intersect are the vertices of the triangle (the blue points), which are (-8, 8), (10, 2) and (-2, 6).



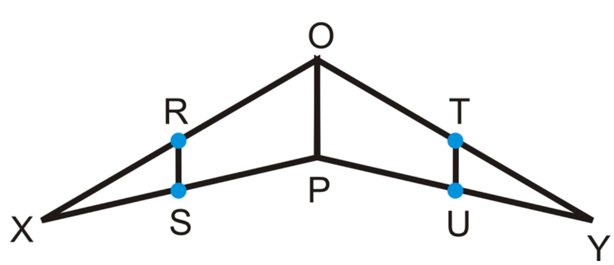
Know What? Revisited To the left is a picture of the 4^{th} figure in the fractal pattern. The number of triangles in each figure is 1, 4, 13, and 40. The pattern is that each term increase by the next power of 3.





Review Questions

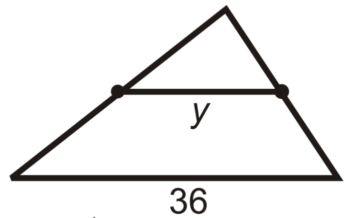
R, S, T, and U are midpoints of the sides of $\triangle XPO$ and $\triangle YPO$.

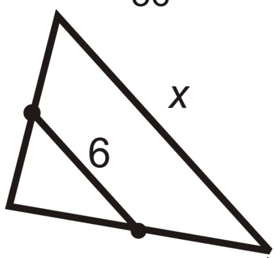


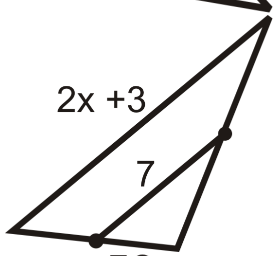
1. If $OP = 12$, find RS and TU.
2. If $RS = 8$, find TU.

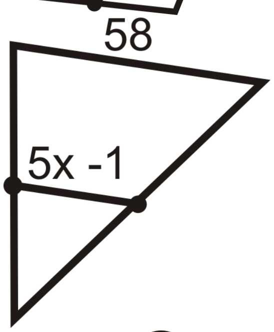
3. If $RS = 2x$, and $OP = 20$, find x and TU.
4. If $OP = 4x$ and $RS = 6x - 8$, find x.
5. Is $\triangle XOP \cong \triangle YOP$? Why or why not?

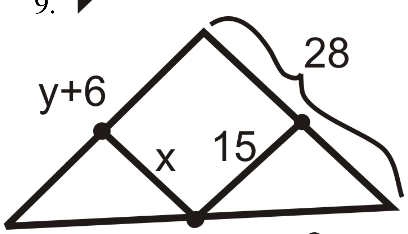
For questions 6-13, find the indicated variable(s). You may assume that all line segments within a triangle are midsegments.

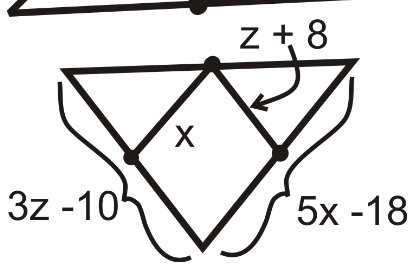
6. 

7. 

8. 

9. 

10. 

11. 

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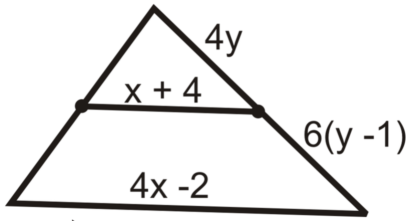
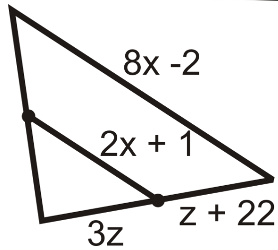


13.


14. The sides of $\triangle XYZ$ are 26, 38, and 42. $\triangle ABC$ is formed by joining the midpoints of $\triangle XYZ$.
    a. Find the perimeter of $\triangle ABC$.
    b. Find the perimeter of $\triangle XYZ$.
    c. What is the relationship between the perimeter of a triangle and the perimeter of the triangle formed by connecting its midpoints?

Coordinate Geometry Given the vertices of $\triangle ABC$ below, find the midpoints of each side.

15. $A(5,-2), B(9,4)$ and $C(-3,8)$
16. $A(-10,1), B(4,11)$ and $C(0,-7)$
17. $A(0,5), B(4,-1)$ and $C(-2,-3)$
18. $A(2,4), B(8,-4)$ and $C(2,-4)$

Multi-Step Problem The midpoints of the sides of $\triangle RST$ are $G(0,-2), H(9,1)$, and $I(6,-5)$. Answer the following questions.

19. Find the slope of GH, HI, and GI.
20. Plot the three midpoints and connect them to form midsegment triangle, $\triangle GHI$.
21. Using the slopes, find the coordinates of the vertices of $\triangle RST$.
22. Find GH using the distance formula. Then, find the length of the sides it is parallel to. What should happen?

More Coordinate Geometry Given the midpoints of the sides of a triangle, find the vertices of the triangle. Refer back to problems 19-21 for help.

23. (-2, 1), (0, -1) and (-2, -3)
24. (1, 4), (4, 1) and (2, 1)

Given the vertices of $\triangle ABC$, find:

a) the midpoints of M, N and O where M is the midpoint of $\overline{AB}$, N is the midpoint of $\overline{BC}$ and C is the midpoint of $\overline{AC}$.

b) Show that midsegments $\overline{MN}, \overline{NO}$ and $\overline{OM}$ are parallel to sides $\overline{AC}, \overline{AB}$ and $\overline{BC}$ respectively.

c) Show that midsegments $\overline{MN}, \overline{NO}$ and $\overline{OM}$ are half the length of sides $\overline{AC}, \overline{AB}$ and $\overline{BC}$ respectively.

25. $A(-3,5), B(3,1)$ and $C(-5,-5)$
26. $A(-2,2), B(4,4)$ and $C(6,0)$

For questions 27-30, $\triangle CAT$ has vertices $C(x_1,y_1), A(x_2,y_2)$ and $T(x_3,y_3)$.

27. Find the midpoints of sides $\overline{CA}$ and $\overline{CT}$. Label them L and M respectively.
28. Find the slopes of $\overline{LM}$ and $\overline{AT}$.
29. Find the lengths of $\overline{LM}$ and $\overline{AT}$.
30. What have you just proven algebraically?

Review Queue Answers

a. $\left(\frac{-4+6}{2}, \frac{1+7}{2}\right) = (1,4)$
b. $\left(\frac{5+11}{2}, \frac{-3+5}{2}\right) = (8,1)$
c. $\left(\frac{0-4}{2}, \frac{-2+14}{2}\right) = (-2,6)$
d. $m = \frac{-3-1}{-2-(-1)} = \frac{-4}{-1} = 4$
$y = mx + b$
$-3 = 4(-2) + b$
$b = 5,\ y = 4x + 5$
e. $-7 = 4(2) + b$
$b = -15,\ y = 4x - 15$

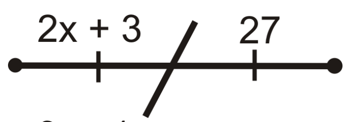
5.2 Perpendicular Bisectors in Triangles

Learning Objectives

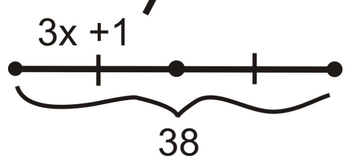
- Understand points of concurrency.
- Apply the Perpendicular Bisector Theorem and its converse to triangles.
- Understand concurrency for perpendicular bisectors.

Review Queue

a. Construct the perpendicular bisector of a 3 inch line. Use Investigation 1-3 from Chapter 1 to help you.
b. Find the value of x.

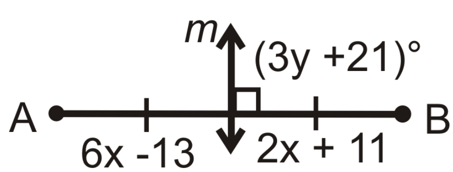


a.



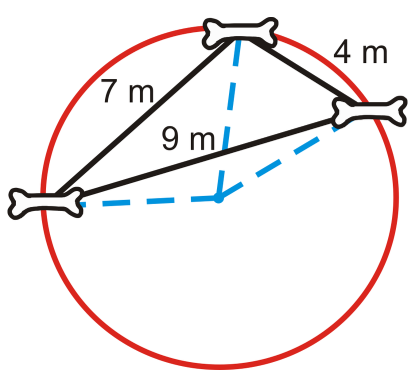
b.

c. Find the value of x and y. Is m the perpendicular bisector of AB? How do you know?



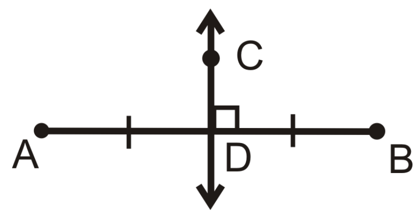
Know What? An archeologist has found three bones in Cairo, Egypt. The bones are 4 meters apart, 7 meters apart and 9 meters apart (to form a triangle). The likelihood that more bones are in this area is very high. The archeologist wants to dig in an appropriate circle around these bones. If these bones are on the edge of the digging circle, where is the center of the circle?

Can you determine how far apart each bone is from the center of the circle? What is this length?



Perpendicular Bisectors

In Chapter 1, you learned that a perpendicular bisector intersects a line segment at its midpoint and is perpendicular. In #1 in the Review Queue above, you constructed a perpendicular bisector of a 3 inch segment. Let's analyze this figure.



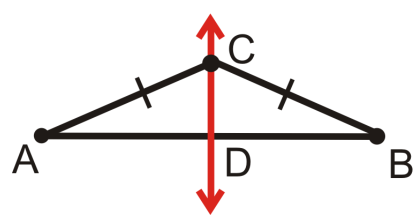
$\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$. If we were to draw in $\overline{AC}$ and $\overline{CB}$, we would find that they are equal. Therefore, any point on the perpendicular bisector of a segment is the same distance from each endpoint.

Perpendicular Bisector Theorem: If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

The proof of the Perpendicular Bisector Theorem is in the exercises for this section. In addition to the Perpendicular Bisector Theorem, we also know that its converse is true.

Perpendicular Bisector Theorem Converse: If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Proof of the Perpendicular Bisector Theorem Converse



Given: $\overline{AC} \cong \overline{CB}$

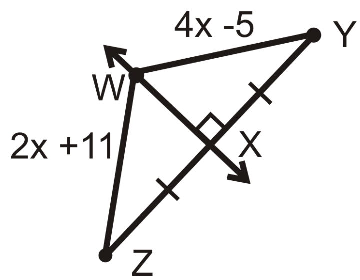
Prove: $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$

TABLE 5.1:

Statement	***Reason***
1. $\overline{AC} \cong \overline{CB}$	Given
2. $\triangle ACB$ is an isosceles triangle	Definition of an isosceles triangle
3. $\angle A \cong \angle B$	Isosceles Triangle Theorem
4. Draw point D, such that D is the midpoint of $\overline{AB}$.	Every line segment has exactly one midpoint
5. $\overline{AD} \cong \overline{DB}$	Definition of a midpoint
6. $\triangle ACD \cong \triangle BCD$	SAS
7. $\angle CDA \cong \angle CDB$	CPCTC
8. $m\angle CDA = m\angle CDB = 90^\circ$	Congruent Supplements Theorem
9. $\overleftrightarrow{CD} \perp \overline{AB}$	Definition of perpendicular lines
10. $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$	Definition of perpendicular bisector

Let's use the Perpendicular Bisector Theorem and its converse in a few examples.

Example 1: ***Algebra Connection*** Find x and the length of each segment.

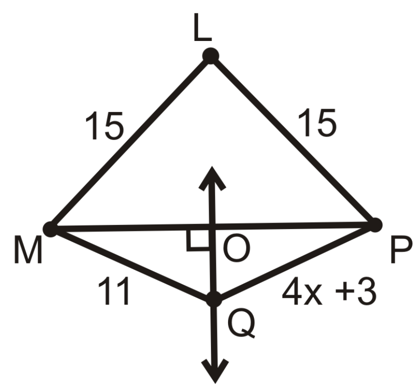


Solution: From the markings, we know that $\overleftrightarrow{WX}$ is the perpendicular bisector of $\overline{XY}$. Therefore, we can use the Perpendicular Bisector Theorem to conclude that $WZ = WY$. Write an equation.

$$\begin{aligned} 2x+11 &= 4x-5 \\ 16 &= 2x \\ 8 &= x \end{aligned}$$

To find the length of WZ and WY, substitute 8 into either expression, $2(8)+11 = 16+11 = 27$.

Example 2: $\overleftrightarrow{OQ}$ is the perpendicular bisector of $\overline{MP}$.



a) Which segments are equal?

b) Find x.

c) Is L on $\overleftrightarrow{OQ}$? How do you know?

Solution:

a) $ML = LP$ because they are both 15.

$MO = OP$ because O is the midpoint of $\overline{MP}$

$MQ = QP$ because Q is on the perpendicular bisector of $\overline{MP}$.

b) $4x+3 = 11$
$4x = 8$
$x = 2$

c) Yes, L is on $\overleftrightarrow{OQ}$ because $ML = LP$ (Perpendicular Bisector Theorem Converse).

Perpendicular Bisectors and Triangles

Two lines intersect at a point. If more than two lines intersect at the same point, it is called a point of concurrency.

Point of Concurrency: When three or more lines intersect at the same point.

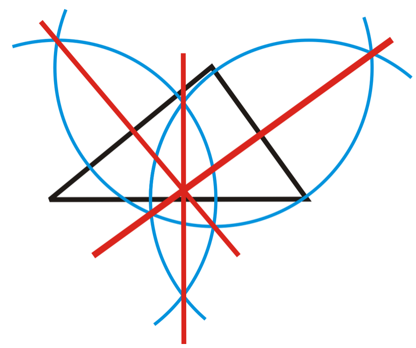
Investigation 5-1: Constructing the Perpendicular Bisectors of the Sides of a Triangle

Tools Needed: paper, pencil, compass, ruler

1. Draw a scalene triangle.

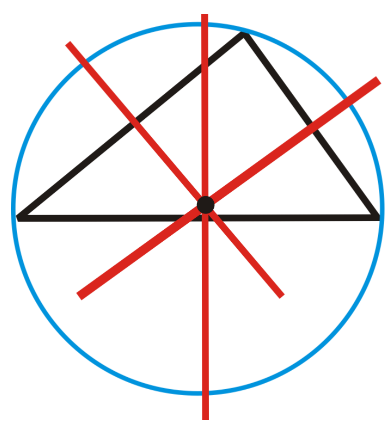
2. Construct the perpendicular bisector (Investigation 1-3) for all three sides.

The three perpendicular bisectors all intersect at the same point, called the circumcenter.

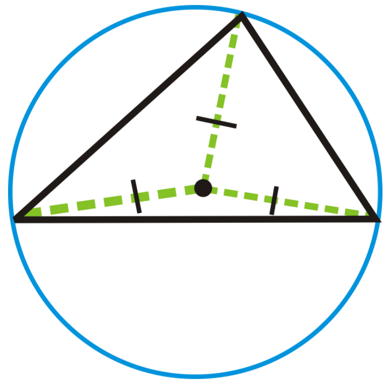


Circumcenter: The point of concurrency for the perpendicular bisectors of the sides of a triangle.

3. Erase the arc marks to leave only the perpendicular bisectors. Put the pointer of your compass on the circumcenter. Open the compass so that the pencil is on one of the vertices. Draw a circle. What happens?

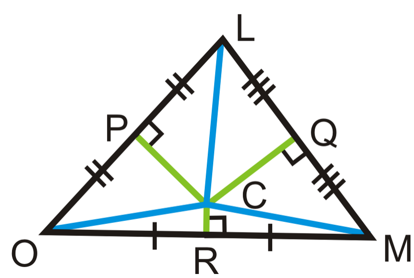


The circumcenter is the center of a circle that passes through all the vertices of the triangle. We say that this circle ***circumscribes*** the triangle. This means that ***the circumcenter is equidistant to the vertices.***



Concurrency of Perpendicular Bisectors Theorem: The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the vertices.

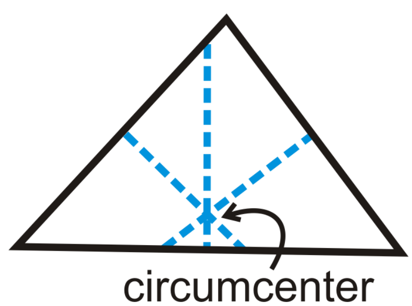
If $\overline{PC}, \overline{QC}$, and $\overline{RC}$ are perpendicular bisectors, then $LC = MC = OC$.



Example 3: For further exploration, try the following:

a. Cut out an acute triangle from a sheet of paper.
b. Fold the triangle over one side so that the side is folded in half. Crease.
c. Repeat for the other two sides. What do you notice?

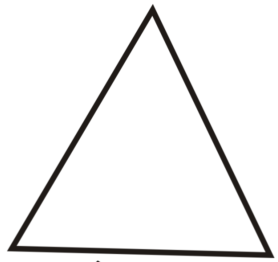
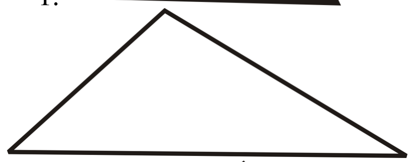
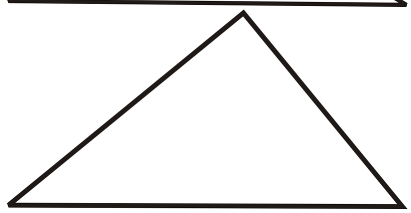
Solution: The folds (blue dashed lines)are the perpendicular bisectors and cross at the circumcenter.



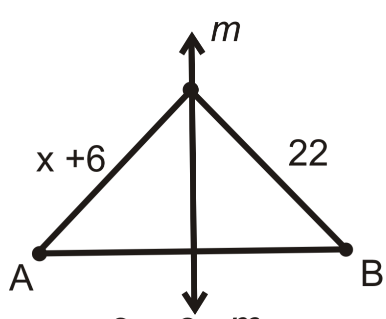
Know What? Revisited The center of the circle will be the circumcenter of the triangle formed by the three bones. Construct the perpendicular bisector of at least two sides to find the circumcenter. After locating the circumcenter, the archeologist can measure the distance from each bone to it, which would be the radius of the circle. This length is approximately 4.7 meters.

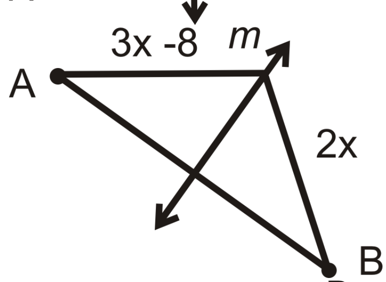
Review Questions

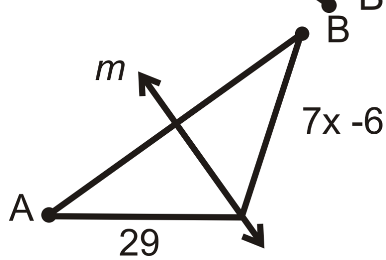
Construction Construct the circumcenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-1.

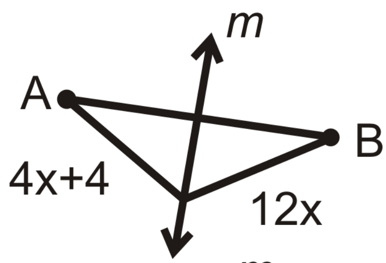
1. 
2. 
3. 
4. Can you use the method in Example 3 to locate the circumcenter for these three triangles?
5. Based on your constructions in 1-3, state a conjecture about the relationship between a triangle and the location of its circumcenter.
6. Construct equilateral triangle $\triangle ABC$ (Investigation 4-6). Construct the perpendicular bisectors of the sides of the triangle and label the circumcenter X. Connect the circumcenter to each vertex. Your original triangle is now divided into six triangles. What can you conclude about the six triangles? Why?

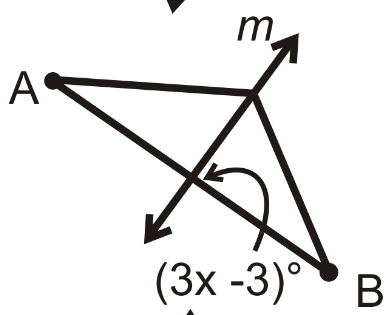
Algebra Connection For questions 7-12, find the value of x. m is the perpendicular bisector of AB.

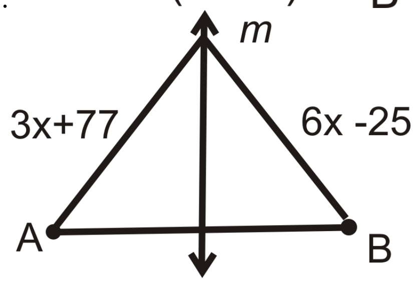
7. 

8. 

9. 

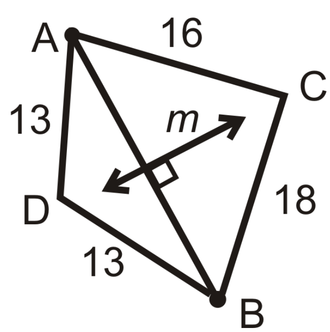
10. 

11. 

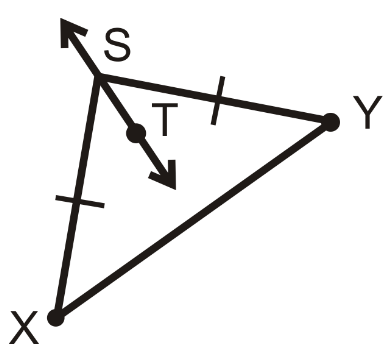
12. 

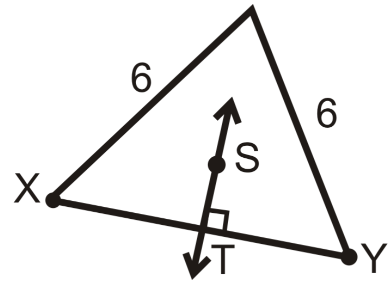
13. m is the perpendicular bisector of $\overline{AB}$.

a. List all the congruent segments.
b. Is C on $\overline{AB}$? Why or why not?
c. Is D on $\overline{AB}$? Why or why not?



For Questions 14 and 15, determine if $\overleftrightarrow{ST}$ is the perpendicular bisector of $\overline{XY}$. Explain why or why not.

14. 



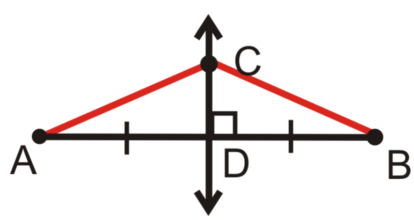
15.

For Questions 16-20, consider line segment $\overline{AB}$ with endpoints $A(2,1)$ and $B(6,3)$.

16. Find the slope of AB.
17. Find the midpoint of AB.
18. Find the equation of the perpendicular bisector of AB.
19. Find AB. Simplify the radical, if needed.
20. Plot A, B, and the perpendicular bisector. Label it m. How could you find a point C on m, such that C would be the third vertex of equilateral triangle $\triangle ABC$? *You do not have to find the coordinates, just describe how you would do it.*

For Questions 21-25, consider $\triangle ABC$ with vertices $A(7,6), B(7,-2)$ and $C(0,5)$. Plot this triangle on graph paper.

21. Find the midpoint and slope of $\overline{AB}$ and use them to draw the perpendicular bisector of $\overline{AB}$. You do not need to write the equation.
22. Find the midpoint and slope of $\overline{BC}$ and use them to draw the perpendicular bisector of $\overline{BC}$. You do not need to write the equation.
23. Find the midpoint and slope of $\overline{AC}$ and use them to draw the perpendicular bisector of $\overline{AC}$. You do not need to write the equation.
24. Are the three lines concurrent? What are the coordinates of their point of intersection (what is the circumcenter of the triangle)?
25. Use your compass to draw the circumscribed circle about the triangle with your point found in question 24 as the center of your circle.
26. Repeat questions 21-25 with $\triangle LMO$ where $L(2,9), M(3,0)$ and $O(-7,0)$.
27. Repeat questions 21-25 with $\triangle REX$ where $R(4,2), E(6,0)$ and $X(0,0)$.
28. Can you explain why the perpendicular bisectors of the sides of a triangle would all pass through the center of the circle containing the vertices of the triangle? Think about the definition of a circle: The set of all point equidistant from a given center.
29. Fill in the blanks: There is exactly __________ circle which contains any ___________ points.
30. Fill in the blanks of the proof of the Perpendicular Bisector Theorem.



Given: $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$Prove: $\overline{AC} \cong \overline{CB}$

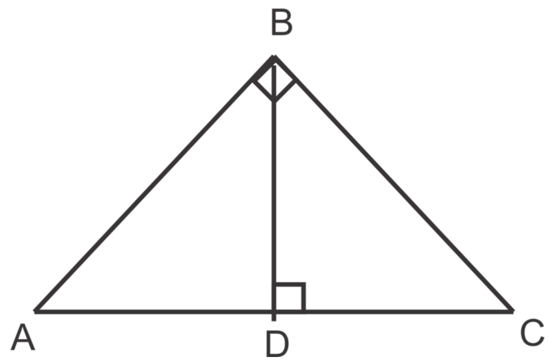
TABLE 5.2:

Statement	Reason
1.	

TABLE 5.2: (continued)

Statement	*Reason*
2. D is the midpoint of $\overline{AB}$	
3.	Definition of a midpoint
4. $\angle CDA$ and $\angle CDB$ are right angles	
5. $\angle CDA \cong \angle CDB$	
6.	Reflexive PoC
7. $\triangle CDA \cong \triangle CDB$	
8. $\overline{AC} \cong \overline{CB}$	

31. Write a two column proof. Given: $\triangle ABC$ is a right isosceles triangle and $\overline{BD}$ is the $\perp$ bisector of $\overline{AC}$ Prove: $\triangle ABD$ and $\triangle CBD$ are congruent.



32. Write a paragraph explaining why the two smaller triangles in question 31 are also isosceles right triangles.

Review Queue Answers

a. Reference Investigation 1-3.

a. $$\begin{aligned} 2x+3 &= 27 \\ 2x &= 24 \\ x &= 12 \end{aligned}$$

b. $$\begin{aligned} 3x+1 &= 19 \\ 3x &= 18 \\ x &= 6 \end{aligned}$$

b. $$\begin{aligned} 6x-13 &= 2x+11 \\ 4x &= 24 \\ x &= 6 \end{aligned} \qquad \begin{aligned} 3y+21 &= 90^\circ \\ 3y &= 69^\circ \\ y &= 23^\circ \end{aligned}$$

Yes, m is the perpendicular bisector of AB because it is perpendicular to AB and passes through the midpoint.

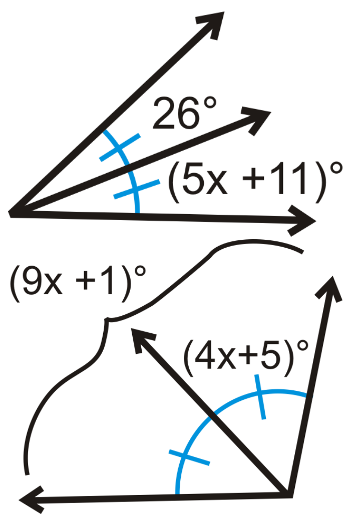
5.3 Angle Bisectors in Triangles

Learning Objectives

- Apply the Angle Bisector Theorem and its converse.
- Understand concurrency for angle bisectors.

Review Queue

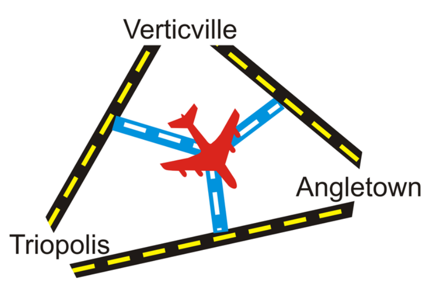
a. Construct the angle bisector of an 80° angle (Investigation 1-4).
b. Draw the following: M is on the interior of $\angle LNO$. O is on the interior of $\angle MNP$. If $\overleftrightarrow{NM}$ and $\overleftrightarrow{NO}$ are the angle bisectors of $\angle LNO$ and $\angle MNP$ respectively, write all the congruent angles.
c. Find the value of x.



a.

b.

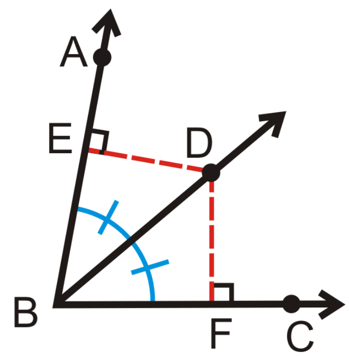
Know What? The cities of Verticville, Triopolis, and Angletown are joining their city budgets together to build a centrally located airport. There are freeways between the three cities and they want to have the freeway on the interior of these freeways. Where is the best location to put the airport so that they have to build the least amount of road?



In the picture to the right, the blue roads are proposed.

Angle Bisectors

In Chapter 1, you learned that an angle bisector cuts an angle exactly in half. In #1 in the Review Queue above, you constructed an angle bisector of an 80° angle. Let's analyze this figure.



$\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. Looking at point D, if we were to draw $\overline{ED}$ and $\overline{DF}$, we would find that they are equal. Recall from Chapter 3 that the shortest distance from a point to a line is the perpendicular length between them. ED and DF are the shortest lengths between D, *which is on the angle bisector*, and each side of the angle.

Angle Bisector Theorem: If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

In other words, if $\overleftrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BE} \perp \overline{ED}$, and $\overrightarrow{BF} \perp \overline{DF}$, then $ED = DF$.

Proof of the Angle Bisector Theorem

Given: $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp \overline{AD}$, and $\overrightarrow{BC} \perp \overline{DC}$

Prove: $\overline{AD} \cong \overline{DC}$

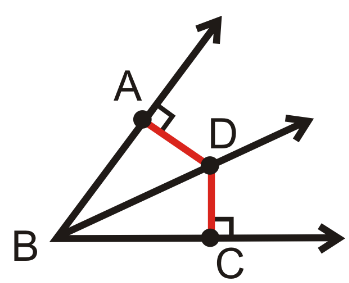


TABLE 5.3:

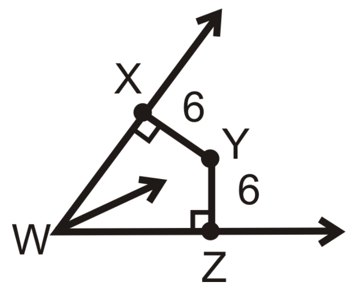
Statement	***Reason***
1. $\overrightarrow{BD}$ bisects $\angle ABC, \overrightarrow{BA} \perp \overline{AD}, \overrightarrow{BC} \perp \overline{DC}$	Given
2. $\angle ABD \cong \angle DBC$	Definition of an angle bisector
3. $\angle DAB$ and $\angle DCB$ are right angles	Definition of perpendicular lines
4. $\angle DAB \cong \angle DCB$	All right angles are congruent
5. $\overline{BD} \cong \overline{BD}$	Reflexive PoC
6. $\triangle ABD \cong \triangle CBD$	AAS
7. $\overline{AD} \cong \overline{DC}$	CPCTC

The converse of this theorem is also true. The proof is in the review questions.

Angle Bisector Theorem Converse: If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

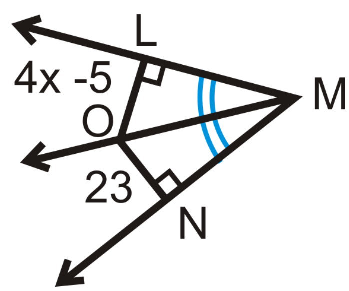
Because the Angle Bisector Theorem and its converse are both true we have a biconditional statement. We can put the two conditional statements together using if and only if. *A point is on the angle bisector of an angle if and only if it is equidistant from the sides of the triangle.*

Example 1: Is Y on the angle bisector of $\angle XWZ$?



Solution: In order for Y to be on the angle bisector XY needs to be equal to YZ and they both need to be perpendicular to the sides of the angle. From the markings we know $\overline{XY} \perp \overrightarrow{WX}$ and $\overline{ZY} \perp \overrightarrow{WZ}$. Second, $XY = YZ = 6$. From this we can conclude that Y is on the angle bisector.

Example 2: $\overrightarrow{MO}$ is the angle bisector of $\angle LMN$. Find the measure of x.



Solution: $LO = ON$ by the Angle Bisector Theorem Converse.

$$\begin{aligned} 4x - 5 &= 23 \\ 4x &= 28 \\ x &= 7 \end{aligned}$$

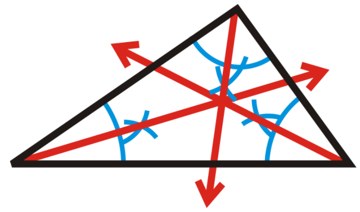
Angle Bisectors in a Triangle

Like perpendicular bisectors, the point of concurrency for angle bisectors has interesting properties.

Investigation 5-2: Constructing Angle Bisectors in Triangles

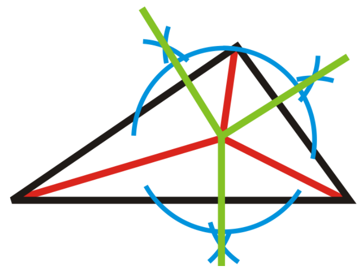
Tools Needed: compass, ruler, pencil, paper

1. Draw a scalene triangle. Construct the angle bisector of each angle. Use Investigation 1-4 and #1 from the Review Queue to help you.

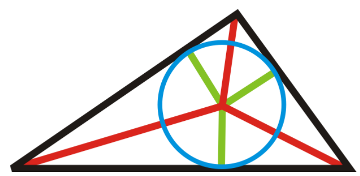


Incenter: The point of concurrency for the angle bisectors of a triangle.

2. Erase the arc marks and the angle bisectors after the incenter. Draw or construct the perpendicular lines to each side, through the incenter.



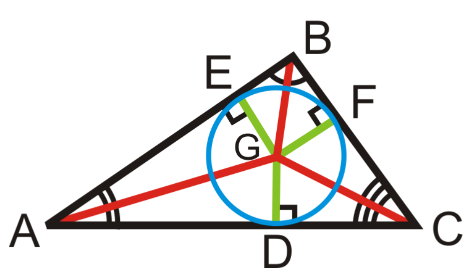
3. Erase the arc marks from #2 and the perpendicular lines beyond the sides of the triangle. Place the pointer of the compass on the incenter. Open the compass to intersect one of the three perpendicular lines drawn in #2. Draw a circle.



Notice that the circle touches all three sides of the triangle. We say that this circle is ***inscribed*** in the triangle because it touches all three sides. The incenter is on all three angle bisectors, so ***the incenter is equidistant from all three sides of the triangle.***

Concurrency of Angle Bisectors Theorem: The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

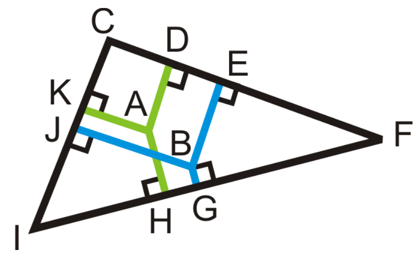
If $\overline{AG}, \overline{BG}$, and $\overline{GC}$ are the angle bisectors of the angles in the triangle, then $EG = GF = GD$.



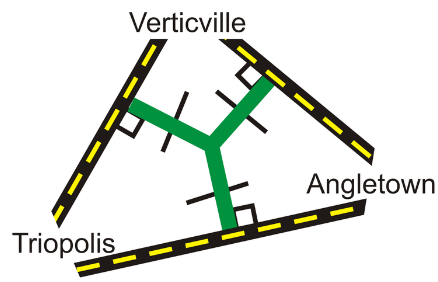
In other words, $\overline{EG}, \overline{FG}$, and $\overline{DG}$ are the radii of the inscribed circle.

Example 3: If J, E, and G are midpoints and $KA = AD = AH$ what are points A and B called?

Solution: A is the incenter because $KA = AD = AH$, which means that it is equidistant to the sides. B is the circumcenter because $\overline{JB}, \overline{BE}$, and $\overline{BG}$ are the perpendicular bisectors to the sides.

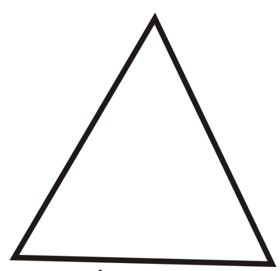


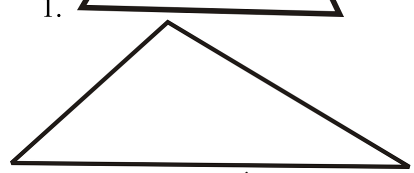
Know What? Revisited The airport needs to be equidistant to the three highways between the three cities. Therefore, the roads are all perpendicular to each side and congruent. The airport should be located at the incenter of the triangle.

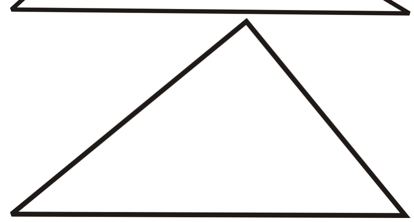


Review Questions

Construction Construct the incenter for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-2. Draw the inscribed circle too.


1.

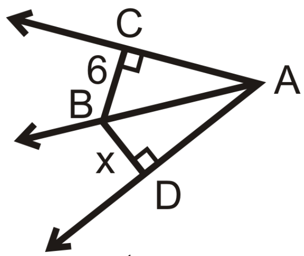

2.


3.

4. Is the incenter always going to be inside of the triangle? Why?
5. For an equilateral triangle, what can you conclude about the circumcenter and the incenter?

For questions 6-11, $\overrightarrow{AB}$ is the angle bisector of $\angle CAD$. Solve for the missing variable.

6.



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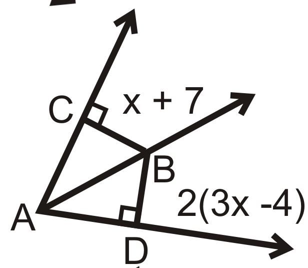

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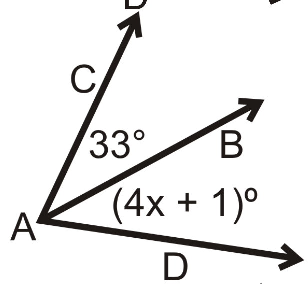

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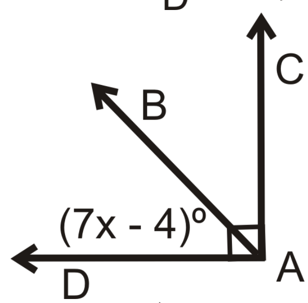

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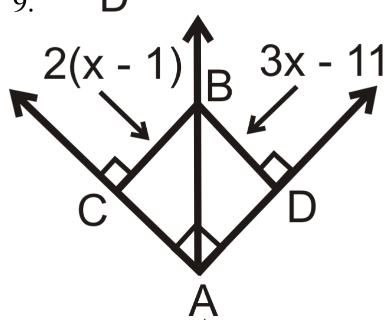

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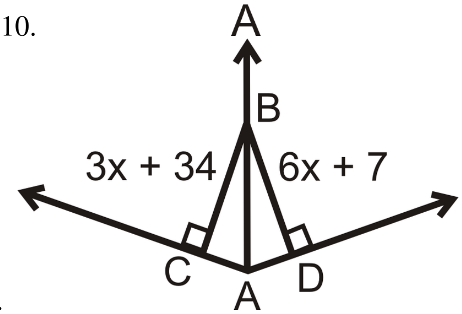
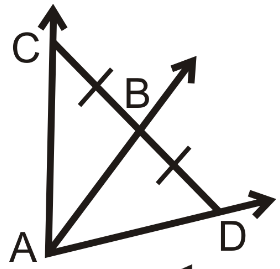
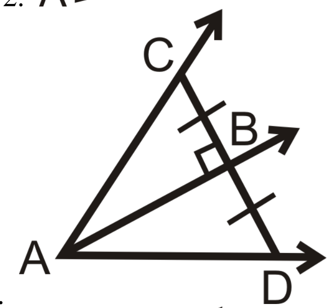
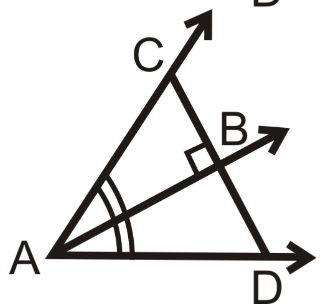


Is there enough information to determine if $\overrightarrow{AB}$ is the angle bisectorof $\angle CAD$? Why or why not?

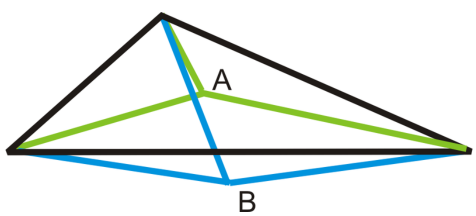
12. 

13. 

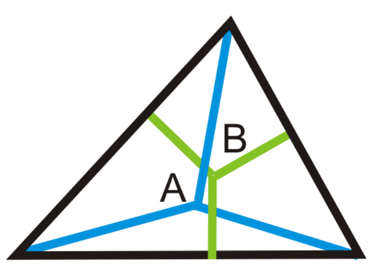
14. 

What are points *A* and *B*? How do you know?

15. The blue lines are congruent The green lines are angle bisectors



16. Both sets of lines are congruent The green lines are perpendicular to the sides



17. Fill in the blanks in the Angle Bisector Theorem Converse. Given: $\overline{AD} \cong \overline{DC}$, such that AD and DC are the shortest distances to $\overrightarrow{BA}$ and $\overrightarrow{BC}$Prove: $\overrightarrow{BD}$ bisects $\angle ABC$

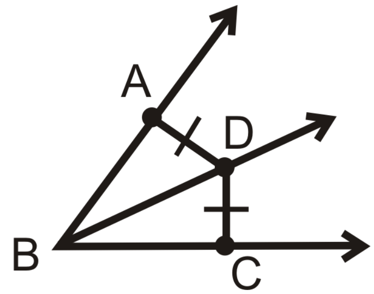


TABLE 5.4:

Statement	*Reason*
1.	
2.	The shortest distance from a point to a line is perpendicular.
3. $\angle DAB$ and $\angle DCB$ are right angles	
4. $\angle DAB \cong \angle DCB$	
5. $\overline{BD} \cong \overline{BD}$	
6. $\triangle ABD \cong \triangle CBD$	
7.	CPCTC
8. $\overrightarrow{BD}$ bisects $\angle ABC$	

Determine if the following descriptions refer to the incenter or circumcenter of the triangle.

18. A lighthouse on a triangular island is equidistant to the three coastlines.
19. A hospital is equidistant to three cities.
20. A circular walking path passes through three historical landmarks.
21. A circular walking path connects three other straight paths.

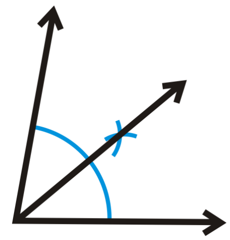
Constructions

22. Construct an equilateral triangle.
23. Construct the angle bisectors of two of the angles to locate the incenter.
24. Construct the perpendicular bisectors of two sides to locate the circumcenter.
25. What do you notice? Use these points to construct an inscribed circle inside the triangle and a circumscribed circle about the triangle.

Multi- Step Problem

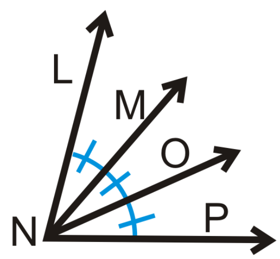
26. Draw $\angle ABC$ through $A(1,3), B(3,-1)$ and $C(7,1)$.
27. Use slopes to show that $\angle ABC$ is a right angle.
28. Use the distance formula to find AB and BC.
29. Construct a line perpendicular to AB through A.
30. Construct a line perpendicular to BC through C.
31. These lines intersect in the interior of $\angle ABC$. Label this point D and draw $\overrightarrow{BD}$.
32. Is $\overrightarrow{BD}$ the angle bisector of $\angle ABC$? Justify your answer.

Review Queue Answers



a.

b. $\angle LNM \cong \angle MNO \cong \angle ONP$
$\angle LNO \cong \angle MNP$



a. $5x + 11 = 26$
$5x = 15$
$x = 3$

b. $9x - 1 = 2(4x + 5)$
$9x - 1 = 8x + 10$
$x = 11^\circ$

5.4 Medians and Altitudes in Triangles

Learning Objectives

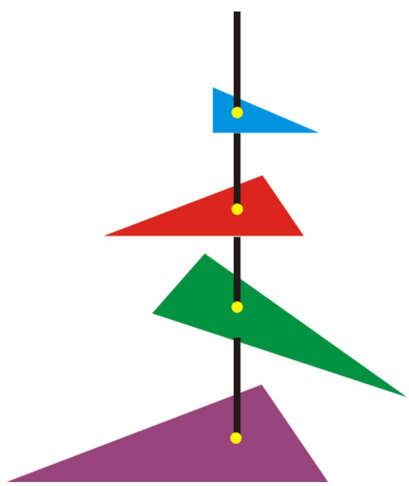
- Define median and find their point of concurrency in a triangle.
- Apply medians to the coordinate plane.
- Construct the altitude of a triangle and find their point of concurrency in a triangle.

Review Queue

a. Find the midpoint between (9, -1) and (1, 15).
b. Find the equation of the line between the two points from #1.
c. Find the equation of the line that is perpendicular to the line from #2 through (-6, 2).

Know What? Triangles are frequently used in art. Your art teacher assigns an art project involving triangles. You decide to make a series of hanging triangles of all different sizes from one long piece of wire. Where should you hang the triangles from so that they balance horizontally?

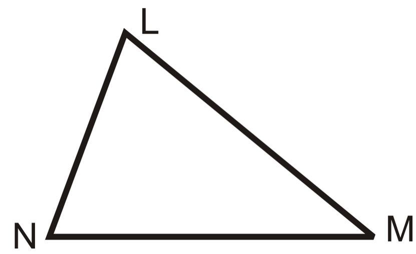
You decide to plot one triangle on the coordinate plane to find the location of this point. The coordinates of the vertices are (0, 0), (6, 12) and (18, 0). What is the coordinate of this point?



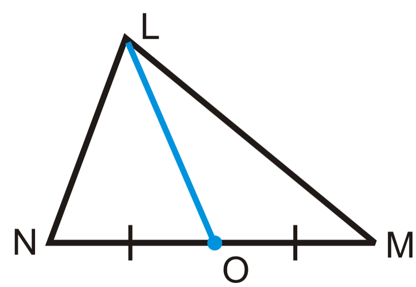
Medians

Median: The line segment that joins a vertex and the midpoint of the opposite side (of a triangle).

Example 1: Draw the median $\overline{LO}$ for $\triangle LMN$ below.



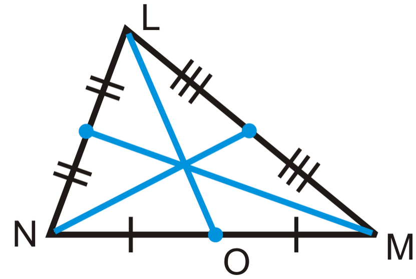
Solution: From the definition, we need to locate the midpoint of $\overline{NM}$. We were told that the median is $\overline{LO}$, which means that it will connect the vertex L and the midpoint of $\overline{NM}$, to be labeled O. Measure NM and make a point halfway between N and M. Then, connect O to L.



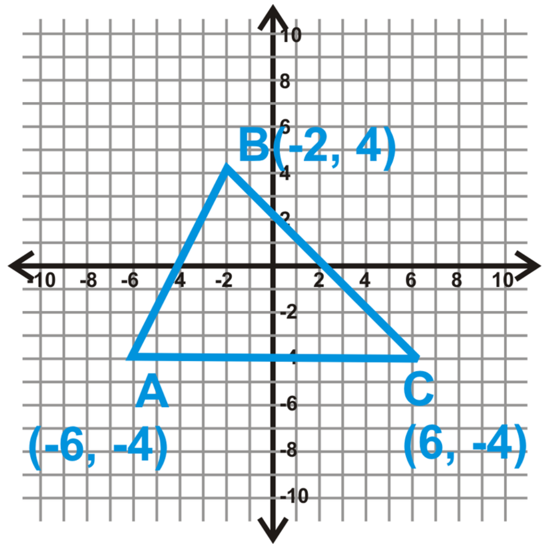
Notice that a median is very different from a perpendicular bisector or an angle bisector. A perpendicular bisector also goes through the midpoint, but it does not necessarily go through the vertex of the opposite side. And, unlike an angle bisector, a median does not necessarily bisect the angle.

Example 2: Find the other two medians of $\triangle LMN$.

Solution: Repeat the process from Example 1 for sides $\overline{LN}$ and $\overline{LM}$. Be sure to always include the appropriate tick marks to indicate midpoints.



Example 3: Find the equation of the median from B to the midpoint of $\overline{AC}$ for the triangle in the $x-y$ plane below.



Solution: To find the equation of the median, first we need to find the midpoint of $\overline{AC}$, using the Midpoint Formula.

$$\left(\frac{-6+6}{2}, \frac{-4+(-4)}{2}\right) = \left(\frac{0}{2}, \frac{-8}{2}\right) = (0, -4)$$

Now, we have two points that make a line, B and the midpoint. Find the slope and $y-$intercept.

$$\begin{aligned} m &= \frac{-4-4}{0-(-2)} = \frac{-8}{2} = -4 \\ y &= -4x + b \\ -4 &= -4(0) + b \\ -4 &= b \end{aligned}$$

The equation of the median is $y = -4x - 4$

Point of Concurrency for Medians

From Example 2, we saw that the three medians of a triangle intersect at one point, just like the perpendicular bisectors and angle bisectors. This point is called the centroid.

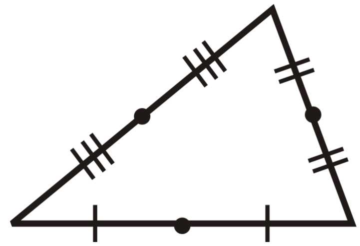
Centroid: The point of concurrency for the medians of a triangle.

Unlike the circumcenter and incenter, the centroid does not have anything to do with circles. It has a different property.

Investigation 5-3: Properties of the Centroid

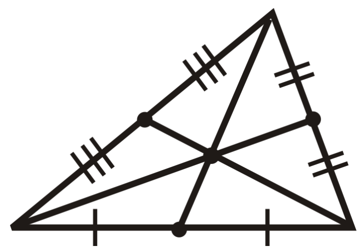
Tools Needed: pencil, paper, ruler, compass

1. Construct a scalene triangle with sides of length 6 cm, 10 cm, and 12 cm (Investigation 4-2). Use the ruler to measure each side and mark the midpoint.

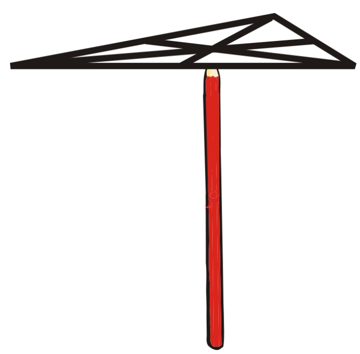


2. Draw in the medians and mark the centroid.

Measure the length of each median. Then, measure the length from each vertex to the centroid and from the centroid to the midpoint. Do you notice anything?



3. Cut out the triangle. Place the centroid on either the tip of the pencil or the pointer of the compass. What happens?



From this investigation, we have discovered the properties of the centroid. They are summarized below.

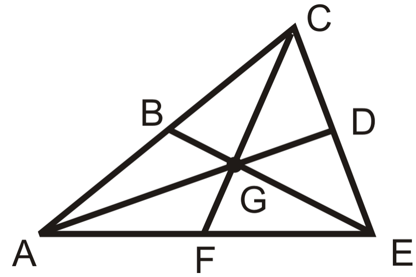
Concurrency of Medians Theorem: The medians of a triangle intersect in a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side. The centroid is also the "balancing point" of a triangle.

If G is the centroid, then we can conclude:

$$AG = \frac{2}{3}AD, CG = \frac{2}{3}CF, EG = \frac{2}{3}BE$$
$$DG = \frac{1}{3}AD, FG = \frac{1}{3}CF, BG = \frac{1}{3}BE$$

And, combining these equations, we can also conclude:

$$DG = \frac{1}{2}AG, FG = \frac{1}{2}CG, BG = \frac{1}{2}EG$$

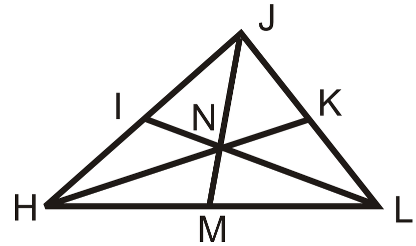


In addition to these ratios, G is also the balance point of $\triangle ACE$. This means that the triangle will balance when placed on a pencil (#3 in Investigation 5-3) at this point.

Example 4: I, K, and M are midpoints of the sides of $\triangle HJL$.

a) If $JM = 18$, find JN and NM.

b) If $HN = 14$, find NK and HK.

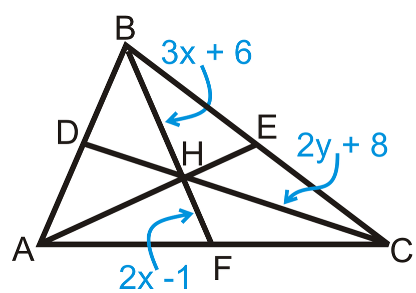


Solution:

a) JN is two-thirds of JM. So, $JN = \frac{2}{3} \cdot 18 = 12$. NM is either half of 12, a third of 18 or $18 - 12$. $NM = 6$.

b) HN is two-thirds of HK. So, $14 = \frac{2}{3} \cdot HK$ and $HK = 14 \cdot \frac{3}{2} = 21$. NK is a third of 21, half of 14, or $21 - 14$. $NK = 7$.

Example 5: ***Algebra Connection*** H is the centroid of $\triangle ABC$ and $DC = 5y - 16$. Find x and y.



Solution: HF is half of BH. Use this information to solve for x. For y, HC is two-thirds of DC. Set up an equation for both.

$$\begin{aligned} \frac{1}{2}BH &= HF \text{ or } BH = 2HF \\ 3x + 6 &= 2(2x - 1) \\ 3x + 6 &= 4x - 2 \\ 8 &= x \end{aligned}$$

$$\begin{aligned} HC &= \frac{2}{3}DC \text{ or } \frac{3}{2}HC = DC \\ \frac{3}{2}(2y + 8) &= 5y - 16 \\ 3y + 12 &= 5y - 16 \\ 28 &= 2y \end{aligned}$$

Altitudes

The last line segment within a triangle is an altitude. It is also called the height of a triangle.

Altitude: A line segment from a vertex and perpendicular to the opposite side.

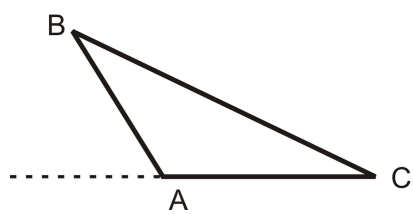
Here are a few examples.

As you can see, an altitude can be a side of a triangle or outside of the triangle. When a triangle is a right triangle, the altitude, or height, is the leg. If the triangle is obtuse, then the altitude will be outside of the triangle. ***To construct an altitude, use Investigation 3-2*** (constructing a perpendicular line through a point not on the given line). Think of the vertex as the point and the given line as the opposite side.

Investigation 5-4: Constructing an Altitude for an Obtuse Triangle

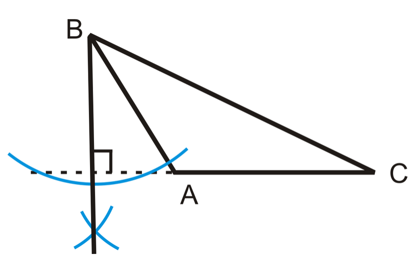
Tools Needed: pencil, paper, compass, ruler

a. Draw an obtuse triangle. Label it $\triangle ABC$, like the picture to the right. Extend side $\overline{AC}$, beyond point A.



b. Using Investigation 3-2, construct a perpendicular line to $\overline{AC}$, through B.

The altitude does not have to extend past side $\overline{AC}$, as it does in the picture. Technically the height is only the vertical distance from the highest vertex to the opposite side.

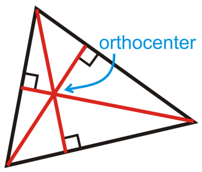
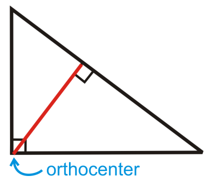
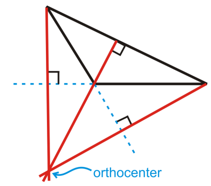


As was true with perpendicular bisectors, angle bisectors, and medians,the altitudes of a triangle are also concurrent. Unlike the other three, the point does not have any special properties.

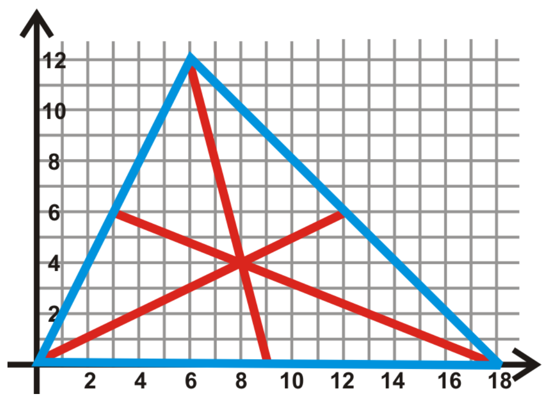
Orthocenter: The point of concurrency for the altitudes of triangle.

Here is what the orthocenter looks like for the three triangles. It has three different locations, much like the perpendicular bisectors.

TABLE 5.5:

Acute Triangle	Right Triangle	Obtuse Triangle
		
The orthocenter is inside the triangle.	The legs of the triangle are two of the altitudes. The orthocenter is the vertex of the right angle.	The orthocenter is outside the triangle.

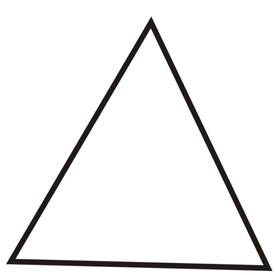
Know What? Revisited The point that you should put the wire through is the centroid. That way, each triangle will balance on the wire.

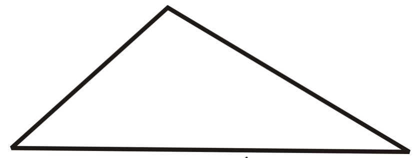
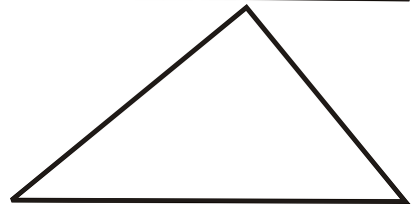


The triangle that we wanted to plot on the $x-y$ plane is to the right. Drawing all the medians, it looks like the centroid is (8, 4). To verify this, you could find the equation of two medians and set them equal to each other and solve for x. Two equations are $y = \frac{1}{2}x$ and $y = -4x + 36$. Setting them equal to each other, we find that $x = 8$ and then $y = 4$.

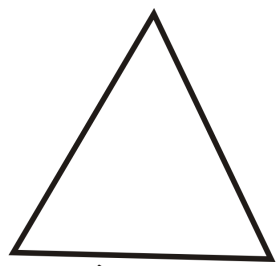
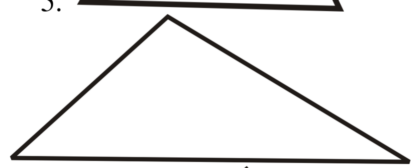
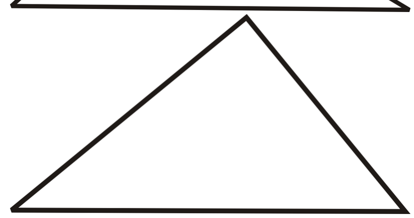
Review Questions

Construction Construct the centroid for the following triangles by tracing each triangle onto a piece of paper and using Investigation 5-3.

1. 

2. 
3. 
4. Is the centroid always going to be inside of the triangle? Why?

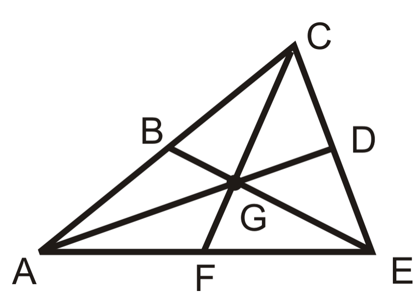
Construction Construct the orthocenter for the following triangles by tracing each triangle onto a piece of paper and using Investigations 3-2 and 5-4.

5. 
6. 
7. 
8. What do you think will happen if the triangle is equilateral? What can we say about the incenter, circumcenter, centroid, and orthocenter? Why do you think this is?
9. How many lines do you actually have to "construct" to find any point of concurrency?

For questions 10-13, find the equation of each median, from vertex A to the opposite side, $\overline{BC}$.

10. $A(9,5), B(2,5), C(4,1)$
11. $A(-2,3), B(-3,-7), C(5,-5)$
12. $A(-1,5), B(0,-1), C(6,3)$
13. $A(6,-3), B(-5,-4), C(-1,-8)$

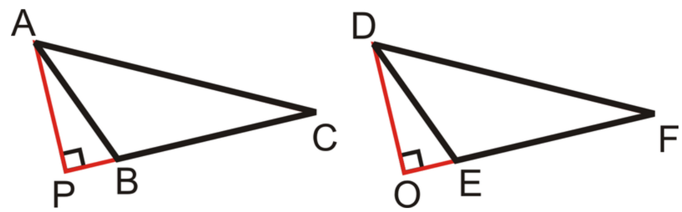
For questions 14-18, B, D, and F are the midpoints of each side and G is the centroid. Find the following lengths.



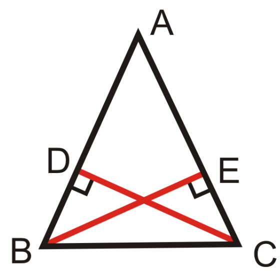
14. If $BG = 5$, find GE and BE
15. If $CG = 16$, find GF and CF
16. If $AD = 30$, find AG and GD
17. If $GF = x$, find GC and CF
18. If $AG = 9x$ and $GD = 5x - 1$, find x and AD.

Write a two-column proof.

19. Given: $\triangle ABC \cong \triangle DEF$ $\overline{AP}$ and $\overline{DO}$ are altitudes Prove: $\overline{AP} \cong \overline{DO}$



20. Given: Isosceles $\triangle ABC$ with legs $\overline{AB}$ and $\overline{AC}\overline{BD} \perp \overline{DC}$ and $\overline{CE} \perp \overline{BE}$Prove: $\overline{BD} \cong \overline{CE}$



Use $\triangle ABC$ with $A(-2,9), B(6,1)$ and $C(-4,-7)$ for questions 21-26.

21. Find the midpoint of $\overline{AB}$ and label it M.
22. Write the equation of $\overleftrightarrow{CM}$.
23. Find the midpoint of $\overline{BC}$ and label it N.
24. Write the equation of $\overleftrightarrow{AN}$.
25. Find the intersection of $\overleftrightarrow{CM}$ and $\overleftrightarrow{AN}$.
26. What is this point called?

Another way to find the centroid of a triangle in the coordinate plane is to find the midpoint of one side and then find the point two thirds of the way from the third vertex to this point. To find the point two thirds of the way from point $A(x_1, y_1)$ to $B(x_2, y_2)$ use the formula: $\left(\frac{x_1+2x_2}{3}, \frac{y_1+2y_2}{3}\right)$. Use this method to find the centroid in the following problems.

27. (-1, 3), (5, -2) and (-1, -4)
28. (1, -2), (-5, 4) and (7, 7)
29. Use the coordinates $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) and the method used in the last two problems to find a formula for the centroid of a triangle in the coordinate plane.
30. Use your formula from problem 29 to find the centroid of the triangle with vertices (2, -7), (-5, 1) and (6, -9).

Review Queue Answers

a. $midpoint = \left(\frac{9+1}{2}, \frac{-1+15}{2}\right) = (5,7)$

b. $m = \frac{15+1}{1-9} = \frac{16}{-8} = -2$ $15 = -2(1) + b$ $y = -2x + 17$
$17 = b$

c. $y = \frac{1}{2}x + b$
$2 = \frac{1}{2}(-6) + b$
$2 = -3 + b$
$5 = b$
$y = \frac{1}{2}x + 5$

5.5 Inequalities in Triangles

Learning Objectives

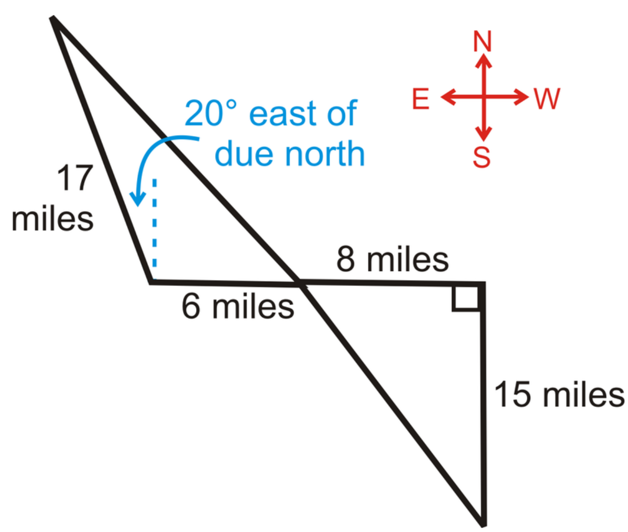
- Determine relationships among the angles and sides of a triangle.
- Understand the Triangle Inequality Theorem.
- Understand the Hinge Theorem and its converse.

Review Queue

Solve the following inequalities.

a. $4x-9 \leq 19$
b. $-5 > -2x+13$
c. $\frac{2}{3}x+1 \geq 13$
d. $-7 < 3x-1 < 14$

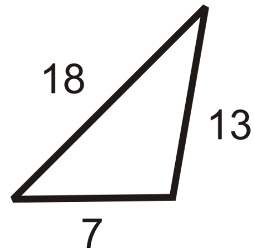
Know What? Two mountain bike riders leave from the same parking lot and headin opposite directions, on two different trails. The first ridergoes 8 miles due west, then rides due south for 15 miles. The second rider goes 6 miles due east, then changes direction and rides 20° east of due north for 17 miles. Both riders have been travelling for 23 miles, but which one is further from the parking lot?



Comparing Angles and Sides

Look at the triangle to the right. The sides of the triangle are given. Can you determine which angle is the largest?

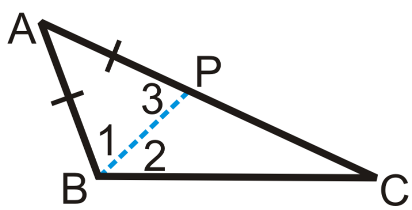
As you might guess, the largest angle will be opposite 18 because it is the longest side. Similarly, the smallest angle will be opposite the shortest side, 7. Therefore, the angle measure in the middle will be opposite 13.



Theorem 5-9: If one side of a triangle is longer than another side, then the angle opposite the longer side will be larger than the angle opposite the shorter side.

Converse of Theorem 5-9: If one angle in a triangle is larger than another angle in a triangle, then the side opposite the larger angle will be longer than the side opposite the smaller angle.

Proof of Theorem 5-9



Given: $AC > AB$

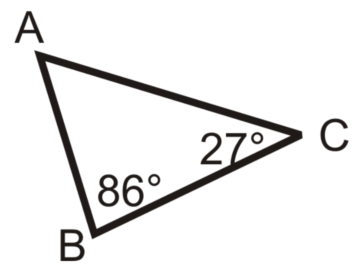
Prove: $m\angle ABC > m\angle C$

TABLE 5.6:

Statement	***Reason***
1. $AC > AB$	Given
2. Locate point P such that $AB = AP$	Ruler Postulate
3. $\triangle ABP$ is an isosceles triangle	Definition of an isosceles triangle
4. $m\angle 1 = m\angle 3$	Base Angles Theorem
5. $m\angle 3 = m\angle 2 + m\angle C$	Exterior Angle Theorem
6. $m\angle 1 = m\angle 2 + m\angle C$	Substitution PoE
7. $m\angle ABC = m\angle 1 + m\angle 2$	Angle Addition Postulate
8. $m\angle ABC = m\angle 2 + m\angle 2 + m\angle C$	Substitution PoE
9. $m\angle ABC > m\angle C$	Definition of "greater than" (from step 8)

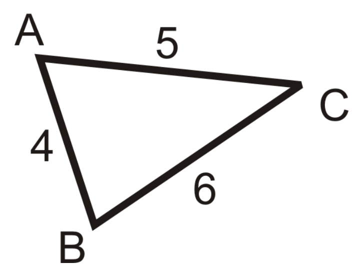
To prove the converse, we will need to do so indirectly. This will be done in the extension at the end of this chapter.

Example 1: List the sides in order, from shortest to longest.



Solution: First, we need to find $m\angle A$. From the Triangle Sum Theorem, $m\angle A + 86^\circ + 27^\circ = 180^\circ$. So, $m\angle A = 67^\circ$. From Theorem 5-9, we can conclude that the longest side is opposite the largest angle. 86° is the largest angle, so AC is the longest side. The next largest angle is 67°, so BC would be the next longest side. 27° is the smallest angle, so AB is the shortest side. In order from shortest to longest, the answer is: AB, BC, AC.

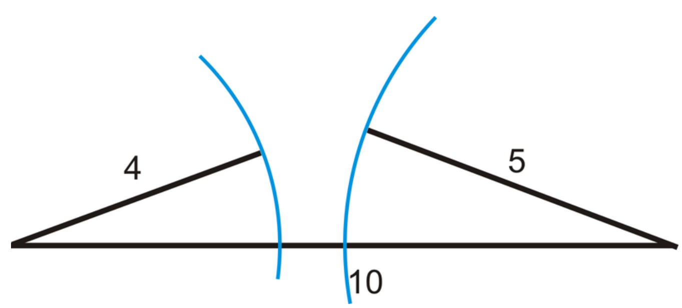
Example 2: List the angles in order, from largest to smallest.



Solution: Just like with the sides, the largest angle is opposite the longest side. The longest side is BC, so the largest angle is $\angle A$. Next would be $\angle B$ and finally $\angle A$ is the smallest angle.

Triangle Inequality Theorem

Can any three lengths make a triangle? The answer is no. There are limits on what the lengths can be. For example, the lengths 1, 2, 3 cannot make a triangle because $1+2=3$, so they would all lie on the same line. The lengths 4, 5, 10 also cannot make a triangle because $4+5=9$.



The arc marks show that the two sides would never meet to form a triangle.

Triangle Inequality Theorem: The sum of the lengths of any two sides of a triangle must be greater than the length of the third.

Example 3: Do the lengths below make a triangle?

a) 4.1, 3.5, 7.5

b) 4, 4, 8

c) 6, 7, 8

Solution: Even though the Triangle Inequality Theorem says "the sum of the length of any two sides," really, it is referring to the sum of the lengths of the two shorter sides must be longer than the third.

a) $4.1+3.5>7.5$ Yes, these lengths could make a triangle.

b) $4+4=8$ No, not a triangle. Two lengths cannot equal the third.

c) $6+7>8$ Yes, these lengths could make a triangle.

Example 4: Find the possible lengths of the third side of a triangle if the other two sides are 10 and 6.

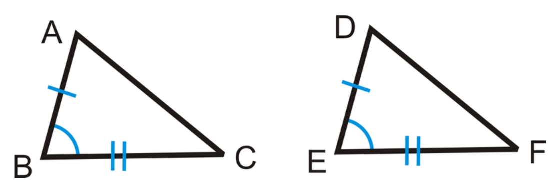
Solution: The Triangle Inequality Theorem can also help you determine the possible range of the third side of a triangle. The two given sides are 6 and 10, so the third side, s, can either be the shortest side or the longest side. For example s could be 5 because $6+5>10$. It could also be 15 because $6+10>15$. Therefore, we write the possible values of s as a range, $4<s<16$.

Notice the range is no less than 4, and not equal to 4. The third side could be 4.1 because $4.1+6$ would be greater than the third side, 10. For the same reason, s cannot be greater than 16, but it could 15.9. In this case, s would be the longest side and $10+6$ must be greater than s to form a triangle.

If two sides are lengths** a **and** b**, then the third side,** s**, has the range** $a-b<s<a+b$**.

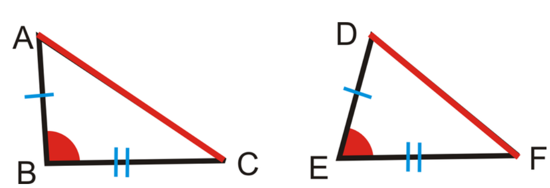
The SAS Inequality Theorem (also called the Hinge Theorem)

The Hinge Theorem is an extension of the Triangle Inequality Theorem using two triangles. If we have two congruent triangles $\triangle ABC$ and $\triangle DEF$, marked below:



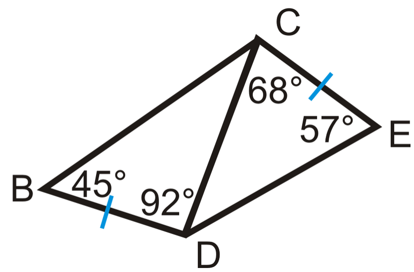
Therefore, if $AB=DE$ and $BC=EF$ and $m\angle B>m\angle E$, then $AC>DF$.

Now, let's adjust $m\angle B>m\angle E$. Would that make $AC>DF$? Yes. See the picture below.



The SAS Inequality Theorem (Hinge Theorem): If two sides of a triangle are congruent to two sides of another triangle, but the included angle of one triangle has greater measure than the included angle of the other triangle, then the third side of the first triangle is longer than the third side of the second triangle.

Example 5: List the sides in order, from least to greatest.



Solution: Let's start with $\triangle DCE$. The missing angle is 55°. By Theorem 5-9, the sides, in order are CE, CD, and DE.

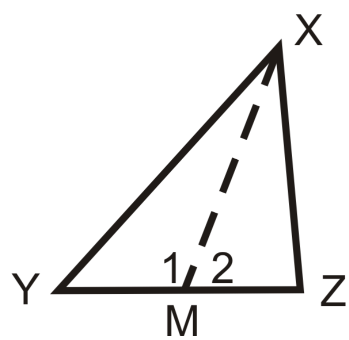
For $\triangle BCD$, the missing angle is 43°. Again, by Theorem 5-9, the order of the sides is BD, CD, and BC.

By the SAS Inequality Theorem, we know that $BC > DE$, so the order of all the sides would be: $BD = CE, CD, DE, BC$.

SSS Inequality Theorem (also called the Converse of the Hinge Theorem)

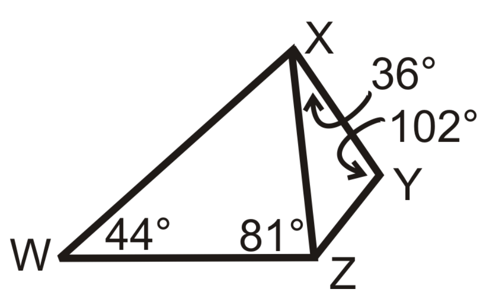
SSS Inequality Theorem: If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.

Example 6: If $\overline{XM}$ is a median of $\triangle XYZ$ and $XY > XZ$, what can we say about $m\angle 1$ and $m\angle 2$? What we can deduce from the following diagrams.



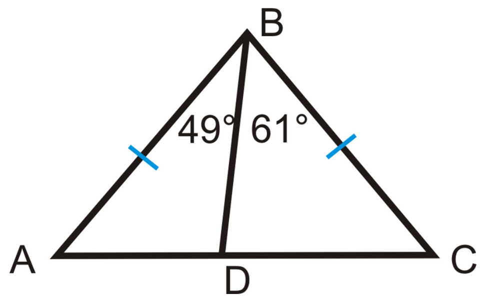
Solution: By the definition of a median, M is the midpoint of $\overline{YZ}$. This means that $YM = MZ$. $MX = MX$ by the Reflexive Property and we know that $XY > XZ$. Therefore, we can use the SSS Inequality Theorem to conclude that $m\angle 1 > m\angle 2$.

Example 7: List the sides of the two triangles in order, from least to greatest.



Solution: Here we have no congruent sides or angles. So, let's look at each triangle separately. Start with $\triangle XYZ$. First the missing angle is 42°. By Theorem 5-9, the order of the sides is YZ, XY, and XZ. For $\triangle WXZ$, the missing angle is 55°. The order of these sides is XZ, WZ, and WX. Because the longest side in $\triangle XYZ$ is the shortest side in $\triangle WXZ$, we can put all the sides together in one list: YZ, XY, XZ, WZ, WX.

Example 8: Below is isosceles triangle $\triangle ABC$. List everything you can about the triangle and why.



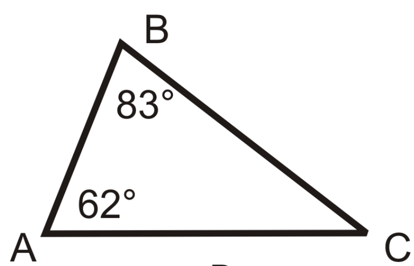
Solution:

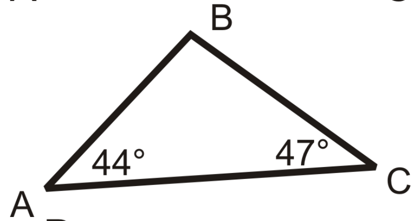
- $AB = BC$ because it is given.
- $m\angle A = m\angle C$ by the Base Angle Theorem.
- $AD < DC$ because $m\angle ABD < m\angle CBD$ and the SAS Triangle Inequality Theorem.

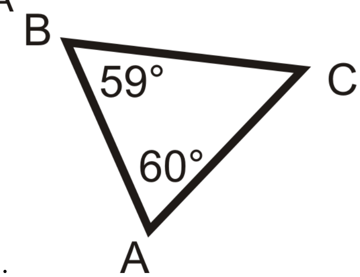
Know What? Revisited Even though the two sets of lengths are not equal, they both add up to 23. Therefore, the second rider is further away from the parking lot because $110^\circ > 90^\circ$.

Review Questions

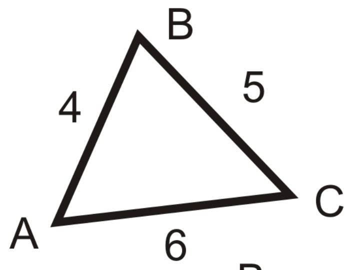
For questions 1-3, list the sides in order from shortest to longest.

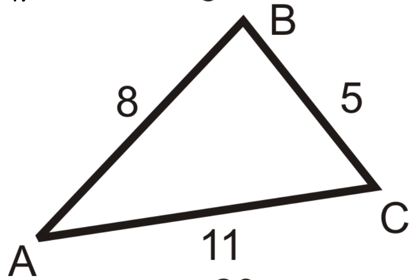
1. 

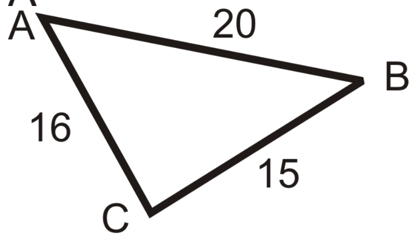
2. 

3. 

For questions 4-6, list the angles from largest to smallest.

4. 

5. 

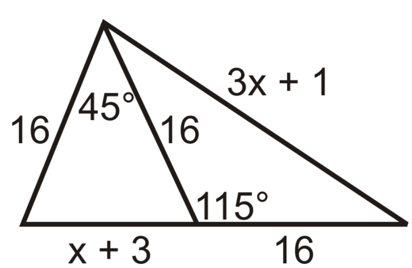
6. 

Determine if the sets of lengths below can make a triangle. If not, state why.

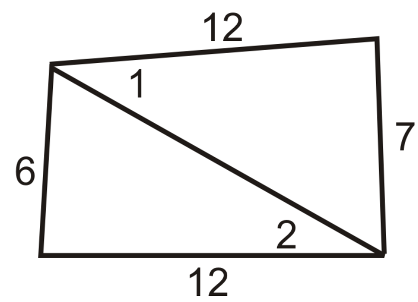
7. 6, 6, 13
8. 1, 2, 3
9. 7, 8, 10
10. 5, 4, 3
11. 23, 56, 85
12. 30, 40, 50

If two lengths of the sides of a triangle are given, determine the range of the length of the third side.

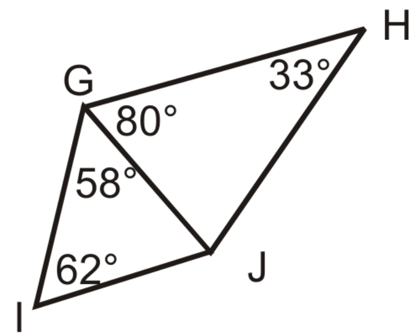
13. 8 and 9
14. 4 and 15
15. 20 and 32
16. The base of an isosceles triangle has length 24. What can you say about the length of each leg?
17. What conclusions can you draw about x?



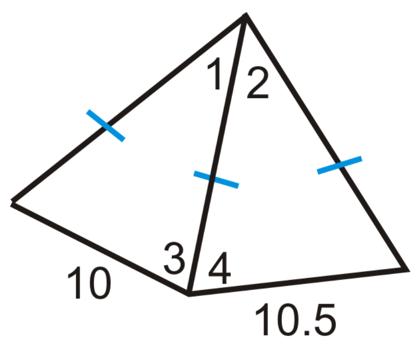
18. Compare $m\angle 1$ and $m\angle 2$.



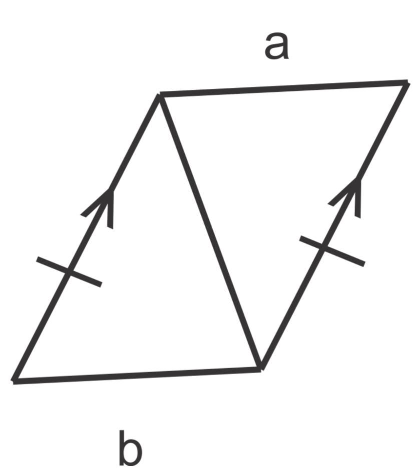
19. List the sides from shortest to longest.



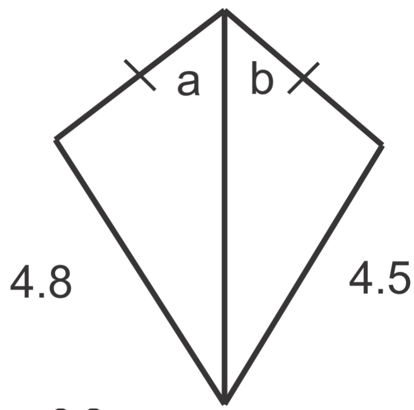
20. Compare $m\angle 1$ and $m\angle 2$. What can you say about $m\angle 3$ and $m\angle 4$?



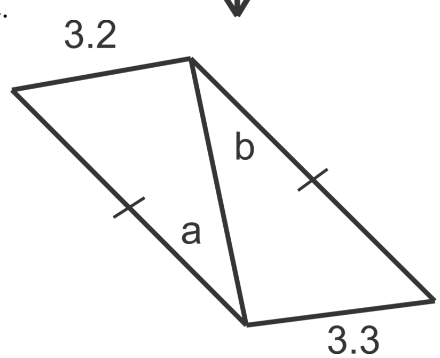
In questions 21-23, compare the measures of *a* and *b*.



21.

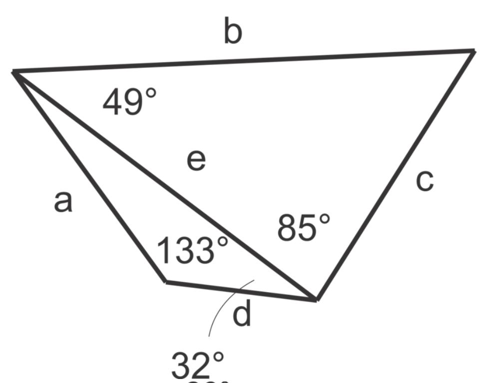


22.

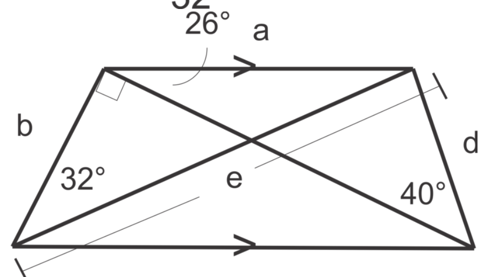


23.

In questions 24 and 25, list the measures of the sides in order from least to greatest

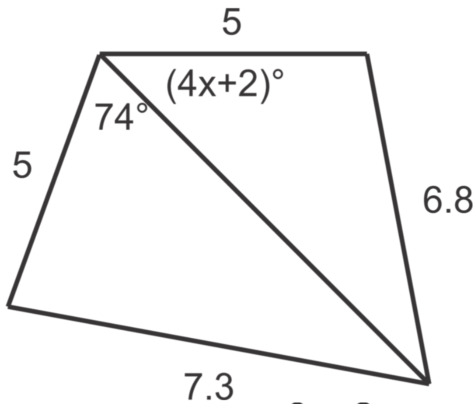


24.

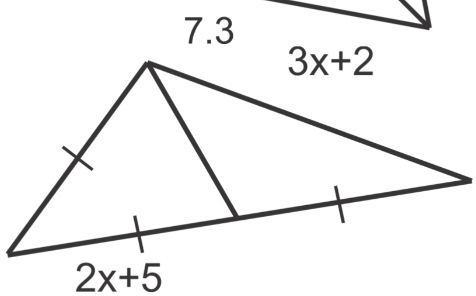


25.

In questions 26 and 27 determine the range of possible values for x.



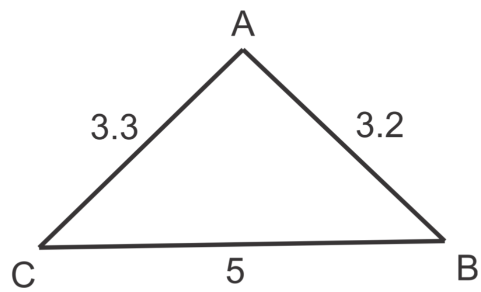
26.



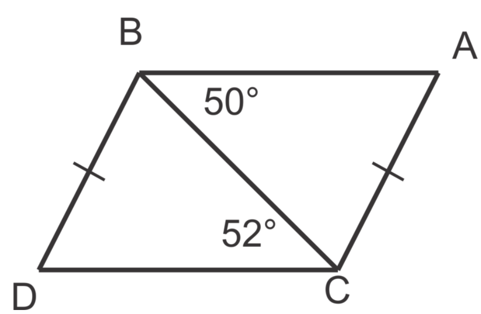
27.

In questions 28 and 29 explain why the conclusion is false.

28. Conclusion: $m\angle C < m\angle B < m\angle A$



29. Conclusion: $AB < DC$



30. If $\overline{AB}$ is a median of $\triangle CAT$ and $CA > AT$, explain why $\angle ABT$ is acute. You may wish to draw a diagram.

Review Queue Answers

a. $4x - 9 \le 19$
$4x \le 28$
$x \le 7$

b. $-5 > -2x + 13$
$-18 > -2x$
$9 < x$
c. $\frac{2}{3}x + 1 \geq 13$
$\frac{2}{3}x \geq 12$
$x \geq 18$
d. $-7 < 3x - 1 < 14$
$-6 < 3x < 15$
$-2 < x < 5$

5.6 Extension: Indirect Proof

The indirect proof or proof by contradiction is a part of 41 out of 50 states' mathematic standards. Depending on the state, the teacher may choose to use none, part or all of this section.

Learning Objectives

- Reason indirectly to develop proofs.

Until now, we have proved theorems true by direct reasoning, where conclusions are drawn from a series of facts and previously proven theorems. However, we cannot always use direct reasoning to prove every theorem.

Indirect Proof: When the conclusion from a hypothesis is assumed false (or opposite of what it states) and then a contradiction is reached from the given or deduced statements.

The easiest way to understand indirect proofs is by example. You may choose to use the two-column format or a paragraph proof. First we will explore indirect proofs with algebra and then geometry.

Indirect Proofs in Algebra

Example 1: If $x = 2$, then $3x - 5 \neq 10$. Prove this statement is true by contradiction.

Solution: In an indirect proof the first thing you do is assume the conclusion of the statement is false. In this case, we will assume the opposite of $3x - 5 \neq 10$

If $x = 2$, then $3x - 5 = 10$

Now, proceed with this statement, as if it is true. Solve for x.

$$\begin{aligned} 3x - 5 &= 10 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

$x = 5$ contradicts the given statement that $x = 2$. Hence, our assumption is incorrect and $3x - 5$ cannot equal 10.

Example 2: If n is an integer and n^2 is odd, then n is odd. Prove this is true indirectly.

Solution: First, assume the opposite of "n is odd."

n is even.

Now, square n and see what happens.

If n is even, then $n = 2a$, where a is any integer.

$$n^2 = (2a)^2 = 4a^2$$

This means that n^2 is a multiple of 4. No odd number can be divided evenly by an even number, so this contradicts our assumption that n is even. Therefore, n must be odd if n^2 is odd.

Indirect Proofs in Geometry

Example 3: If $\triangle ABC$ is isosceles, then the measure of the base angles cannot be 92°. Prove this indirectly.

Solution: Assume the opposite of the conclusion.

The measure of the base angles <u>is</u> 92°.

If the base angles are 92°, then they add up to 184°. This contradicts the Triangle Sum Theorem that says all triangles add up to 180°. Therefore, the base angles cannot be 92°.

Example 4: Prove the SSS Inequality Theorem is true by contradiction.

Solution: The SSS Inequality Theorem says: "If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle." First, assume the opposite of the conclusion.

The included angle of the first triangle is <u>less than or equal to</u> the included angle of the second triangle.

If the included angles are equal then the two triangles would be congruent by SAS and the third sides would be congruent by CPCTC. This contradicts the hypothesis of the original statement "the third side of the first triangle is longer than the third side of the second." Therefore, the included angle of the first triangle must be larger than the included angle of the second.

To summarize:

- Assume the ***opposite*** of the conclusion (second half) of the statement.
- Proceed as if this assumption is true to find the ***contradiction***.
- Once there is a contradiction, the original statement is true.
- ***DO NOT use specific examples.*** Use variables so that the contradiction can be generalized.

Review Questions

Prove the following statements true indirectly.

1. If n is an integer and n^2 is even, then n is even.
2. If $m\angle A \neq m\angle B$ in $\triangle ABC$, then $\triangle ABC$ is not equilateral.
3. If $x > 3$, then $x^2 > 9$.
4. The base angles of an isosceles triangle are congruent.
5. If x is even and y is odd, then $x + y$ is odd.
6. In $\triangle ABE$, if $\angle A$ is a right angle, then $\angle B$ cannot be obtuse.
7. If A, B, and C are collinear, then $AB + BC = AC$ (Segment Addition Postulate).
8. If a collection of nickels and dimes is worth 85 cents, then there must be an odd number of nickels.
9. Hugo is taking a true/false test in his Geometry class. There are five questions on the quiz. The teacher gives her students the following clues: The last answer on the quiz is not the same as the fourth answer. The third answer is true. If the fourth answer is true, then the one before it is false. Use an indirect proof to prove that the last answer on the quiz is true.

10. On a test of 15 questions, Charlie claims that his friend Suzie must have gotten at least 10 questions right. Another friend, Larry, does not agree and suggests that Suzie could not have gotten that many correct. Rebecca claims that Suzie certainly got at least one question correct. If *only one* of these statements is true, how many questions did Suzie get right?

5.7 Chapter 5 Review

Keywords, Theorems and Postulates

Midsegment
A line segment that connects two midpoints of adjacent sides of a triangle.

Midsegment Theorem
The midsegment of a triangle is half the length of the side it is parallel to.

Perpendicular Bisector Theorem
If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Perpendicular Bisector Theorem Converse
If a point is equidistant from the endpoints of a segment, then the point is on the perpendicular bisector of the segment.

Point of Concurrency
When three or more lines intersect at the same point.

Circumcenter
The point of concurrency for the perpendicular bisectors of the sides of a triangle.

Concurrency of Perpendicular Bisectors Theorem
The perpendicular bisectors of the sides of a triangle intersect in a point that is equidistant from the vertices.

Angle Bisector Theorem
If a point is on the bisector of an angle, then the point is equidistant from the sides of the angle.

Angle Bisector Theorem Converse
If a point is in the interior of an angle and equidistant from the sides, then it lies on the bisector of the angle.

Incenter
The point of concurrency for the angle bisectors of a triangle.

Concurrency of Angle Bisectors Theorem
The angle bisectors of a triangle intersect in a point that is equidistant from the three sides of the triangle.

Median
The line segment that joins a vertex and the midpoint of the opposite side (of a triangle).

Centroid
The point of concurrency for the medians of a triangle.

Concurrency of Medians Theorem
The medians of a triangle intersect in a point that is two-thirds of the distance from the vertices to the midpoint of the opposite side.

Altitude
A line segment from a vertex and perpendicular to the opposite side.

Orthocenter
The point of concurrency for the altitudes of triangle.

Theorem 5-9
If one side of a triangle is longer than another side, then the angle opposite the longer side will be larger than the angle opposite the shorter side.

Converse of Theorem 5-9
If one angle in a triangle is larger than another angle in a triangle, then the side opposite the larger angle will be longer than the side opposite the smaller angle.

Triangle Inequality Theorem
The sum of the lengths of any two sides of a triangle must be greater than the length of the third.

SAS Inequality Theorem
If two sides of a triangle are congruent to two sides of another triangle, but the included angle of one triangle has greater measure than the included angle of the other triangle, then the third side of the first triangle is longer than the third side of the second triangle.

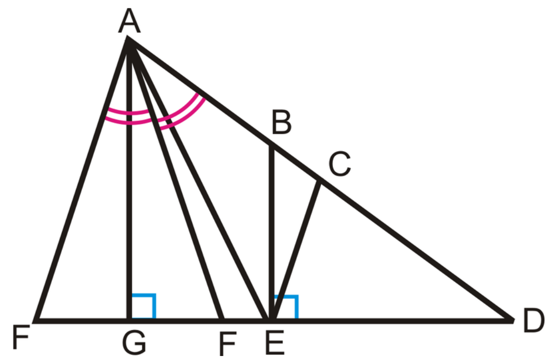
SSS Inequality Theorem
If two sides of a triangle are congruent to two sides of another triangle, but the third side of the first triangle is longer than the third side of the second triangle, then the included angle of the first triangle is greater in measure than the included angle of the second triangle.

Indirect Proof
When the conclusion from a hypothesis is assumed false (or opposite of what it states) and then a contradiction is reached from the given or deduced statements.

Review

If C and E are the midpoints of the sides they lie on, find:

1. The perpendicular bisector of $\overline{FD}$.
2. The median of $\overline{FD}$.
3. The angle bisector of $\angle FAD$.
4. A midsegment.
5. An altitude.



6. Trace $\triangle FAD$ onto a piece of paper with the perpendicular bisector. Construct another perpendicular bisector. What is the point of concurrency called? Use this information to draw the appropriate circle.
7. Trace $\triangle FAD$ onto a piece of paper with the angle bisector. Construct another angle bisector. What is the point of concurrency called? Use this information to draw the appropriate circle.
8. Trace $\triangle FAD$ onto a piece of paper with the median. Construct another median. What is the point of concurrency called? What are its properties?
9. Trace $\triangle FAD$ onto a piece of paper with the altitude. Construct another altitude. What is the point of concurrency called? Which points of concurrency can lie outside a triangle?
10. A triangle has sides with length $x+6$ and $2x-1$. Find the range of the third side.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9690 .

CHAPTER 6 Polygons and Quadrilaterals

Chapter Outline

This chapter starts with the properties of polygons and narrows to focus on quadrilaterals. We will study several different types of quadrilaterals: parallelograms, rhombi, rectangles, squares, kites and trapezoids. Then, we will prove that different types of quadrilaterals are parallelograms or something more specific.

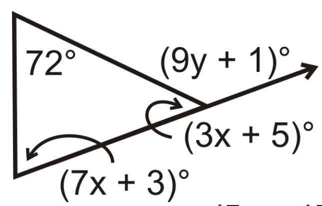
6.1 Angles in Polygons

Learning Objectives

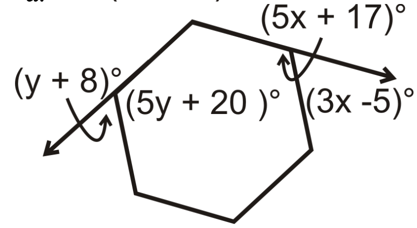
- Extend the concept of interior and exterior angles from triangles to convex polygons.
- Find the sums of interior angles in convex polygons.
- Identify the special properties of interior angles in convex quadrilaterals.

Review Queue

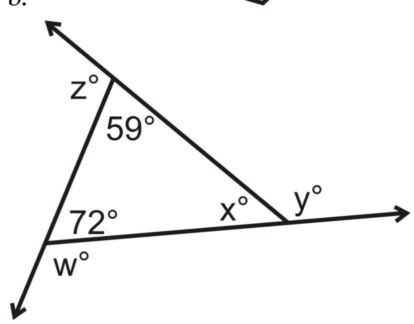
a. Find the measure of x and y.



a.



b.



b.

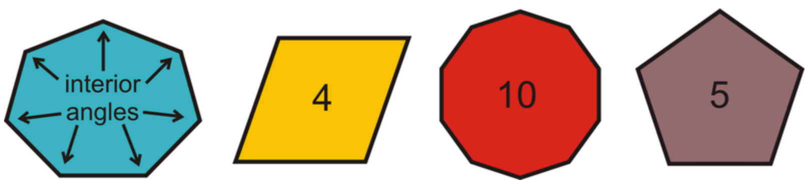
a. Find $w^\circ, x^\circ, y^\circ$, and z°.
b. What is $w^\circ + y^\circ + z^\circ$?
c. What two angles add up to y°?
d. What are $72^\circ, 59^\circ$, and x° called? What are w°, y°, and z° called?

Know What? To the right is a picture of Devil's Post pile, near Mammoth Lakes, California. These posts are cooled lava (called columnar basalt) and as the lava pools and cools, it ideally would form regular hexagonal columns. However, variations in cooling caused some columns to either not be perfect or pentagonal.

First, define regular in your own words. Then, what is the sum of the angles in a regular hexagon? What would each angle be?

Interior Angles in Convex Polygons

Recall from Chapter 4, that interior angles are the angles inside a closed figure with straight sides. Even though this concept was introduced with triangles, it can be extended to any polygon. As you can see in the images below, a polygon has the same number of interior angles as it does sides.

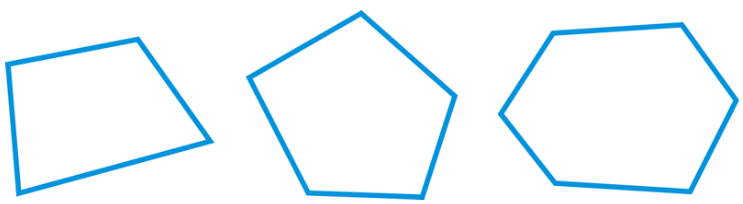


From Chapter 1, we learned that a diagonal connects two non-adjacent vertices of a convex polygon. Also, recall that the sum of the angles in a triangle is 180°. What about other polygons?

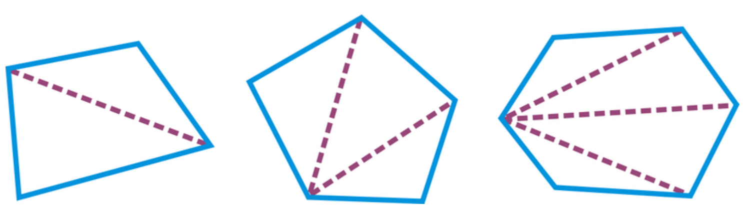
Investigation 6-1: Polygon Sum Formula

Tools Needed: paper, pencil, ruler, colored pencils (optional)

1. Draw a quadrilateral, pentagon, and hexagon.



2. Cut each polygon into triangles by drawing all the diagonals from one vertex. Count the number of triangles.



Make sure none of the triangles overlap.

3. Make a table with the information below.

TABLE 6.1:

Name of Polygon	*Number of Sides*	*Number of $\triangle$s from one vertex*	*(Column 3) $\times$ ($^\circ$ in a $\triangle$)*	*Total Number of Degrees*
Quadrilateral	4	2	$2 \times 180^\circ$	360°
Pentagon	5	3	$3 \times 180^\circ$	540°
Hexagon	6	4	$4 \times 180^\circ$	720°

4. Do you see a pattern? Notice that the total number of degrees goes up by 180°. So, if the number sides is n, then the number of triangles from one vertex is $n-2$. Therefore, the formula would be $(n-2) \times 180^\circ$.

Polygon Sum Formula: For any $n-$gon, the sum of the interior angles is $(n-2) \times 180^\circ$.

Example 1: Find the sum of the interior angles of an octagon.

Solution: Use the Polygon Sum Formula and set $n = 8$.

$$(8-2) \times 180^\circ = 6 \times 180^\circ = 1080^\circ$$

Example 2: The sum of the interior angles of a polygon is 1980°. How many sides does this polygon have?

Solution: Use the Polygon Sum Formula and solve for n.

$$\begin{aligned}(n-2) \times 180^\circ &= 1980^\circ \\ 180^\circ n - 360^\circ &= 1980^\circ \\ 180^\circ n &= 2340^\circ \\ n &= 13 \qquad \text{The polygon has 13 sides.}\end{aligned}$$

Example 3: How many degrees does each angle in an equiangular nonagon have?

Solution: First we need to find the sum of the interior angles in a nonagon, set $n = 9$.

$$(9-2) \times 180^\circ = 7 \times 180^\circ = 1260^\circ$$

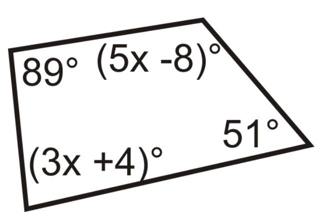
Second, because the nonagon is equiangular, every angle is equal. Dividing 1260° by 9 we get each angle is 140°.

Equiangular Polygon Formula: For any equiangular $n-$gon, the measure of each angle is $\frac{(n-2) \times 180^\circ}{n}$.

Regular Polygon: When a polygon is equilateral and equiangular.

It is important to note that in the Equiangular Polygon Formula, the word *equiangular* can be substituted with *regular*.

Example 4: ***Algebra Connection*** Find the measure of x.

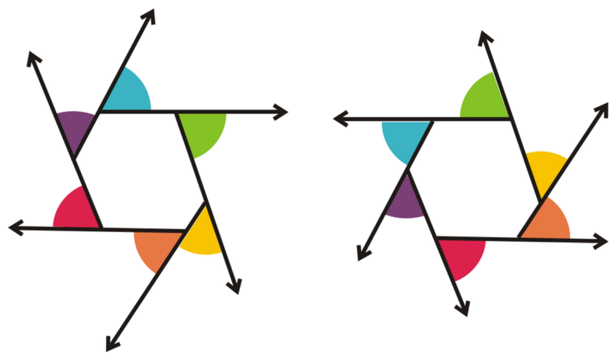


Solution: From our investigation, we found that a quadrilateral has 360°. We can write an equation to solve for x.

$$\begin{aligned} 89^\circ + (5x-8)^\circ + (3x+4)^\circ + 51^\circ &= 360^\circ \\ 8x &= 224^\circ \\ x &= 28^\circ \end{aligned}$$

Exterior Angles in Convex Polygons

Recall that an exterior angle is an angle on the outside of a polygon and is formed by extending a side of the polygon (Chapter 4).



As you can see, there are two sets of exterior angles for any vertex on a polygon. It does not matter which set you use because one set is just the vertical angles of the other, making the measurement equal. In the picture to the left, the color-matched angles are vertical angles and congruent.

In Chapter 4, we introduced the Exterior Angle Sum Theorem, which stated that the exterior angles of a triangle add up to 360°. Let's extend this theorem to all polygons.

Investigation 6-2: Exterior Angle Tear-Up

Tools Needed: pencil, paper, colored pencils, scissors

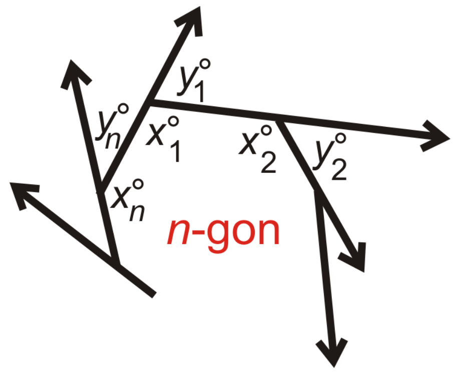
a. Draw a hexagon like the hexagons above. Color in the exterior angles as well.
b. Cut out each exterior angle and label them 1-6.

c. Fit the six angles together by putting their vertices together. What happens?

The angles all fit around a point, meaning that the exterior angles of a hexagon add up to 360°, just like a triangle. We can say this is true for all polygons.

Exterior Angle Sum Theorem: The sum of the exterior angles of any polygon is 360°.

Proof of the Exterior Angle Sum Theorem



Given: Any $n-$gon with n sides, n interior angles and n exterior angles.

Prove: n exterior angles add up to 360°

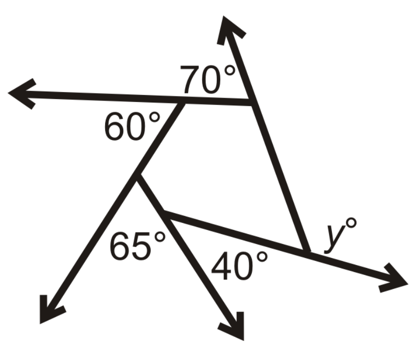
NOTE: The interior angles are $x_1, x_2, \dots x_n$.

The exterior angles are $y_1, y_2, \dots y_n$.

TABLE 6.2:

Statement	*Reason*
1. Any $n-$gon with n sides, n interior angles and n exterior angles.	Given
2. x_n° *and* y_n° are a linear pair	Definition of a linear pair
3. x_n° *and* y_n° are supplementary	Linear Pair Postulate
4. $x_n^\circ + y_n^\circ = 180^\circ$	Definition of supplementary angles
5. $(x_1^\circ + x_2^\circ + \dots + x_n^\circ) + (y_1^\circ + y_2^\circ + \dots + y_n^\circ) = 180^\circ n$	Sum of all interior and exterior angles in an $n-$gon
6. $(n-2)180^\circ = (x_1^\circ + x_2^\circ + \dots + x_n^\circ)$	Polygon Sum Formula
7. $180^\circ n = (n-2)180^\circ + (y_1^\circ + y_2^\circ + \dots + y_n^\circ)$	Substitution PoE
8. $180^\circ n = 180^\circ n - 360^\circ + (y_1^\circ + y_2^\circ + \dots + y_n^\circ)$	Distributive PoE
9. $360^\circ = (y_1^\circ + y_2^\circ + \dots + y_n^\circ)$	Subtraction PoE

Example 5: What is y?



Solution: y is an exterior angle, as well as all the other given angle measures. Exterior angles add up to 360°, so set up an equation.

$$70^\circ + 60^\circ + 65^\circ + 40^\circ + y = 360^\circ$$
$$y = 125^\circ$$

Example 6: What is the measure of each exterior angle of a regular heptagon?

Solution: Because the polygon is regular, each interior angle is equal. This also means that all the exterior angles are equal. The exterior angles add up to 360°, so each angle is $\frac{360^\circ}{7} \approx 51.43^\circ$.

Know What? Revisited A regular polygon has congruent sides and angles. A regular hexagon has $(6-2)180^\circ = 4 \cdot 180^\circ = 720^\circ$ total degrees. Each angle would be 720° divided by 6 or 120°.

Review Questions

1. Fill in the table.

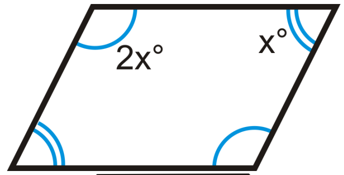
TABLE 6.3:

# of sides	**# of $\triangle s$ from one vertex**	**$\triangle s \times 180^\circ$ (sum)**	**Each angle in a *regular* $n-$gon**	**Sum of the *exterior* angles**
3	1	180°	60°	
4	2	360°	90°	
5	3	540°	108°	
6	4	720°	120°	
7				
8				
9				
10				
11				
12				

2. What is the sum of the angles in a 15-gon?
3. What is the sum of the angles in a 23-gon?
4. The sum of the interior angles of a polygon is 4320°. How many sides does the polygon have?
5. The sum of the interior angles of a polygon is 3240°. How many sides does the polygon have?
6. What is the measure of each angle in a regular 16-gon?
7. What is the measure of each angle in an equiangular 24-gon?
8. What is the measure of each exterior angle of a dodecagon?
9. What is the measure of each exterior angle of a 36-gon?
10. What is the sum of the exterior angles of a 27-gon?
11. If the measure of one interior angle of a regular polygon is 160°, how many sides does it have?
12. How many sides does a regular polygon have if the measure of one of its interior angles is 168°?
13. If the measure of one interior angle of a regular polygon is $158\frac{14}{17}^\circ$, how many sides does it have?
14. How many sides does a regular polygon have if the measure of one exterior angle is 15°?
15. If the measure of one exterior angle of a regular polygon is 36°, how many sides does it have?
16. How many sides does a regular polygon have if the measure of one exterior angle is $32\frac{8}{11}^\circ$?

For questions 11-20, find the measure of the missing variable(s).

17.



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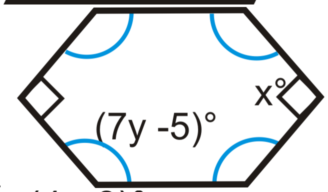

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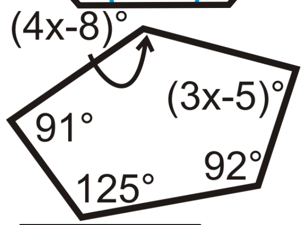

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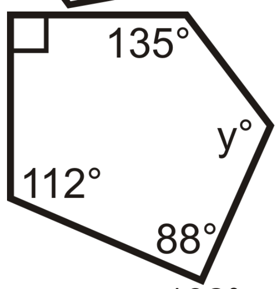

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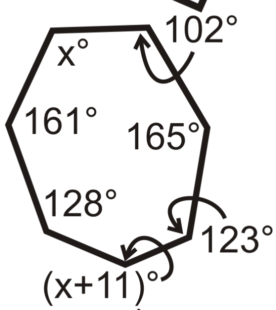

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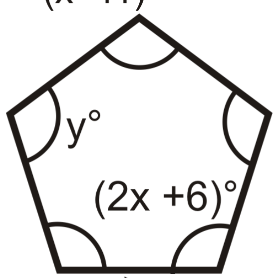

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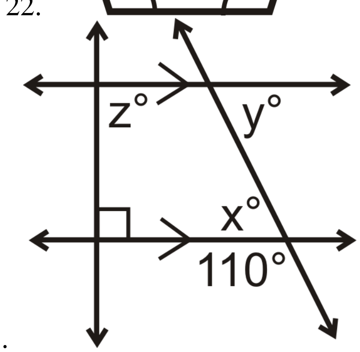
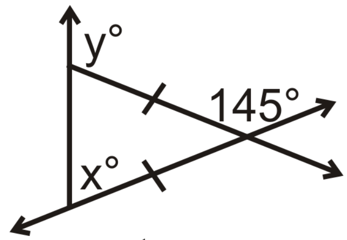
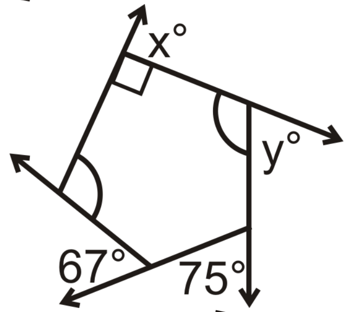
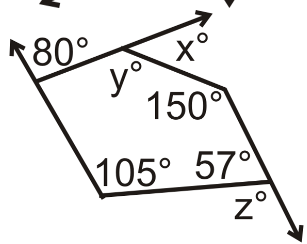
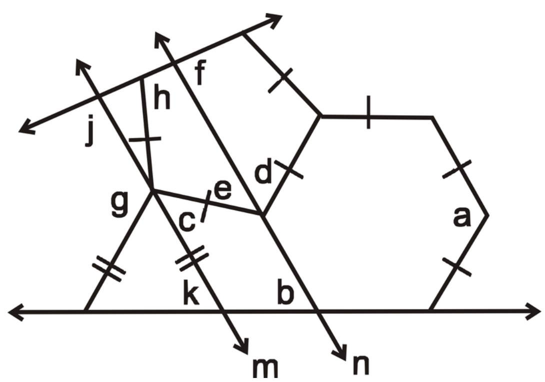


24. 

25. 

26. 

27. The interior angles of a pentagon are $x^\circ, x^\circ, 2x^\circ, 2x^\circ$, and $2x^\circ$. What is the measure of the larger angles?
28. The exterior angles of a quadrilateral are $x^\circ, 2x^\circ, 3x^\circ$, and $4x^\circ$. What is the measure of the smallest angle?
29. The interior angles of a hexagon are $x^\circ, (x+1)^\circ, (x+2)^\circ, (x+3)^\circ, (x+4)^\circ$, and $(x+5)^\circ$. What is x?
30. ***Challenge*** Each interior angle forms a linear pair with an exterior angle. In a regular polygon you can use two different formulas to find the measure of each exterior angle. One way is $\frac{360^\circ}{n}$ and the other is $180^\circ - \frac{(n-2)180^\circ}{n}$ (180° minus Equiangular Polygon Formula). Use algebra to show these two expressions are equivalent.
31. ***Angle Puzzle*** Find the measures of the lettered angles below given that $m \,||\, n$.



Review Queue Answers

a. $$\begin{aligned} 72^\circ + (7x+3)^\circ + (3x+5)^\circ &= 180^\circ \\ 10x + 80^\circ &= 180^\circ \\ 10x &= 100^\circ \\ x &= 10^\circ \end{aligned}$$

b. $$\begin{aligned} (5x+17)^\circ + (3x-5)^\circ &= 180^\circ \\ 8x + 12^\circ &= 180^\circ \end{aligned}$$

$$8x = 168^\circ$$
$$x = 21^\circ$$

a. $w = 108^\circ, x = 49^\circ, y = 131^\circ, z = 121^\circ$
b. 360°
c. $59^\circ + 72^\circ$
d. interior angles, exterior angles

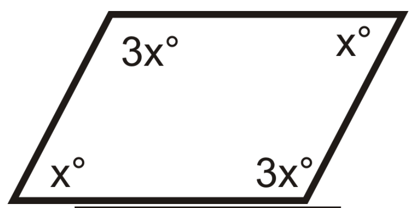
6.2 Properties of Parallelograms

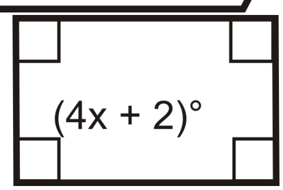
Learning Objectives

- Define a parallelogram.
- Understand the properties of a parallelogram
- Apply theorems about a parallelogram's sides, angles and diagonals.

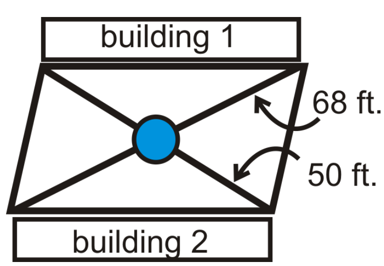
Review Queue

a. Draw a quadrilateral with one set of parallel sides.
b. Draw a quadrilateral with two sets of parallel sides.
c. Find the measure of the missing angles in the quadrilaterals below.

a.


b.


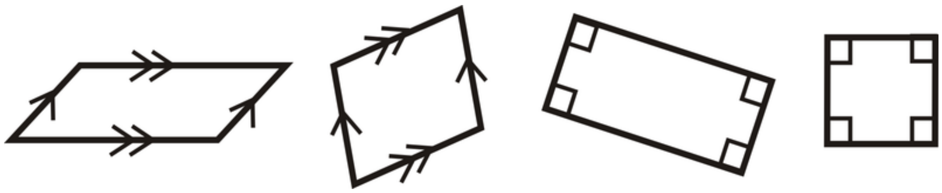
Know What? A college has a parallelogram-shaped courtyard between two buildings. The school wants to build two walkways on the diagonals of the parallelogram with a fountain where they intersect. The walkways are going to be 50 feet and 68 feet long. Where would the fountain be?



What is a Parallelogram?

Parallelogram: A quadrilateral with two pairs of parallel sides.

Here are some examples:

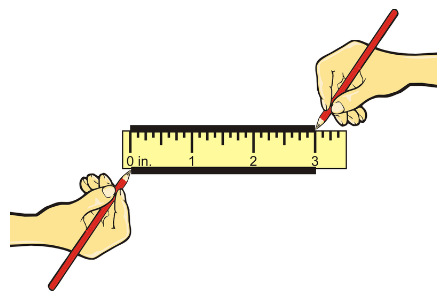


Notice that each pair of sides is marked parallel. As is the case with the rectangle and square, recall that two lines are parallel when they are perpendicular to the same line. Once we know that a quadrilateral is a parallelogram, we can discover some additional properties.

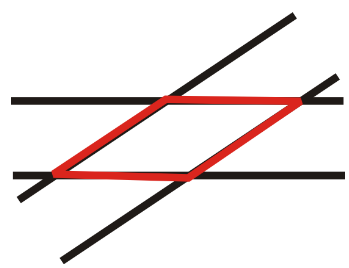
Investigation 6-2: Properties of Parallelograms

Tools Needed: Paper, pencil, ruler, protractor

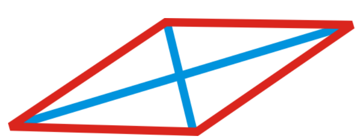
a. Draw a set of parallel lines by placing your ruler on the paper and drawing a line on either side of it. Make your lines 3 inches long.



b. Rotate the ruler and repeat this so that you have a parallelogram. Your second set of parallel lines can be any length. If you have colored pencils, outline the parallelogram in another color.



c. Measure the four interior angles of the parallelogram as well as the length of each side. Can you conclude anything about parallelograms, other than opposite sides are parallel?
d. Draw the diagonals. Measure each and then measure the lengths from the point of intersection to each vertex.



To continue to explore the properties of a parallelogram, see the website:

http://www.mathwarehouse.com/geometry/quadrilaterals/parallelograms/interactive-parallelogram.php

In the above investigation, we drew a parallelogram. From this investigation we can conclude:

- The sides that are parallel are also congruent.
- Opposite angles are congruent.
- Consecutive angles are supplementary.

- The diagonals bisect each other.

Opposite Sides Theorem: If a quadrilateral is a parallelogram, then the opposite sides are congruent.

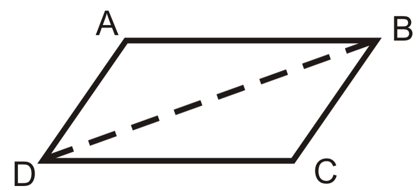
Opposite Angles Theorem: If a quadrilateral is a parallelogram, then the opposite angles are congruent.

Consecutive Angles Theorem: If a quadrilateral is a parallelogram, then the consecutive angles are supplementary.

Parallelogram Diagonals Theorem: If a quadrilateral is a parallelogram, then the diagonals bisect each other.

To prove the first three theorems, one of the diagonals must be added to the figure and then the two triangles can be proved congruent.

Proof of Opposite Sides Theorem



Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$

Prove: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

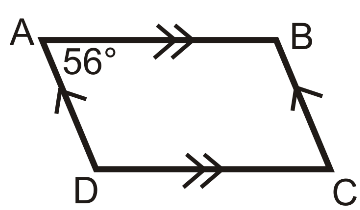
TABLE 6.4:

Statement	***Reason***
1. $ABCD$ is a parallelogram with diagonal $\overline{BD}$	Given
2. $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$	Definition of a parallelogram
3. $\angle ABD \cong BDC$, $\angle ADB \cong DBC$	Alternate Interior Angles Theorem
4. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
5. $\triangle ABD \cong \triangle CDB$	ASA
6. $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	CPCTC

The proof of the Opposite Angles Theorem is almost identical. For the last step, the angles are congruent by CPCTC. You will prove the other three theorems in the review questions.

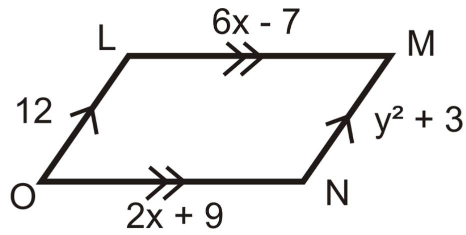
Example 1: $ABCD$ is a parallelogram. If $m\angle A = 56^\circ$, find the measure of the other three angles.

Solution: Draw a picture. When labeling the vertices, the letters are listed, in order, clockwise.



If $m\angle A = 56^\circ$, then $m\angle C = 56^\circ$ because they are opposite angles. $\angle B$ and $\angle D$ are consecutive angles with $\angle A$, so they are both supplementary to $\angle A$. $m\angle A + m\angle B = 180^\circ, 56^\circ + m\angle B = 180^\circ, m\angle B = 124^\circ$. $m\angle D = 124^\circ$.

Example 2: ***Algebra Connection*** Find the values of x and y.



Solution: Opposite sides are congruent, so we can set each pair equal to each other and solve both equations.

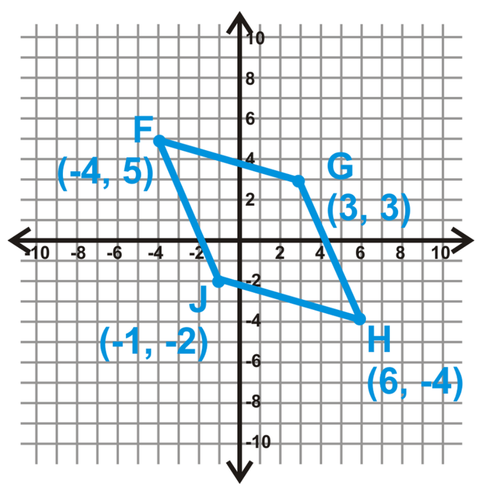
$$\begin{aligned} 6x - 7 &= 2x + 9 \\ 4x &= 16 \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} y^2 + 3 &= 12 \\ y^2 &= 9 \\ y &= 3 \ or \ -3 \end{aligned}$$

Even though $y = 3$ or -3, lengths cannot be negative, so $y = 3$.

Diagonals in a Parallelogram

From the Parallelogram Diagonals Theorem, we know that the diagonals of a parallelogram bisect each other.

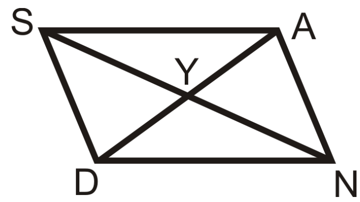
Example 3: Show that the diagonals of $FGHJ$ bisect each other.



Solution: The easiest way to show this is to find the midpoint of each diagonal. If it is the same point, you know they intersect at each other's midpoint and, by definition, cuts a line in half.

$$\text{Midpoint of } \overline{FH}: \left(\frac{-4+6}{2}, \frac{5-4}{2}\right) = (1, 0.5)$$
$$\text{Midpoint of } \overline{GJ}: \left(\frac{3-1}{2}, \frac{3-2}{2}\right) = (1, 0.5)$$

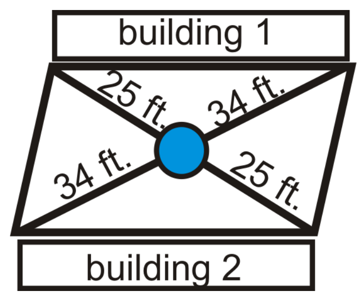
Example 4: ***Algebra Connection*** $SAND$ is a parallelogram and $SY = 4x - 11$ and $YN = x + 10$. Solve for x.



Solution: $\overline{AD}$ and $\overline{SN}$ bisect each other, so $SY = YN$.

$$\begin{aligned} 4x - 11 &= x + 10 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Know What? Revisited By the Parallelogram Diagonals Theorem, the fountain is going to be 34 feet from either endpoint on the 68 foot diagonal and 25 feet from either endpoint on the 50 foot diagonal.



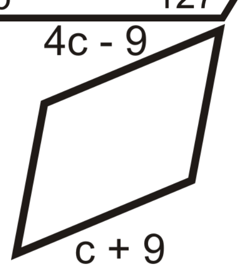
Review Questions

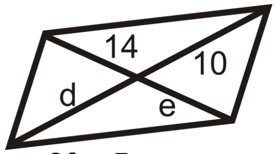
1. If $m\angle B = 72^\circ$ in parallelogram $ABCD$, find the other three angles.
2. If $m\angle S = 143^\circ$ in parallelogram $PQRS$, find the other three angles.
3. If $\overline{AB} \perp \overline{BC}$ in parallelogram $ABCD$, find the measure of all four angles.
4. If $m\angle F = x^\circ$ in parallelogram $EFGH$, find expressions for the other three angles in terms of x.

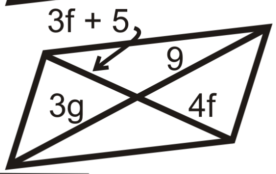
For questions 5-13, find the measures of the variable(s). All the figures below are parallelograms.

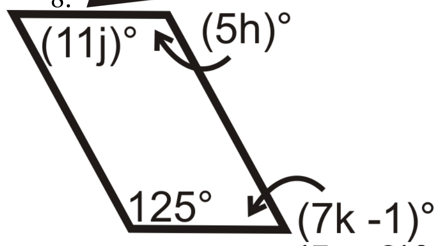
5.

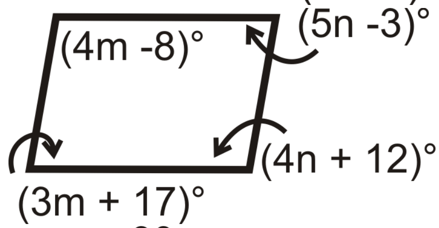
6.

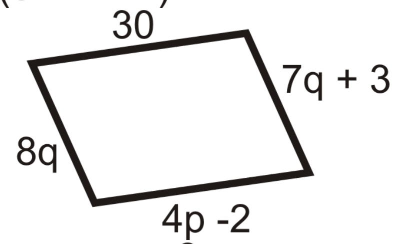


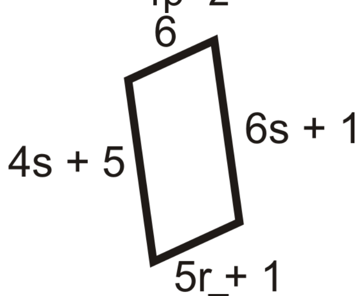
7. 

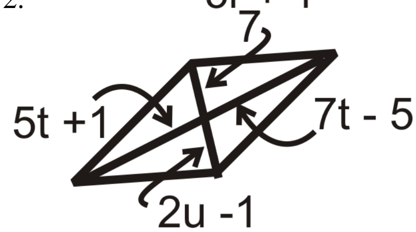
8. 

9. 

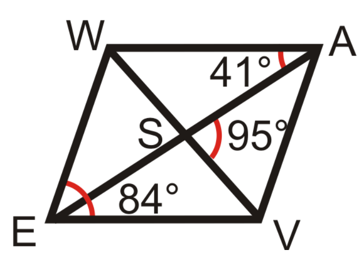
10. 

11. 

12. 

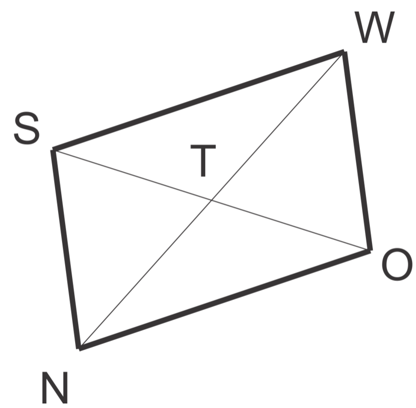
13. 

Use the parallelogram $WAVE$ to find:



14. $m\angle AWE$
15. $m\angle ESV$
16. $m\angle WEA$
17. $m\angle AVW$

In the parallelogram $SNOW, ST = 6, NW = 4, m\angle OSW = 36^\circ, m\angle SNW = 58^\circ$ and $m\angle NTS = 80^\circ$. (*diagram is not drawn to scale*)



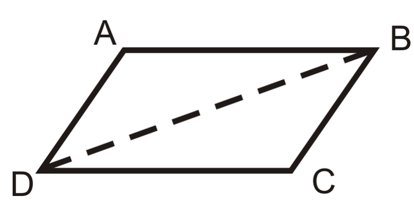
18. SO
19. NT
20. $m\angle NWS$
21. $m\angle SOW$

Plot the points $E(-1,3), F(3,4), G(5,-1), H(1,-2)$ and use parallelogram $EFGH$ for problems 22-25.

22. Find the coordinates of the point at which the diagonals intersect. How did you do this?
23. Find the slopes of all four sides. What do you notice?
24. Use the distance formula to find the lengths of all four sides. What do you notice?
25. Make a conjecture about how you might determine whether a quadrilateral in the coordinate is a parallelogram.

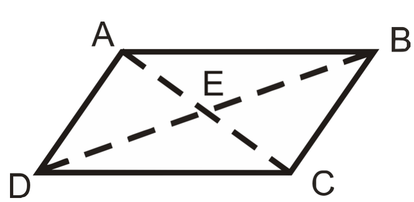
Write a two-column proof.

26. ***Opposite Angles Theorem***



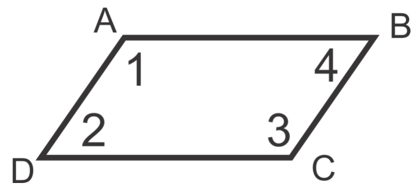
Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$Prove: $\angle A \cong \angle C$

27. ***Parallelogram Diagonals Theorem***



Given: $ABCD$ is a parallelogram with diagonals $\overline{BD}$ and $\overline{AC}$Prove: $\overline{AE} \cong \overline{EC}$, $\overline{DE} \cong \overline{EB}$

28. Fill in the blanks for the proof of the ***Consecutive Angles Theorem***

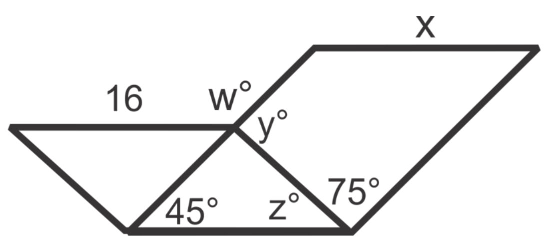


Given: $ABCD$ is a parallelogram Prove: $m\angle 1 + m\angle 2 = 180^\circ$

TABLE 6.5:

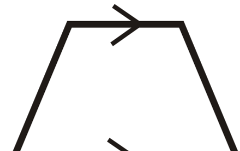
Statements	***Reasons***
1.	Given
2. $m\angle 1 = m\angle 3$ and ________	
3. $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 360^\circ$	
4. $m\angle 1 + m\angle 2 + m\angle 1 + m\angle 2 = 360^\circ$	
5. $2(m\angle 1 + m\angle 2) = 360^\circ$	
6.	Division POE

Use the diagram below to find the indicated lengths or angle measures for problems 29-32. The two quadrilaterals that share a side are parallelograms.



29. w
30. x
31. y
32. z

Review Queue Answers



a.

b.

c. $$\begin{aligned} 3x+x+3x+x &= 360^\circ \\ 8x &= 360^\circ \\ x &= 45^\circ \end{aligned}$$

d. $$\begin{aligned} 4x+2 &= 90^\circ \\ 4x &= 88^\circ \\ x &= 22^\circ \end{aligned}$$

6.3 Proving Quadrilaterals are Parallelograms

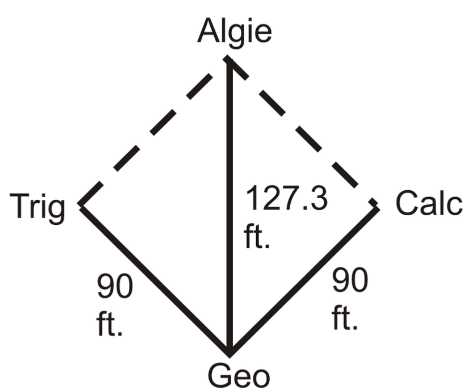
Learning Objectives

- Prove a quadrilateral is a parallelogram using the converses of the theorems from the previous section.
- Prove a quadrilateral is a parallelogram in the coordinate plane.

Review Queue

a. Write the converses of: the Opposite Sides Theorem, Opposite Angles Theorem, Consecutive Angles Theorem and the Parallelogram Diagonals Theorem.
b. Are any of these converses true? If not, find a counterexample.
c. Plot the points $A(2,2)$, $B(4,-2)$, $C(-2,-4)$, and $D(-6,-2)$.
    a. Find the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$. Is $ABCD$ a parallelogram?
    b. Find the point of intersection of the diagonals. Does this go along with what you found in part a?

Know What? Four friends, Geo, Trig, Algie, and Calc are marking out a baseball diamond. Geo is standing at home plate. Trig is 90 feet away at 3^{rd} base, Algie is 127.3 feet away at 2^{nd} base, and Calc is 90 feet away at 1^{st} base. The angle at home plate is 90°, from 1^{st} to 3^{rd} is 90°. Find the length of the other diagonal and determine if the baseball diamond is a parallelogram. If it is, what kind of parallelogram is it?



Determining if a Quadrilateral is a Parallelogram

In the last section, we introduced the Opposite Sides Theorem, Opposite Angles Theorem, Consecutive Angles Theorem and the Parallelogram Diagonals Theorem. #1 in the Review Queue above, had you write the converses of each of these:

Opposite Sides Theorem Converse: If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

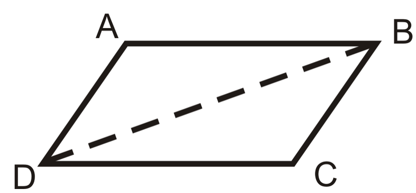
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse: If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

Are these converses true? The answer is yes. Each of these converses can be a way to show that a quadrilateral is a parallelogram. However, the Consecutive Angles Converse can be a bit tricky, considering you would have to show that each angle is supplementary to its neighbor ($\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle A$ and $\angle D$). We will not use this converse.

Proof of the Opposite Sides Theorem Converse



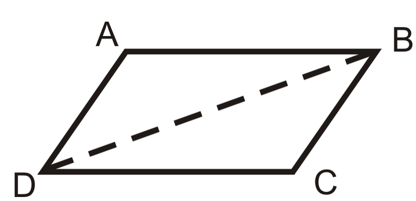
Given: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

Prove: $ABCD$ is a parallelogram

TABLE 6.6:

Statement	***Reason***
1. $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	Given
2. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
3. $\triangle ABD \cong \triangle CDB$	SSS
4. $\angle ABD \cong \angle BDC$, $\angle ADB \cong \angle DBC$	CPCTC
5. $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$	Alternate Interior Angles Converse
6. $ABCD$ is a parallelogram	Definition of a parallelogram

Example 1: Write a two-column proof.



Given: $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$

Prove: $ABCD$ is a parallelogram

Solution:

TABLE 6.7:

Statement	***Reason***
1. $\overline{AB} \parallel \overline{DC}$ and $\overline{AB} \cong \overline{DC}$	Given
2. $\angle ABD \cong \angle BDC$	Alternate Interior Angles

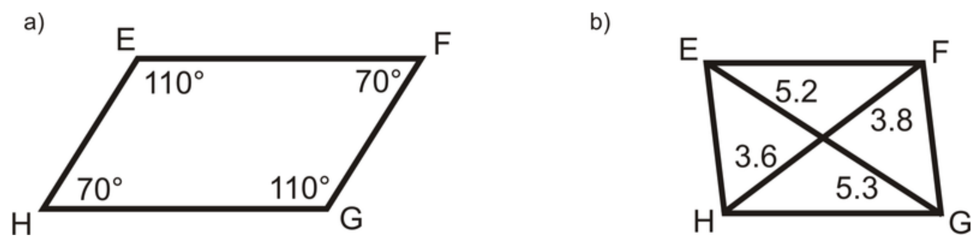
TABLE 6.7: (continued)

Statement	Reason
3. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
4. $\triangle ABD \cong \triangle CDB$	SAS
5. $\overline{AD} \cong \overline{BC}$	CPCTC
6. $ABCD$ is a parallelogram	Opposite Sides Converse

Example 1 proves an additional way to show that a quadrilateral is a parallelogram.

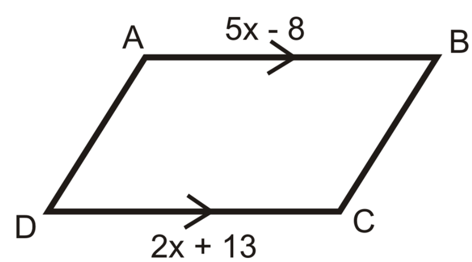
Theorem 5-10: If a quadrilateral has one set of parallel lines that are also congruent, then it is a parallelogram.

Example 2: Is quadrilateral $EFGH$ a parallelogram? How do you know?



Solution: For part a, the opposite angles are equal, so by the Opposite Angles Theorem Converse, $EFGH$ is a parallelogram. In part b, the diagonals do not bisect each other, so $EFGH$ is not a parallelogram.

Example 3: ***Algebra Connection*** What value of x would make $ABCD$ a parallelogram?



Solution: $\overline{AB} \,||\, \overline{DC}$ from the markings. By Theorem 5-10, $ABCD$ would be a parallelogram if $AB = DC$ as well.

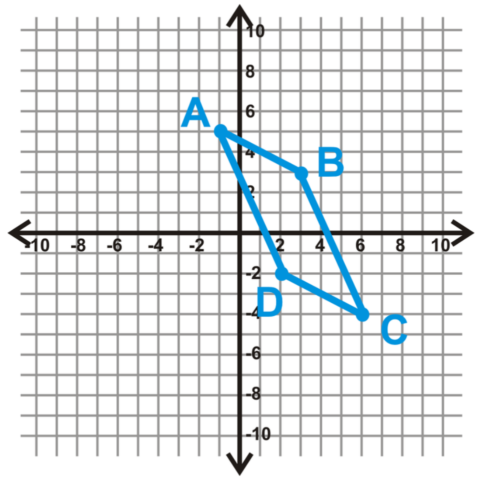
$$\begin{aligned} 5x - 8 &= 2x + 13 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

In order for $ABCD$ to be a parallelogram, x must equal 7.

Showing a Quadrilateral is a Parallelogram in the Coordinate Plane

To show that a quadrilateral is a parallelogram in the $x-y$ plane, you will need to use a combination of the slope formulas, the distance formula and the midpoint formula. For example, to use the Definition of a Parallelogram, you would need to ***find the slope of all four sides*** to see if the opposite sides are parallel. To use the Opposite Sides Converse, you would have to find the length (***using the distance formula***) of each side to see if the opposite sides are congruent. To use the Parallelogram Diagonals Converse, you would need to use the ***midpoint formula*** for each diagonal to see if the midpoint is the same for both. Finally, you can use Theorem 5-10 in the coordinate plane. To use this theorem, you would need to show that one pair of opposite sides has the same slope (***slope formula***) and the same length (***distance formula***).

Example 4: Is the quadrilateral $ABCD$ a parallelogram?



Solution: We have determined there are four different ways to show a quadrilateral is a parallelogram in the $x-y$ plane. Let's use Theorem 5-10. First, find the length of AB and CD.

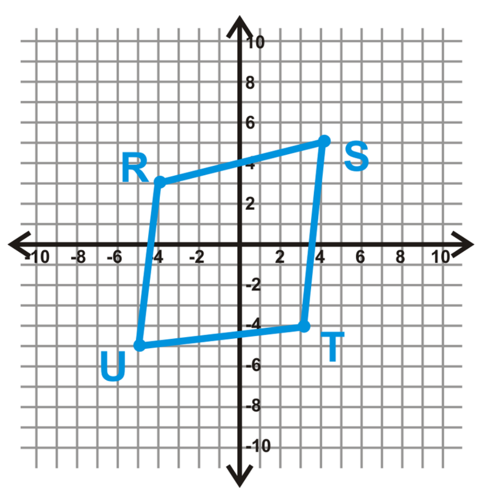
$$\begin{aligned} AB &= \sqrt{(-1-3)^2+(5-3)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned} \qquad \begin{aligned} CD &= \sqrt{(2-6)^2+(-2+4)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned}$$

$AB = CD$, so if the two lines have the same slope, $ABCD$ is a parallelogram.

Slope $AB = \frac{5-3}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$ Slope $CD = \frac{-2+4}{2-6} = \frac{2}{-4} = -\frac{1}{2}$

By Theorem 5-10, $ABCD$ is a parallelogram.

Example 5: Is the quadrilateral $RSTU$ a parallelogram?



Solution: Let's use the Parallelogram Diagonals Converse to determine if $RSTU$ is a parallelogram. Find the midpoint of each diagonal.

Midpoint of $RT = \left(\frac{-4+3}{2}, \frac{3-4}{2}\right) = (-0.5, -0.5)$

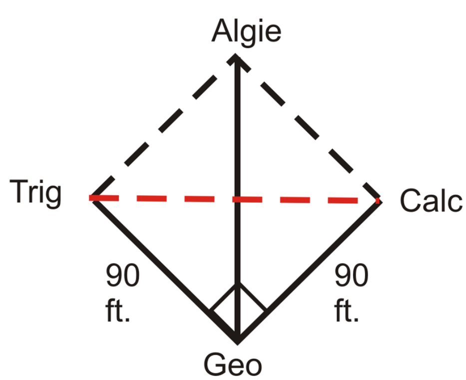
Midpoint of $SU = \left(\frac{4-5}{2}, \frac{5-5}{2}\right) = (-0.5, 0)$

Because the midpoint is not the same, $RSTU$ is not a parallelogram.

Know What? Revisited First, we can use the Pythagorean Theorem to find the length of the second diagonal.

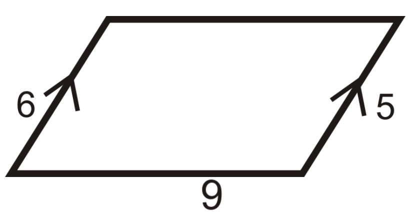
$$\begin{aligned}90^2 + 90^2 &= d^2\\ 8100 + 8100 &= d^2\\ 16200 &= d^2\\ d &= 127.3\end{aligned}$$

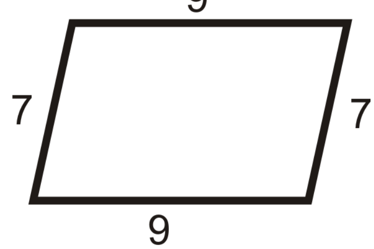
This means that the diagonals are equal. If the diagonals are equal, the other two sides of the diamond are also 90 feet. Therefore, the baseball diamond is a parallelogram, and more specifically, it is a square.

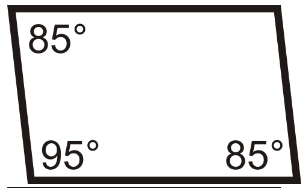


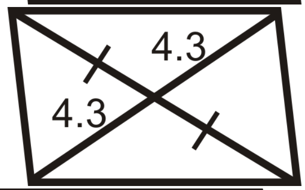
Review Questions

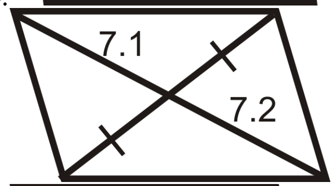
For questions 1-12, determine if the quadrilaterals are parallelograms. If they are, write a reason.

1. 

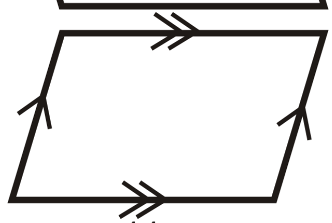
2. 

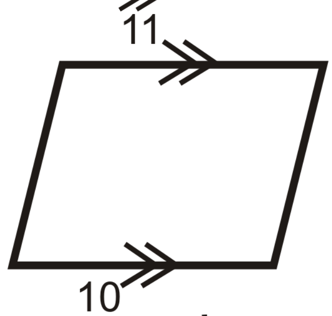
3. 

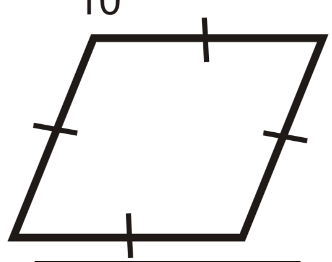
4. 

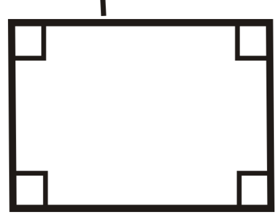
5. 

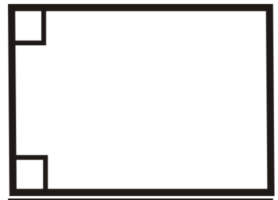
6. 

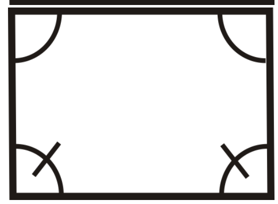
7. 

8. 

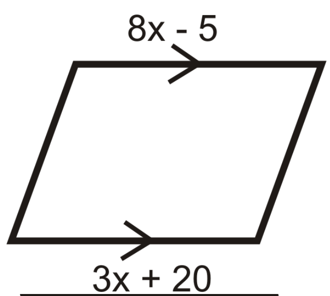
9. 

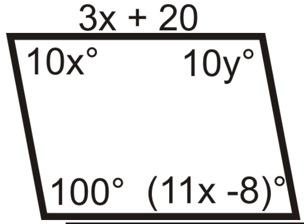
10. 

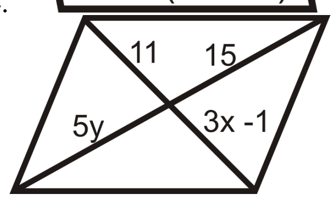
11. 

12. 

For questions 13-15, determine the value of x and y that would make the quadrilateral a parallelogram.

13. 

14. 

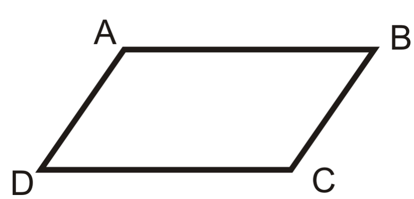
15. 

For questions 16-18, determine if $ABCD$ is a parallelogram.

16. $A(8,-1),\ B(6,5),\ C(-7,2),\ D(-5,-4)$
17. $A(-5,8),\ B(-2,9),\ C(3,4),\ D(0,3)$
18. $A(-2,6),\ B(4,-4),\ C(13,-7),\ D(4,-10)$

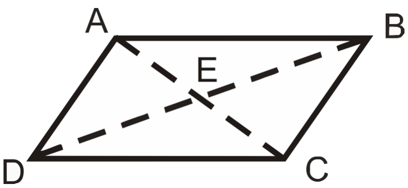
Write a two-column proof.

19. ***Opposite Angles Theorem Converse***

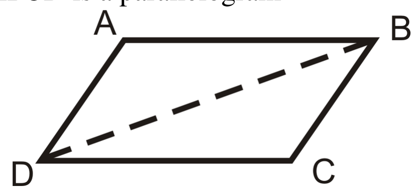


Given: $\angle A \cong \angle C,\ \angle D \cong \angle B$Prove: $ABCD$ is a parallelogram

20. ***Parallelogram Diagonals Theorem Converse***



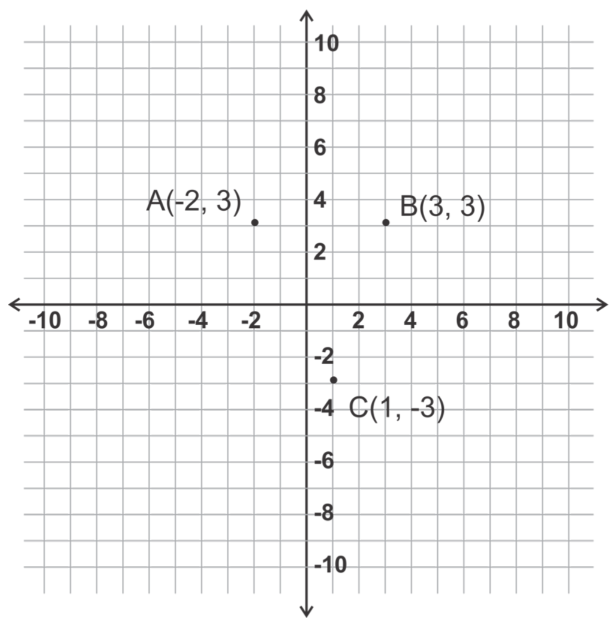
Given: $\overline{AE} \cong \overline{EC}$, $\overline{DE} \cong \overline{EB}$Prove: $ABCD$ is a parallelogram



21.

Given: $\angle ADB \cong CBD$, $\overline{AD} \cong \overline{BC}$Prove: $ABCD$ is a parallelogram

Suppose that $A(-2,3)$, $B(3,3)$ and $C(1,-3)$ are three of four vertices of a parallelogram.



22. Depending on where you choose to put point D, the name of the parallelogram you draw will change. Sketch a picture to show all possible parallelograms. How many can you draw?
23. If you know the parallelogram is named $ABDC$, what is the slope of side parallel to $\overline{AC}$?
24. Again, assuming the parallelogram is named $ABDC$, what is the length of $\overline{BD}$?
25. Find the points of intersection of the diagonals of the three parallelograms formed. Label them X in parallelogram $ABCD$, Y in parallelogram $ADBC$ and Z in parallelogram $ABDC$.
26. Connect the points X, Y and Z to form a triangle. What do you notice about this triangle?

The points $Q(-1,1)$, $U(7,1)$, $A(1,7)$ and $D(-1,5)$ are the vertices of quadrilateral $QUAD$. Plot the points on graph paper to complete problems 27-30.

27. Find the midpoints of sides $\overline{QU}$, $\overline{UA}$, $\overline{AD}$ and $\overline{DQ}$. Label them W, X, Y and Z respectively.
28. Connect the midpoints to form quadrilateral $WXYZ$. What does this quadrilateral appear to be?
29. Use slopes to verify your answer to problem 29.
30. Use midpoints to verify your answer to problem 29.
31. This phenomenon occurs in all quadrilaterals. Describe how you might prove this fact. (Hint: each side of quadrilateral $WXYZ$ is a midsegment in a triangle formed by two sides of the parallelogram and a diagonal.)

Review Queue Answers

1. **Opposite Sides Theorem Converse:** If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

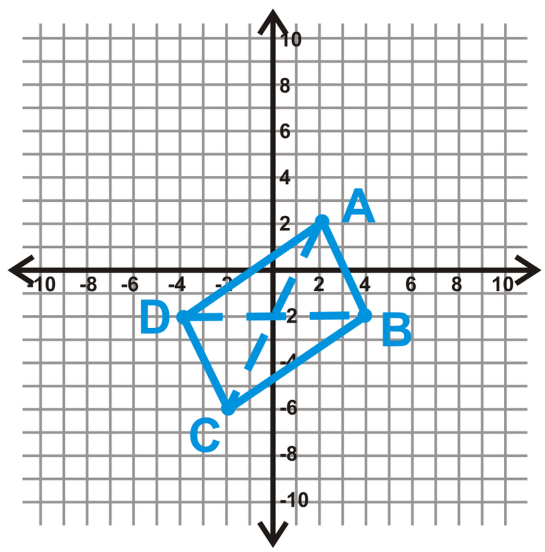
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse: If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

2. All the converses are true.

3.



a) Slope AB = Slope $CD = -\frac{1}{2}$
Slope AD = Slope $BC = \frac{2}{3}$

ABCD is a parallelogram because the opposite sides are parallel.

b) Midpoint of $BD = (0, -2)$
Midpoint of $AC = (0, -2)$

Yes, the midpoint of the diagonals are the same, so they bisect each other. This corresponds with what we found in part a.

6.4 Rectangles, Rhombuses and Squares

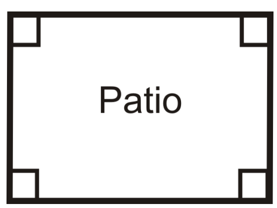
Learning Objectives

- Define and analyze a rectangle, rhombus, and square.
- Determine if a parallelogram is a rectangle, rhombus, or square in the coordinate plane.
- Analyze the properties of the diagonals of a rectangle, rhombus, and square.

Review Queue

a. Define rectangle in your own words. Is a rectangle a parallelogram?
b. Define square in your own words. Is a square a parallelogram? Is it a rectangle?
c. List five examples where you might see a square, rectangle, or rhombus in real life.

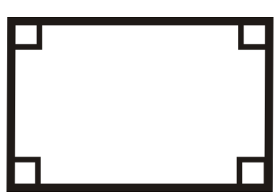
Know What? You are designing a patio for you backyard. You decide to mark it off using your tape measure. Two sides are ***21 feet*** long and two sides are ***28 feet*** long. Explain how you would only use the tape measure to make your patio a rectangle.



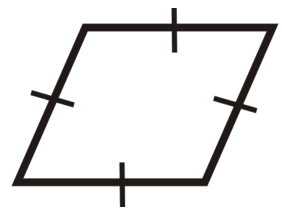
Defining Special Parallelograms

Rectangles, Rhombuses (the plural is also Rhombi) and Squares are all more specific versions of parallelograms, also called special parallelograms. Taking the theorems we learned in the previous two sections, we have three more new theorems.

Rectangle Theorem: A quadrilateral is a rectangle if and only if it has four right (congruent) angles.

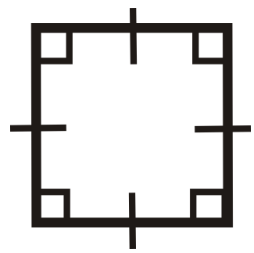


Rhombus Theorem: A quadrilateral is a rhombus if and only if it has four congruent sides.



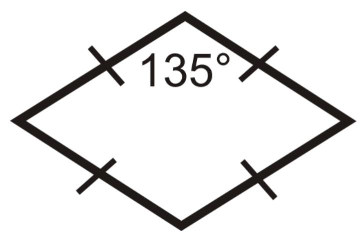
SquareTheorem: A quadrilateral is a square if and only if it has four right angles and four congruent sides.

From the Square Theorem, we can also conclude that a ***square is a rectangle and a rhombus***.

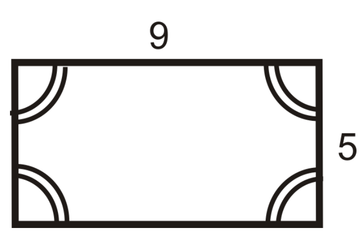


Example 1: What type of parallelogram are the ones below?

a)



b)



Solution:

a) All sides are congruent and one angle is 135°, meaning that the angles are not congruent. By the Rhombus Theorem, this is a rhombus.

b) This quadrilateral has four congruent angles and all the sides are not congruent. By the Rectangle Theorem, this is a rectangle.

Example 2: Is a rhombus SOMETIMES, ALWAYS, or NEVER a square? Explain your reasoning.

Solution: A rhombus has four congruent sides, while a square has four congruent sides and angles. Therefore, a rhombus is only a square when it also has congruent angles. So, a rhombus is SOMETIMES a square.

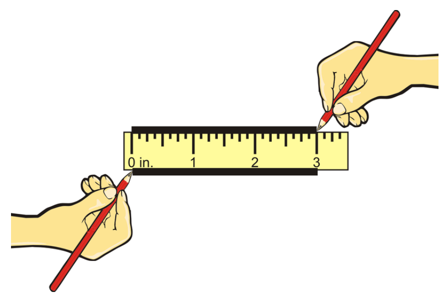
Diagonals in Special Parallelograms

Recall from previous lessons that the ***diagonals in a parallelogram bisect each other***. Therefore, the diagonals of a rectangle, square and rhombus also bisect each other. The diagonals of these parallelograms also have additional properties.

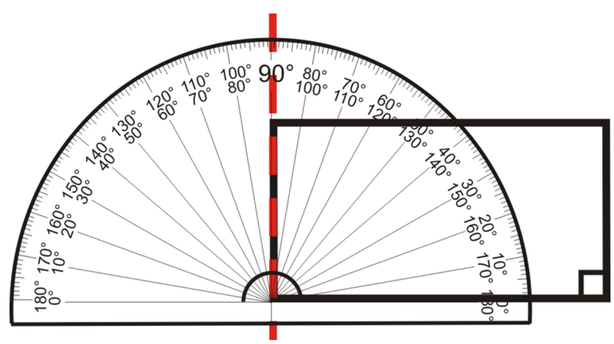
Investigation 6-3: Drawing a Rectangle

Tools Needed: pencil, paper, protractor, ruler

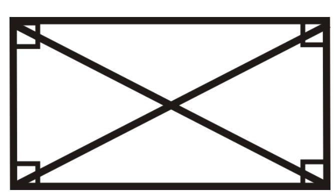
a. Like with Investigation 6-2, draw two lines on either side of your ruler, to ensure they are parallel. Make these lines 3 inches long.



b. Remove the ruler and mark two 90° angles, 2.5 inches apart on the bottom line drawn in Step 1. Then, draw the angles to intersect the top line. This will ensure that all four angles are 90°. Depending on your ruler, the sides should be 2.5 inches and 1 inch.



c. Draw in the diagonals and measure them. What do you discover?



Theorem 6-14: A parallelogram is a rectangle if and only if the diagonals are congruent.

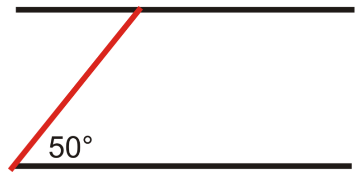
Notice, we did not say *any quadrilateral*. There are quadrilaterals that have congruent diagonals and are not parallelograms.

Investigation 6-4: Drawing a Rhombus

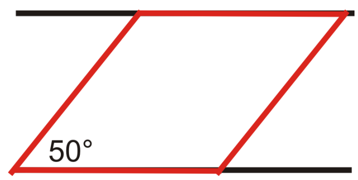
Tools Needed: pencil, paper, protractor, ruler

a. Like with Investigation 6-2 and 6-3, draw two lines on either side of your ruler, to ensure they are parallel. Make these lines 3 inches long.

b. Remove the ruler and mark a 50° angle, at the left end of the bottom line drawn in Step 1. Draw the other side of the angle and make sure it intersects the top line. Measure the length of this side.

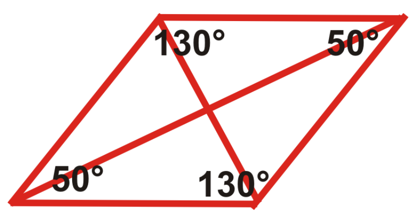


c. The measure of the diagonal (red) side should be about 1.3 inches (if your ruler is 1 inch wide). Mark this length on the bottom line and the top line from the point of intersection with the 50° angle. Draw in the fourth side. It will connect the two endpoints of these lengths.



d. By the way we drew this parallelogram; it is a rhombus because all four sides are 1.3 inches long. Draw in the diagonals.

Measure the angles created by the diagonals: the angles at their point of intersection and the angles created by the sides and each diagonal. You should find the measure of 12 angles total. What do you discover?

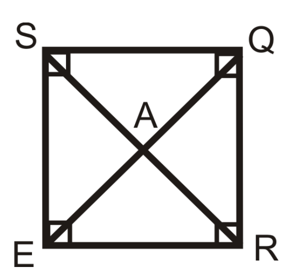


Theorem 6-15: A parallelogram is a rhombus if and only if the diagonals are perpendicular.

Theorem 6-16: A parallelogram is a rhombus if and only if the diagonals bisect each angle.

There are no theorems about the diagonals of a square. We know that a square is a rhombus and a rectangle. So, the diagonals of a square have the properties of a rhombus and a rectangle.

Example 3: List everything you know about the square $SQRE$.



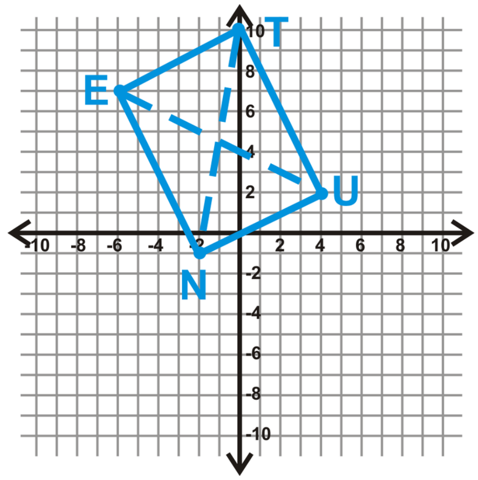
Solution: A square has all the properties of a parallelogram, rectangle and rhombus.

TABLE 6.8:

Properties of Parallelograms	*Properties of Rhombuses*	*Properties of Rectangles*
• $\overline{SQ} \parallel \overline{ER}$	• $\overline{SQ} \cong \overline{ER} \cong \overline{SE} \cong \overline{QR}$	• $\angle SER \cong \angle SQR \cong \angle QSE \cong \angle QRE$
• $\overline{SE} \parallel \overline{QR}$	• $\overline{SR} \perp \overline{QE}$	• $\overline{SR} \cong \overline{QE}$
• $\overline{SQ} \cong \overline{ER}$	• $\angle SEQ \cong \angle QER \cong \angle SQE \cong \angle EQR$	• $\overline{SA} \cong \overline{AR} \cong \overline{QA} \cong \overline{AE}$
• $\overline{SE} \cong \overline{QR}$	• $\angle QSR \cong \angle RSE \cong \angle QRS \cong \angle SRE$	
• $\overline{SA} \cong \overline{AR}$		
• $\overline{QA} \cong \overline{AE}$		
• $\angle SER \cong \angle SQR$		
• $\angle QSE \cong \angle QRE$		

Parallelograms in the Coordinate Plane

Example 4: Determine what type of parallelogram $TUNE$ is: $T(0,10)$, $U(4,2)$, $N(-2,-1)$, and $E(-6,7)$.



Solution: First, let's double-check and make sure the diagonals bisect each other.

Midpoint of $EU = \left(\frac{-6+4}{2}, \frac{7+2}{2}\right) = (-1, 4.5)$

Midpoint of $TN = \left(\frac{0-2}{2}, \frac{10-1}{2}\right) = (-1, 4.5)$

Now, let's see if the diagonals are equal. If they are, then $TUNE$ is a rectangle.

$$\begin{aligned} EU &= \sqrt{(-6-4)^2+(7-2)^2} \\ &= \sqrt{(-10)^2+5^2} \\ &= \sqrt{100+25} \\ &= \sqrt{125} \end{aligned} \qquad \begin{aligned} TN &= \sqrt{(0+2)^2+(10+1)^2} \\ &= \sqrt{2^2+11^2} \\ &= \sqrt{4+121} \\ &= \sqrt{125} \end{aligned}$$

If the diagonals are perpendicular, then $TUNE$ is a square.

Slope of $EU = \frac{7-2}{-6-4} = -\frac{5}{10} = -\frac{1}{2}$ Slope of $TN = \frac{10+1}{0+2} = \frac{11}{2}$

The slope of EU is not the opposite reciprocal of the slope of TN, so we can conclude that $TUNE$ is not a square, it is a rectangle.

Here are the steps to determine if a quadrilateral is a parallelogram, rectangle, rhombus, or square.

1. See if the diagonals bisect each otherby using the midpoint formula.

Yes: ***Parallelogram***, continue to #2. **No:** A ***quadrilateral***, done.

2. Determine if the diagonals are equalby using the distance formula.

Yes: ***Rectangle***, skip to #4. **No:** Could be a rhombus, continue to #3.

3. Determine if the sides are congruentby using the distance formula.

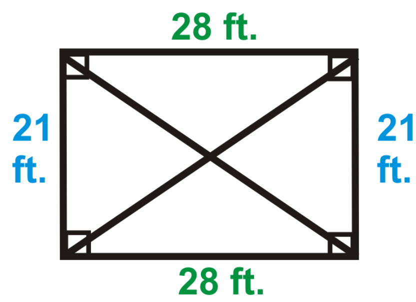
Yes: ***Rhombus***, done. **No:** ***Parallelogram***, done.

4. See if the diagonals are perpendicularby finding their slopes.

Yes: ***Square***, done. **No:** ***Rectangle***, done.

NOTE: This is just one list of steps to take to determine what type of parallelogram a quadrilateral is. There are several other steps that you could take based on the theorems we have learned.

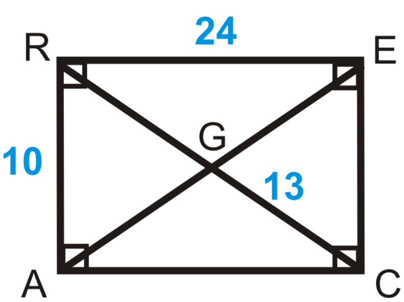
Know What? Revisited In order for the patio to be a rectangle, first the opposite sides must be congruent. So, two sides are 21ft and two are 28 ft. To ensure that the parallelogram is a rectangle *without* measuring the angles, the diagonals must be equal. You can find the length of the diagonals by using the Pythagorean Theorem.



$$d^2 = 21^2 + 28^2 = 441 + 784 = 1225$$
$$d = \sqrt{1225} = 35\ ft$$

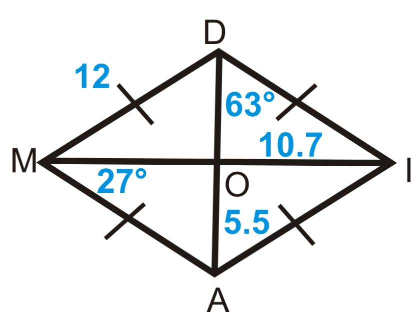
Review Questions

1. $RACE$ is a rectangle. Find:



   a. RG
   b. AE
   c. AC
   d. EC
   e. $m\angle RAC$

2. $DIAM$ is a rhombus. Find:

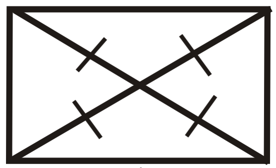


   a. MA
   b. MI
   c. DA
   d. $m\angle DIA$
   e. $m\angle MOA$

3. Draw a square and label it $CUBE$. Mark the point of intersection of the diagonals Y. Find:

a. $m\angle UCE$
b. $m\angle EYB$
c. $m\angle UBY$
d. $m\angle UEB$

For questions 4-12, determine if the quadrilateral is a parallelogram, rectangle, rhombus, square or none. Explain your reasoning.

4.


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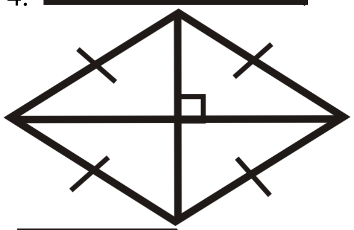

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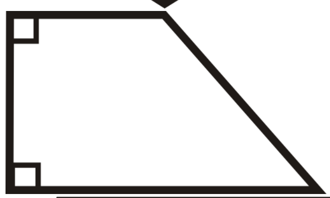

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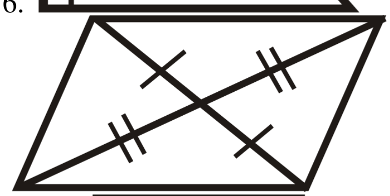

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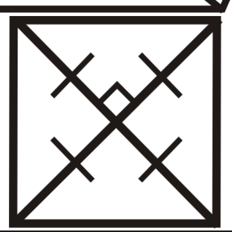

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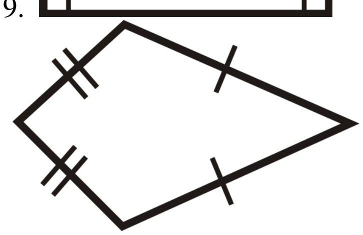
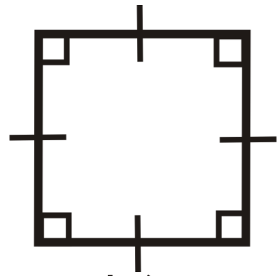
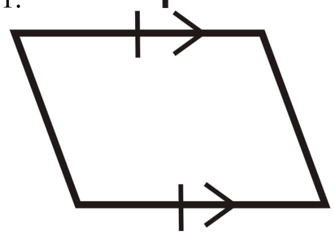


11. 

12. 

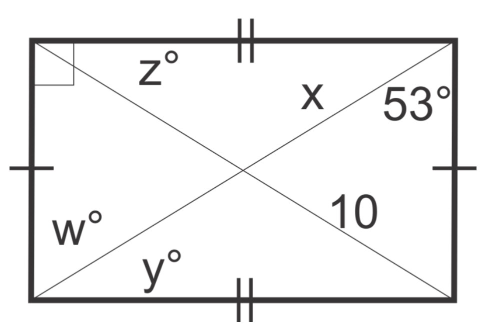
For questions 13-18 determine if the following are ALWAYS, SOMETIME, or NEVER true. Explain your reasoning.

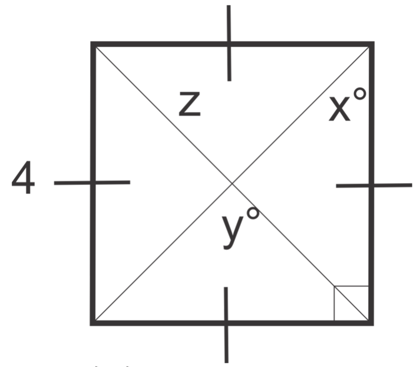
13. A rectangle is a rhombus.
14. A square is a parallelogram.
15. A parallelogram is regular.
16. A square is a rectangle.
17. A rhombus is equiangular.
18. A quadrilateral is a pentagon.

For questions 19-22, determine what type of quadrilateral *ABCD* is.

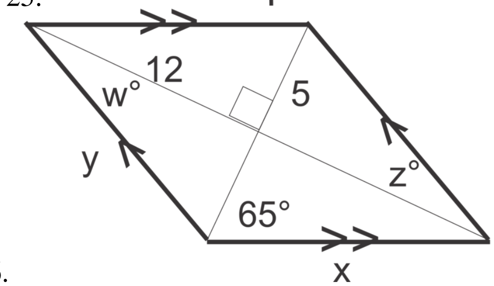
19. $A(-2,4),\ B(-1,2),\ C(-3,1),\ D(-4,3)$
20. $A(-2,3),\ B(3,4),\ C(2,-1),\ D(-3,-2)$
21. $A(1,-1),\ B(7,1),\ C(8,-2),\ D(2,-4)$
22. $A(10,4),\ B(8,-2),\ C(2,2),\ D(4,8)$
23. ***Writing*** From 19-22, you used the coordinate plane to determine what type of quadrilateral *ABCD* is. Other than the method described in this section, describe another way to determine if *ABCD* is a square.

For problems 24-26, find the value of each variable in the figures.

24. 



25.



26.

27. Given: $ABCD$ is a rectangle W, X, Y and Z are midpoints of $\overline{BC}$, $\overline{AB}$, $\overline{AD}$ and $\overline{CD}$ respectively Prove: quadrilateral $WXYZ$ is a rhombus

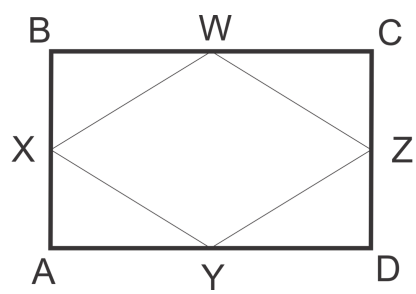


TABLE 6.9:

Statements	***Reasons***
7. $ABCD$ is a rectangle	1. Given
8. $\overline{BW} \cong \overline{WC}$, ____, ____, ____	2. Definition of a midpoint
9. $BD = AC$	3.
10. $\overline{XY}$ is a midsegment in $\triangle ABD$ _______, _______, _______	4. Definition of a midsegment in a triangle
11. $XY = \frac{1}{2}BD = WZ$ and _______	5. Midsegment in a triangle is half the length of the parallel side.
12. $\frac{1}{2}BD = \frac{1}{2}AC$	6.
13. $XY = WZ = YZ = XW$	7.
14. $WXYZ$ is a rhombus	8.

28. Explain why a quadrilateral formed by the segments connecting the midpoints of the sides in a rhombus is always a rectangle.
29. Explain why a quadrilateral formed by the segments connecting the midpoints of the sides in a square is always a square.
30. Construct a rhombus with diagonals of lengths 2 inches and 1.5 inches.
31. Construct a rectangle with diagonal length inches.

Review Queue Answers

a. A rectangle has all equal angles and opposite sides are congruent. It is a parallelogram.
b. A square has equal angles and sides. It is a parallelogram and a rectangle (and a rhombus).
c. Possibilities: picture frame, door, baseball diamond, windows, walls, floor tiles, book cover, pages/paper, table/desk top, black/white board, the diamond suit (in a deck of cards).

6.5 Trapezoids and Kites

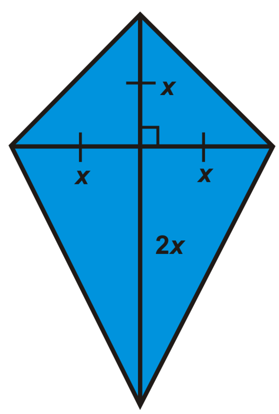
Learning Objectives

- Define and find the properties of trapezoids, isosceles trapezoids, and kites.
- Discover the properties of midsegments of trapezoids.
- Plot trapezoids, isosceles trapezoids, and kites in the coordinate plane.

Review Queue

a. Draw a quadrilateral with one set of parallel lines.
b. Draw a quadrilateral with one set of parallel lines and two right angles.
c. Draw a quadrilateral with one set of parallel lines and two congruent sides.
d. Draw a quadrilateral with one set of parallel lines and three congruent sides.
e. Draw a quadrilateral with two sets of congruent sides and no parallel sides.

Know What? A traditional kite, seen at the right, is made by placing two pieces of wood perpendicular to each other and one piece of wood is bisected by the other. The typical dimensions are included in the picture. If you have two pieces of wood, 36 inches and 54 inches, determine the values of x and $2x$. Then, determine how large a piece of canvas you would need to make the kite (find the perimeter of the kite).



Trapezoids

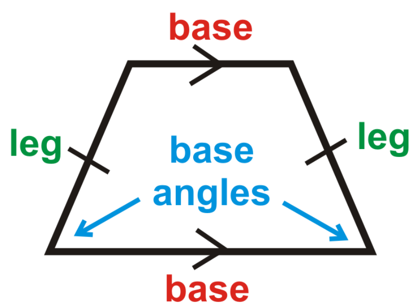
Unlike parallelograms, trapezoids have only one set of parallel lines. The other two sides have no restrictions.

Trapezoid: A quadrilateral with exactly one pair of parallel sides.

Examples look like:

Isosceles Trapezoid: A trapezoid where the non-parallel sides are congruent.

The third trapezoid above is an example of an isosceles trapezoid. Think of it as an isosceles triangle with the top cut off. Isosceles trapezoids also have parts that are labeled much like an isosceles triangle. Both parallel sides are called bases.

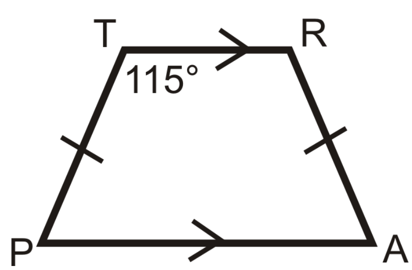


Isosceles Trapezoids

Previously, we introduced the Base Angles Theorem with isosceles triangles. The theorem states that in an isosceles triangle, the two base angles are congruent. This property holds true for isosceles trapezoids. ***The two angles along the same base in an isosceles trapezoid will also be congruent.*** This creates two pairs of congruent angles.

Theorem 6-17: The base angles of an isosceles trapezoid are congruent.

Example 1: Look at trapezoid $TRAP$ below. What is $m\angle A$?

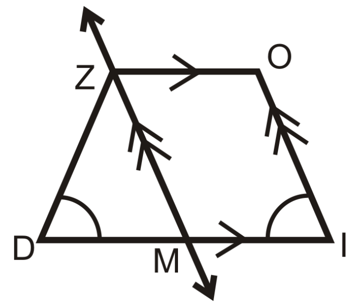


Solution: $TRAP$ is an isosceles trapezoid. So, $m\angle R = 115^\circ$, by Theorem 6-17. To find $m\angle A$, set up an equation.

$$
\begin{aligned}
115^\circ + 115^\circ + m\angle A + m\angle P &= 360^\circ \\
230^\circ + 2m\angle A &= 360^\circ \rightarrow m\angle A = m\angle P \\
2m\angle A &= 130^\circ \\
m\angle A &= 65^\circ
\end{aligned}
$$

Notice that $m\angle R + m\angle A = 115^\circ + 65^\circ = 180^\circ$. These angles will always be supplementary because of the Consecutive Interior Angles Theorem from Chapter 3. Therefore, the two angles along the same leg (or non-parallel side) are always going to be supplementary. Only in isosceles trapezoids will opposite angles also be supplementary.

Example 2: Write a two-column proof.



Given: Trapezoid $ZOID$ and parallelogram $ZOIM$

$\angle D \cong \angle I$

Prove: $\overline{ZD} \cong \overline{OI}$

Solution:

TABLE 6.10:

Statement	***Reason***
1. Trapezoid $ZOID$ and parallelogram $ZOIM$, $\angle D \cong \angle I$	Given
2. $\overline{ZM} \cong \overline{OI}$	Opposite Sides Theorem
3. $\angle I \cong \angle ZMD$	Corresponding Angles Postulate
4. $\angle D \cong \angle ZMD$	Transitive PoC
5. $\overline{ZM} \cong \overline{ZD}$	Base Angles Converse
6. $\overline{ZD} \cong \overline{OI}$	Transitive PoC

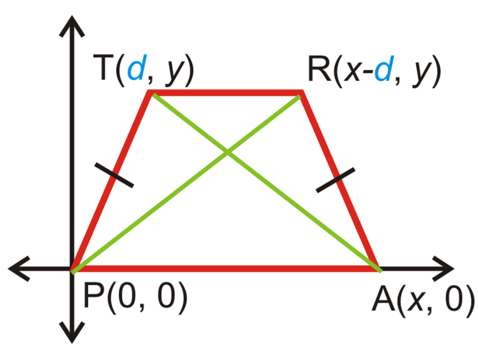
In this example we proved the converse of Theorem 6-17.

Theorem 6-17 Converse: If a trapezoid has congruent base angles, then it is an isosceles trapezoid.

Next, we will investigate the diagonals of an isosceles triangle. Recall, that the diagonals of a rectangle are congruent AND they bisect each other. The diagonals of an isosceles trapezoid are also congruent, but they do NOT bisect each other.

Isosceles Trapezoid Diagonals Theorem: The diagonals of an isosceles trapezoid are congruent.

Example 3: Show $TA = RP$.



Solution: This is an example of a coordinate proof. Here, we will use the distance formula to show that $TA = RP$, but with letters instead of numbers for the coordinates.

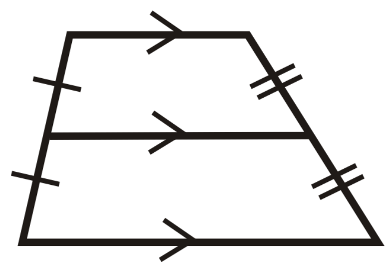
$$TA = \sqrt{(x-d)^2+(0-y)^2} \qquad RP = \sqrt{(x-d-0)^2+(y-0)^2}$$
$$= \sqrt{(x-d)^2+(-y)^2} \qquad = \sqrt{(x-d)^2+y^2}$$
$$= \sqrt{(x-d)^2+y^2}$$

Notice that we end up with the same thing for both diagonals. This means that the diagonals are equal and we have proved the theorem.

Midsegment of a Trapezoid

Midsegment (of a trapezoid): A line segment that connects the midpoints of the non-parallel sides.

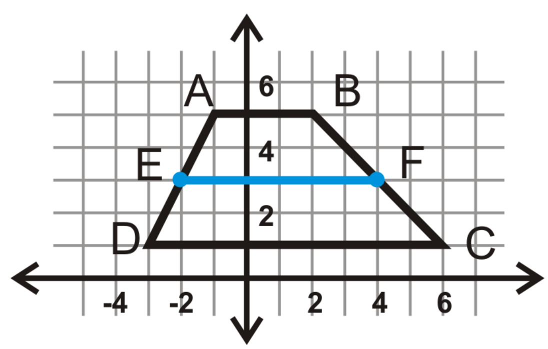
There is only one midsegment in a trapezoid. It will be parallel to the bases because it is located halfway between them. Similar to the midsegment in a triangle, where it is half the length of the side it is parallel to, the midsegment of a trapezoid also has a link to the bases.



Investigation 6-5: Midsegment Property

Tools Needed: graph paper, pencil, ruler

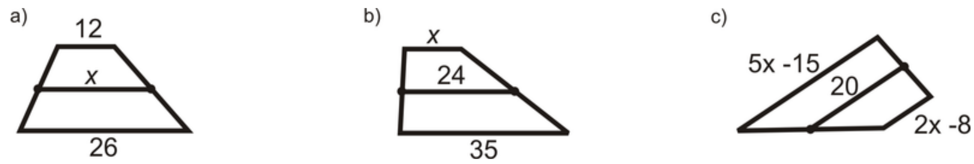
a. Draw a trapezoid on your graph paper with vertices $A(-1,5)$, $B(2,5)$, $C(6,1)$ and $D(-3,1)$. Notice this is NOT an isosceles trapezoid.



b. Find the midpoint of the non-parallel sides either by using slopes or the midpoint formula. Label them E and F. Connect the midpoints to create the midsegment.
c. Find the lengths of AB, EF, and CD. Can you write a formula to find the midsegment?

Midsegment Theorem: The length of the midsegment of a trapezoid is the average of the lengths of the bases, or $EF = \frac{AB+CD}{2}$.

Example 4: ***Algebra Connection*** Find x. All figures are trapezoids with the midsegment.



Solution:

a) x is the average of 12 and 26. $\frac{12+26}{2} = \frac{38}{2} = 19$

b) 24 is the average of x and 35.

$$\begin{aligned}\frac{x+35}{2} &= 24\\ x+35 &= 48\\ x &= 13\end{aligned}$$

c) 20 is the average of $5x-15$ and $2x-8$.

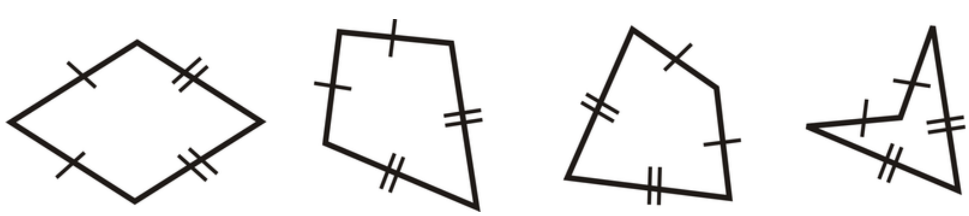
$$\begin{aligned}\frac{5x-15+2x-8}{2} &= 20\\ 7x-23 &= 40\\ 7x &= 63\\ x &= 9\end{aligned}$$

Kites

The last quadrilateral we will study is a kite. Like you might think, it looks like a traditional kite that is flown in the air.

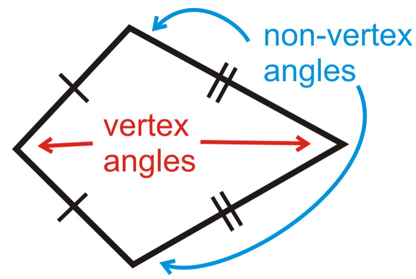
Kite: A quadrilateral with two sets of adjacent congruent sides.

A few examples:



From the definition, a kite is the only quadrilateral that we have discussed that could be concave, as with the case of the last kite. If a kite is concave, it is called a ***dart***.

The angles between the congruent sides are called ***vertex angles***. The otherangles are called ***non-vertex angles***. If we draw the diagonal through the vertex angles, we would have two congruent triangles.



Given: $KITE$ with $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$

Prove: $\angle K \cong \angle T$

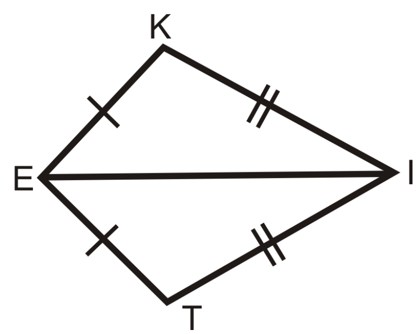


TABLE 6.11:

Statement	***Reason***
1. $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$	Given
2. $\overline{EI} \cong \overline{EI}$	Reflexive PoC
3. $\triangle EKI \cong \triangle ETI$	SSS
4. $\angle K \cong \angle T$	CPCTC

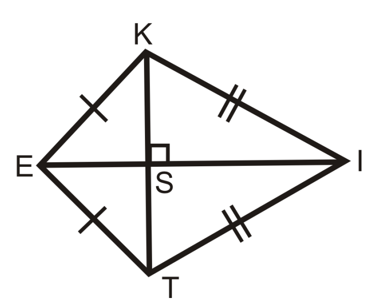
Theorem 6-21: The non-vertex angles of a kite are congruent.

Theorem 6-22: The diagonal through the vertex angles is the angle bisector for both angles.

The proof of Theorem 6-22 is very similar to the proof above for Theorem 6-21. If we draw in the other diagonal in $KITE$ we find that the two diagonals are perpendicular.

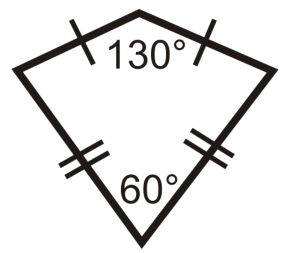
Kite Diagonals Theorem: The diagonals of a kite are perpendicular.

To prove that the diagonals are perpendicular, look at $\triangle KET$ and $\triangle KIT$. Both of these triangles are isosceles triangles, which means $\overline{EI}$ is the perpendicular bisector of $\overline{KT}$ (the Isosceles Triangle Theorem, Chapter 4). Use this information to help you prove the diagonals are perpendicular in the review questions.

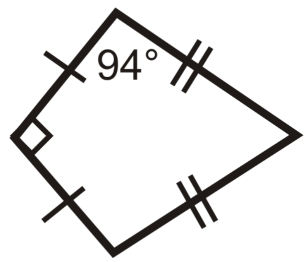


Example 5: Find the other two angle measures in the kites below.

a)



b)



Solution:

a) The two angles left are the non-vertex angles, which are congruent.

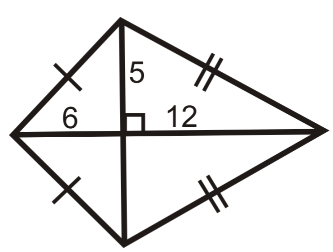
$$\begin{aligned} 130^\circ + 60^\circ + x + x &= 360^\circ \\ 2x &= 170^\circ \\ x &= 85^\circ \qquad \text{Both angles are } 85^\circ. \end{aligned}$$

b) The other non-vertex angle is also 94°. To find the fourth angle, subtract the other three angles from 360°.

$$\begin{aligned} 90^\circ + 94^\circ + 94^\circ + x &= 360^\circ \\ x &= 82^\circ \end{aligned}$$

Be careful with the definition of a kite. The congruent pairs are distinct. This means that ***a rhombus and square cannot be a kite***.

Example 6: Use the Pythagorean Theorem to find the length of the sides of the kite.

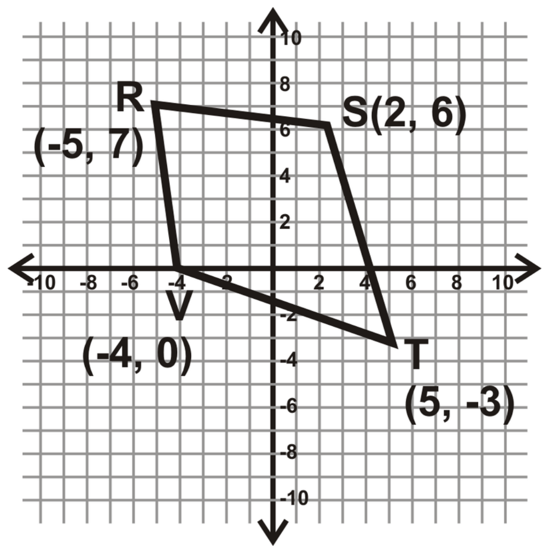


Solution: Recall that the Pythagorean Theorem is $a^2 + b^2 = c^2$, where c is the hypotenuse. In this kite, the sides are all hypotenuses.

$$\begin{aligned} 6^2+5^2 &= h^2 \\ 36+25 &= h^2 \\ 61 &= h^2 \\ \sqrt{61} &= h \end{aligned} \qquad \begin{aligned} 12^2+5^2 &= j^2 \\ 144+25 &= j^2 \\ 169 &= j^2 \\ 13 &= j \end{aligned}$$

Kites and Trapezoids in the Coordinate Plane

Example 7: Determine what type of quadrilateral $RSTV$ is. Simplify all radicals.



Solution: There are two directions you could take here. First, you could determine if the diagonals bisect each other. If they do, then it is a parallelogram and you could proceed like the previous section. Or, you could find the lengths of all the sides. Let's do this option.

$$\begin{aligned} RS &= \sqrt{(-5-2)^2+(7-6)^2} \\ &= \sqrt{(-7)^2+1^2} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(2-5)^2+(6-(-3))^2} \\ &= \sqrt{(-3)^2+9^2} \\ &= \sqrt{90} = 3\sqrt{10} \end{aligned}$$

$$\begin{aligned} RV &= \sqrt{(-5-(-4))^2+(7-0)^2} \\ &= \sqrt{(-1)^2+7^2} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned} \qquad \begin{aligned} VT &= \sqrt{(-4-5)^2+(0-(-3))^2} \\ &= \sqrt{(-9)^2+3^2} \\ &= \sqrt{90} = 3\sqrt{10} \end{aligned}$$

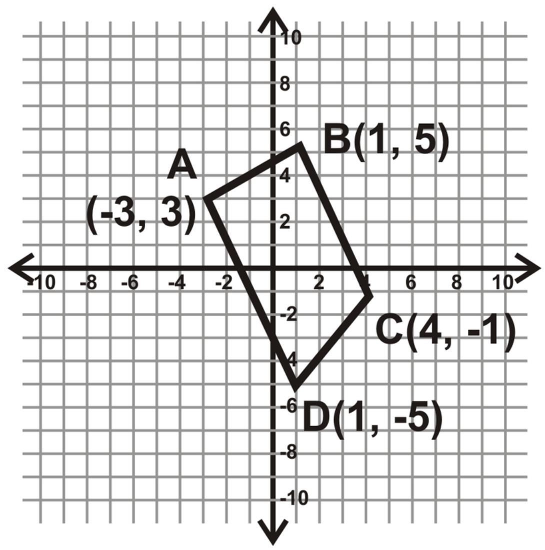
From this we see that the adjacent sides are congruent. Therefore, $RSTV$ is a kite.

Algebra Review: From now on, this text will ask you to "simplify the radical." From Algebra, this means that you pull all square numbers (1, 4, 9, 16, 25, ...) out of the radical. Above $\sqrt{50} = \sqrt{25 \cdot 2}$. We know $\sqrt{25} = 5$, so $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

Hint: If you are only given a set of points when determining what type of quadrilateral a figure is, always plot the points and graph. The visual will help you decide which direction to go.

Example 8: Determine what type of quadrilateral $ABCD$ is. $A(-3,3),\ B(1,5),\ C(4,-1),\ D(1,-5)$. Simplify all radicals.

Solution: First, graph $ABCD$. This will make it easier to figure out what type of quadrilateral it is. From the graph, we can tell this is not a parallelogram. Find the slopes of $\overline{BC}$ and $\overline{AD}$ to see if they are parallel.



Slope of $\overline{BC} = \frac{5-(-1)}{1-4} = \frac{6}{-3} = -2$

Slope of $\overline{AD} = \frac{3-(-5)}{-3-1} = \frac{8}{-4} = -2$

We now know $\overline{BC} \parallel \overline{AD}$. This is a trapezoid. To determine if it is an isosceles trapezoid, find AB and CD.

$$\begin{aligned} AB &= \sqrt{(-3-1)^2+(3-5)^2} \\ &= \sqrt{(-4)^2+(-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(4-1)^2+(-1-(-5))^2} \\ &= \sqrt{3^2+4^2} \\ &= \sqrt{25} = 5 \end{aligned}$$

$AB \neq CD$, therefore this is only a trapezoid.

Example 9: Determine what type of quadrilateral $EFGH$ is.

$$E(5,-1),\ F(11,-3),\ G(5,-5),\ H(-1,-3)$$

Solution: To contrast with Example 8, we will not graph this example. Let's find the length of all four sides.

$$\begin{aligned} EF &= \sqrt{(5-11)^2+(-1-(-3))^2} \\ &= \sqrt{(-6)^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \end{aligned} \qquad \begin{aligned} FG &= \sqrt{(11-5)^2+(-3-(-5))^2} \\ &= \sqrt{6^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \end{aligned}$$

$$GH = \sqrt{(5-(-1))^2+(-5-(-3))^2} \qquad HE = \sqrt{(-1-5)^2+(-3-(-1))^2}$$
$$= \sqrt{6^2+(-2)^2} \qquad = \sqrt{(-6)^2+(-2)^2}$$
$$= \sqrt{40} = 2\sqrt{10} \qquad = \sqrt{40} = 2\sqrt{10}$$

All four sides are equal. That means, this quadrilateral is either a rhombus or a square. The difference between the two is that a square has four 90° angles and congruent diagonals. Let's find the length of the diagonals.

$$EG = \sqrt{(5-5)^2+(-1-(-5))^2} \qquad FH = \sqrt{(11-(-1))^2+(-3-(-3))^2}$$
$$= \sqrt{0^2+4^2} \qquad = \sqrt{12^2+0^2}$$
$$= \sqrt{16} = 4 \qquad = \sqrt{144} = 12$$

The diagonals are not congruent, so $EFGH$ is a rhombus.

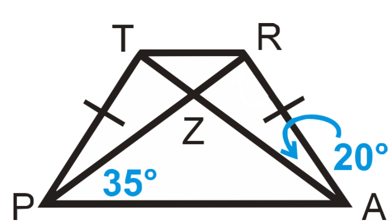
Know What? Revisited If the diagonals (pieces of wood) are 36 inches and 54 inches, x is half of 36, or 18 inches. Then, $2x$ is 36. To determine how large a piece of canvas to get, find the length of each side of the kite using the Pythagorean Theorem.

$$18^2+18^2 = s^2 \qquad 18^2+36^2 = t^2$$
$$324 = s^2 \qquad 1620 = t^2$$
$$18\sqrt{2} \approx 25.5 \approx s \qquad 18\sqrt{5} \approx 40.25 \approx t$$

The perimeter of the kite would be $25.5+25.5+40.25+40.25 = 131.5$ inches or 11 ft, 10.5 in.

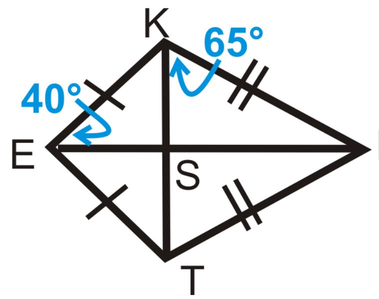
Review Questions

1. $TRAP$ an isosceles trapezoid. Find:



a. $m\angle TPA$
b. $m\angle PTR$
c. $m\angle ZRA$
d. $m\angle PZA$

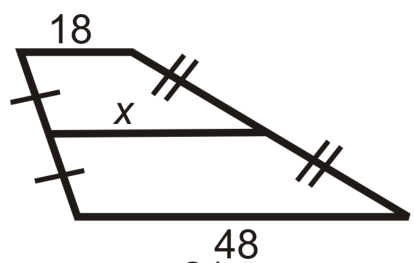
2. $KITE$ is a kite. Find:

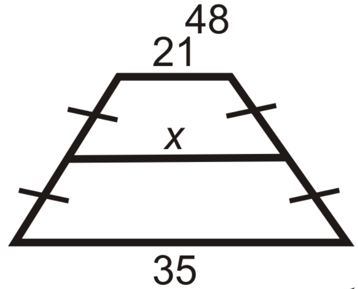


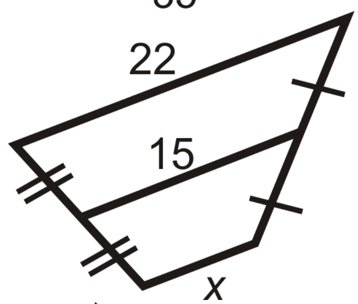
a. $m\angle ETS$
b. $m\angle KIT$
c. $m\angle IST$
d. $m\angle SIT$
e. $m\angle ETI$

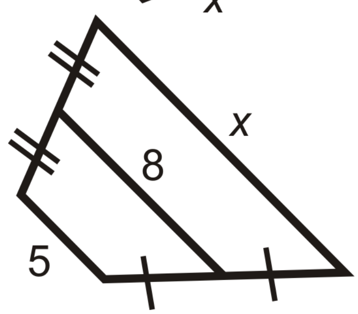
3. ***Writing*** Can the parallel sides of a trapezoid be congruent? Why or why not?
4. ***Writing*** Besides a kite and a rhombus, can you find another quadrilateral with perpendicular diagonals? Explain and draw a picture.
5. ***Writing*** Describe how you would draw or construct a kite.

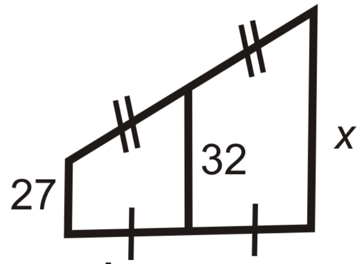
For questions 6-11, find the length of the midsegment or missing side.

6. 

7. 

8. 

9. 



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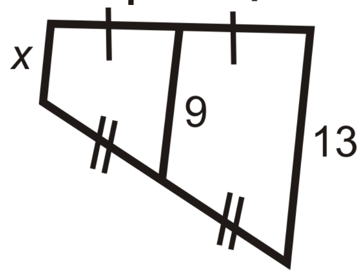
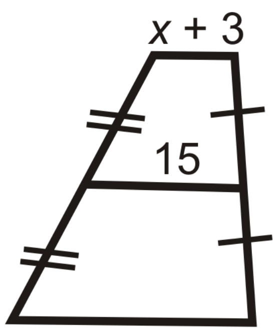


11.

Algebra Connection For questions 12-17, find the value of the missing variable(s). Simplify all radicals.



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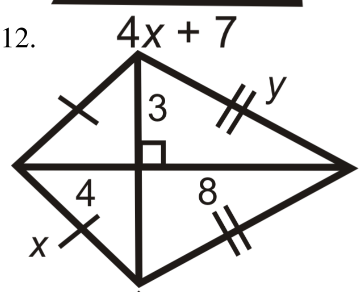

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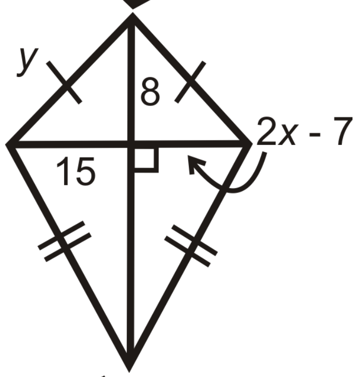

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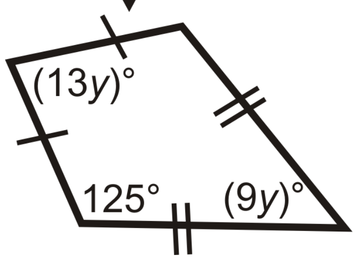

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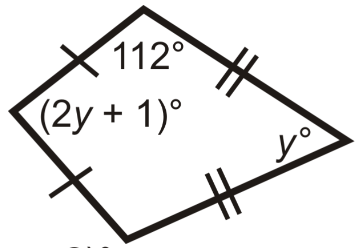

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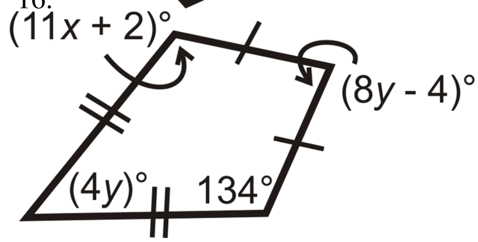
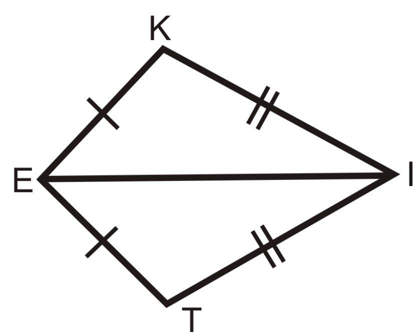


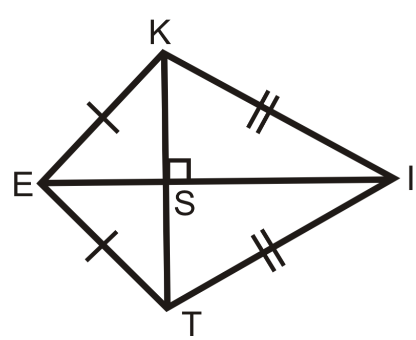
17.

For questions 18-21, determine what type of quadrilateral *ABCD* is. *ABCD* could be any quadrilateral that we have learned in this chapter. If it is none of these, write *none*.

18. $A(1,-2)$, $B(7,-5)$, $C(4,-8)$, $D(-2,-5)$
19. $A(6,6)$, $B(10,8)$, $C(12,4)$, $D(8,2)$
20. $A(-1,8)$, $B(1,4)$, $C(-5,-4)$, $D(-5,6)$
21. $A(5,-1)$, $B(9,-4)$, $C(6,-10)$, $D(3,-5)$
22. $A(-2,2)$, $B(0,1)$, $C(2,2)$, $D(1,5)$
23. $A(-7,4)$, $B(-4,4)$, $C(0,0)$, $D(0,-3)$
24. $A(3,3)$, $B(5,-1)$, $C(7,0)$, $D(5,4)$
25. $A(-4,4)$, $B(-1,2)$, $C(2,4)$, $D(-1,6)$
26. Write a two-column proof of Theorem 6-22. Given: $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$Prove: $\overline{EI}$ is the angle bisector of $\angle KET$ and $\angle KIT$

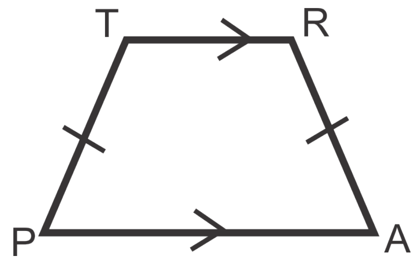


27. Write a two-column proof of the Kite Diagonal Theorem. Given: $\overline{EK} \cong \overline{ET}$, $\overline{KI} \cong \overline{IT}$Prove: $\overline{KT} \perp \overline{EI}$



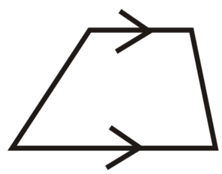
* Use the hint given earlier in this section.

28. Write a two-column proof of the Isosceles Trapezoid Diagonals Theorem using congruent triangles. Given: $TRAP$ is an isosceles trapezoid with $\overline{TR} \parallel \overline{AP}$. Prove: $\overline{TA} \cong \overline{RP}$



29. Explain why the segments connecting the midpoints of the consecutive sides in a kite will always form a rectangle.
30. Explain why the segments connecting the midpoints of the consecutive sides in an isosceles trapezoid will always form a rhombus.

Review Queue Answers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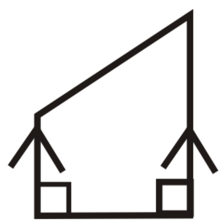
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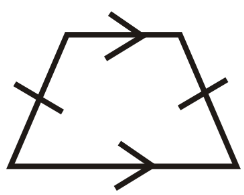
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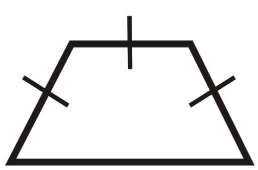
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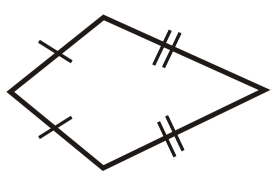

e.

6.6 Chapter 6 Review

Keywords and Theorems

Polygon Sum Formula

For any $n-$gon, the sum of the interior angles is $(n-2)\times 180^\circ$.

Equiangular Polygon Formula

For any equiangular $n-$gon, the measure of each angle is $\frac{(n-2)\times 180^\circ}{n}$.

Regular Polygon

When a polygon is equilateral and equiangular.

Exterior Angle Sum Theorem

The sum of the exterior angles of any polygon is 360°.

Parallelogram

A quadrilateral with two pairs of parallel sides.

Opposite Sides Theorem

If a quadrilateral is a parallelogram, then the opposite sides are congruent.

Opposite Angles Theorem

If a quadrilateral is a parallelogram, then the opposite angles are congruent.

Consecutive Angles Theorem

If a quadrilateral is a parallelogram, then the consecutive angles are supplementary.

Parallelogram Diagonals Theorem

If a quadrilateral is a parallelogram, then the diagonals bisect each other.

Opposite Sides Theorem Converse

If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

Opposite Angles Theorem Converse

If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse

If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse

If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

Theorem 6-10

Rectangle Theorem
A quadrilateral is a rectangle if and only if it has four right (congruent) angles.

Rhombus Theorem
A quadrilateral is a rhombus if and only if it has four congruent sides.

Square Theorem
A quadrilateral is a square if and only if it has four right angles and four congruent sides.

Theorem 6-14
A parallelogram is a rectangle if and only if the diagonals are congruent.

Theorem 6-15
A parallelogram is a rhombus if and only if the diagonals are perpendicular.

Theorem 6-16
A parallelogram is a rhombus if and only if the diagonals bisect each angle.

Trapezoid
A quadrilateral with exactly one pair of parallel sides.

Isosceles Trapezoid
A trapezoid where the non-parallel sides are congruent.

Theorem 6-17
The base angles of an isosceles trapezoid are congruent.

Theorem 6-17 Converse
If a trapezoid has congruent base angles, then it is an isosceles trapezoid.

Isosceles Trapezoid Diagonals Theorem
The diagonals of an isosceles trapezoid are congruent.

Midsegment (of a trapezoid)
A line segment that connects the midpoints of the non-parallel sides.

Midsegment Theorem
The length of the midsegment of a trapezoid is the average of the lengths of the bases

Kite
A quadrilateral with two sets of adjacent congruent sides.

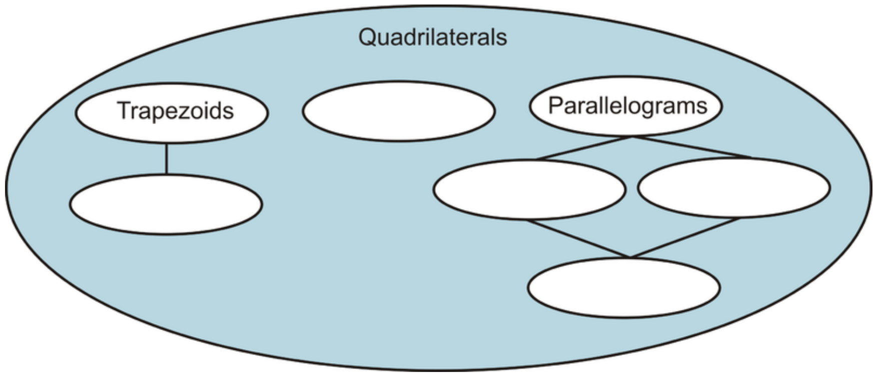
Theorem 6-21
The non-vertex angles of a kite are congruent.

Theorem 6-22
The diagonal through the vertex angles is the angle bisector for both angles.

Kite Diagonals Theorem
The diagonals of a kite are perpendicular.

Quadrilateral Flow Chart

Fill in the flow chart according to what you know about the quadrilaterals we have learned in this chapter.



Sometimes, Always, Never

Determine if the following statements are sometimes, always or never true.

a. A trapezoid is a kite.
b. A square is a parallelogram.
c. An isosceles trapezoid is a quadrilateral.
d. A rhombus is a square.
e. A parallelogram is a square.
f. A square is a kite.
g. A square is a rectangle.
h. A quadrilateral is a rhombus.

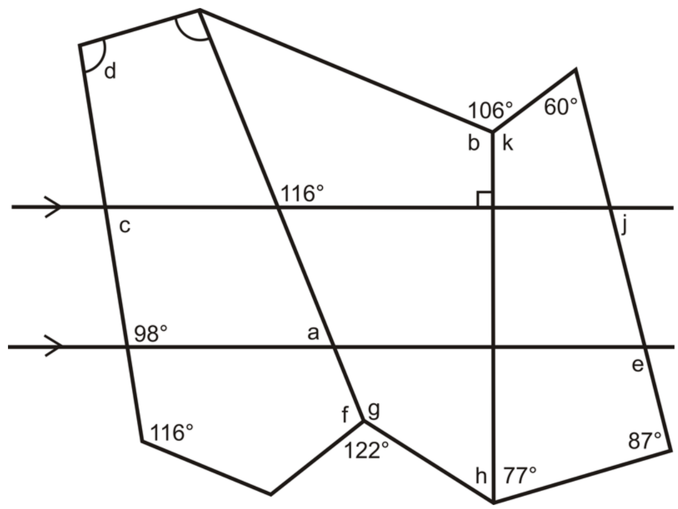
Table Summary

Determine if each quadrilateral has the given properties. If so, write yes or state how many sides (or angles) are congruent, parallel, or perpendicular.

TABLE 6.12:

	Opposite sides $\parallel$	**Diagonals bisect each other**	**Diagonals** $\perp$	**Opposite sides** $\cong$	**Opposite angles** $\cong$	**Consecutive Angles add up to** 180°
Trapezoid						
Isosceles Trapezoid						
Kite						
Parallelogram						
Rectangle						
Rhombus						
Square						

Find the measure of all the lettered angles below. The bottom angle in the pentagon (at the bottom of the drawing) is 138°.



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9691 .

CHAPTER 7 Similarity

Chapter Outline

In this chapter, we will start with a review of ratios and proportions. Second, we will introduce the concept of similarity. Two figures are similar if they have the same shape, but not the same size. We will apply similarity to polygons, quadrilaterals and triangles. Then, we will extend this concept to proportionality with parallel lines and dilations. Finally, there is an extension about self-similarity, or fractals, at the end of the chapter.

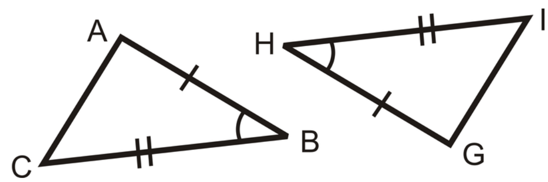
7.1 Ratios and Proportions

Learning Objectives

- Write, simplify, and solve ratios and proportions.
- Use ratios and proportions in problem solving.

Review Queue

a. Are the two triangles congruent? If so, how do you know?



b. If $AC = 5$, what is GI? What is the reason?
c. How many inches are in a foot? In a yard? In 3 yards?
d. How many cups are in a pint? In a quart? In a gallon? In 7 quarts?

Know What? You want to make a scale drawing of your room and furniture for a little redecorating. Your room measures 12 feet by 12 feet. Also in your room is a twin bed (36 in by 75 in), a desk (4 feet by 2 feet), and a chair (3 feet by 3 feet). You decide to scale down your room to 8 in by 8 in, so the drawing fits on a piece of paper. What size should the bed, desk and chair be? Draw an appropriate layout for the furniture within the room. *Do not round your answers.*

Using Ratios

Ratio: A way to compare two numbers. Ratios can be written: $\frac{a}{b}, a:b$, and a to b.

Example 1: The total bagel sales at a bagel shop for Monday is in the table below. What is the ratio of cinnamon raisin bagels to plain bagels?

TABLE 7.1:

Type of Bagel	*Number Sold*
Plain	80
Cinnamon Raisin	30
Sesame	25
Jalapeno Cheddar	20
Everything	45
Honey Wheat	50

Solution: The ratio is $\frac{30}{80}$, 30:80, or 30 to 80. Depending on the problem, ratios are usually written in simplest form, which means to reduce the ratio. The answer is then $\frac{3}{8}$, 3:8, or 3 to 8.

Example 2: What is the ratio, in simplest form, of Honey Wheat bagels to total bagels sold?

Solution: Remember that order matters. Because the Honey Wheat is listed first, that is the number that comes first in the ratio (on in the numerator of the fraction). Find the total number of bagels sold.

$$80+30+25+20+45+50=250$$

The ratio is then $\frac{50}{250}=\frac{1}{5}$, 1:5, or 1 to 5.

We call the ratio 50:250 and 1:5 ***equivalent*** because one reduces to the other.

In some problems you may need to write a ratio of more than two numbers. For example, the ratioof the number of cinnamon raisin bagels to sesame bagels to jalapeno cheddar bagels is 30:25:20 or 6:5:4.

Measurements are used a lot with ratios and proportions. For example, how many feet are in 2 miles? How many inches are in 4 feet? You will need to know these basic measurements.

Example 3: Simplify the following ratios.

a) $\frac{7\ ft}{14\ in}$

b) $9m : 900cm$

c) $\frac{4\ gal}{16\ gal}$

Solution: Change these so that they are in the same units.

a) $\frac{7\cancel{ft}}{14\cancel{in}} \cdot \frac{12\cancel{in}}{1\cancel{ft}} = \frac{84}{14} = \frac{6}{1}$

Notice that the inches cancel each other out. ***All ratios should not have units once simplified.***

b) It is easier to simplify ratios when they are written as a fraction. $\frac{9\ m}{900\ cm} \cdot \frac{100\ cm}{1\ m} = \frac{900}{900} = \frac{1}{1}$

c) $\frac{4\ gal}{16\ gal} = \frac{1}{4}$

Example 4: A talent show features dancers and singers. The ratio of dancers to singers is 3:2. There are 30 performers total, how many singers are there?

Solution: 3:2 is a reduced ratio, so there is a whole number, n, that we can multiply both by to find the total number in each group.

$$\begin{aligned} \text{dancers} = 3n, \text{singers} = 2n \longrightarrow\ 3n+2n &= 30 \\ 5n &= 30 \\ n &= 6 \end{aligned}$$

Therefore, there are $3 \cdot 6 = 18$ dancers and $2 \cdot 6 = 12$ singers. To double-check, $18+12=30$ total performers.

Proportions

Proportion: When two ratios are set equal to each other.

Example 4: Solve the proportions.

a) $\frac{4}{5} = \frac{x}{30}$

b) $\frac{y+1}{8} = \frac{5}{20}$

c) $\frac{6}{5} = \frac{2x+5}{x-2}$

Solution: To solve a proportion, you need to ***cross-multiply***.

a)

$$\begin{aligned} \frac{4}{5} &= \frac{x}{30} \\ 4 \cdot 30 &= 5 \cdot x \\ 120 &= 5x \\ 24 &= x \end{aligned}$$

b)

$$\begin{aligned} \frac{y+1}{8} &= \frac{5}{20} \\ (y+1) \cdot 20 &= 5 \cdot 8 \\ 20y + 20 &= 40 \\ 20y &= 20 \\ y &= 1 \end{aligned}$$

c)

$$\begin{aligned} \frac{6}{5} &= \frac{2x+4}{x\text{-}2} \\ 6 \cdot (x\text{-}2) &= 5 \cdot (2x+4) \\ 6x\text{-}12 &= 10x+20 \\ \text{-}32 &= 4x \\ \text{-}8 &= x \end{aligned}$$

In proportions, the blue numbers are called the ***means*** and the orange numbers are called the ***extremes***. For the proportion to be true, the product of the means must equal the product of the extremes. This can be generalized in the Cross-Multiplication Theorem.

Cross-Multiplication Theorem: Let a, b, c, and d be real numbers, with $b \neq 0$ and $d \neq 0$. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

The proof of the Cross-Multiplication Theorem is an algebraic proof. Recall that multiplying by $\frac{2}{2}, \frac{b}{b}$, or $\frac{d}{d} = 1$ because it is the same number divided by itself $(b \div b = 1)$.

Proof of the Cross-Multiplication Theorem

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} && \text{Multiply the left side by } \frac{d}{d} \text{ and the right side by } \frac{b}{b}. \\ \frac{a}{b} \cdot \frac{d}{d} &= \frac{c}{d} \cdot \frac{b}{b} \\ \frac{ad}{bd} &= \frac{bc}{bd} && \text{The denominators are the same, so the numerators are equal.} \\ ad &= bc \end{aligned}$$

Think of the Cross-Multiplication Theorem as a shortcut. Without this theorem, you would have to go through all of these steps every time to solve a proportion.

Example 5: Your parents have an architect's drawing of their home. On the paper, the house's dimensions are 36 in by 30 in. If the shorter length of your parents' house is actually 50 feet, what is the longer length?

Solution: Set up a proportion. If the shorter length is 50 feet, then it will line up with 30 in. It does not matter which numbers you put in the numerators of the fractions, as long as they line up correctly.

$$\frac{30}{36} = \frac{50}{x} \longrightarrow 1800 = 30x$$
$$60 = x$$

So, the dimension of your parents' house is 50 ft by 60 ft.

Properties of Proportions

The Cross-Multiplication Theorem has several sub-theorems that follow from its proof. The formal term is ***corollary.***

Corollary: A theorem that follows quickly, easily, and directly from another theorem.

Below are three corollaries that are immediate results of the Cross Multiplication Theorem and the fundamental laws of algebra.

Corollary 7-1: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Corollary 7-2: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

Corollary 7-3: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

In other words, a true proportion is also true if you switch the means, switch the extremes, or flip it upside down. Notice that you will still end up with $ad = bc$ after cross-multiplying for all three of these corollaries.

Example 6: Suppose we have the proportion $\frac{2}{5} = \frac{14}{35}$. Write down the other three true proportions that follow from this one.

Solution: First of all, we know this is a true proportion because you would multiply $\frac{2}{5}$ by $\frac{7}{7}$ to get $\frac{14}{35}$. Using the three corollaries, we would get:

a. $\frac{2}{14} = \frac{5}{35}$
b. $\frac{35}{5} = \frac{14}{2}$
c. $\frac{5}{2} = \frac{35}{14}$

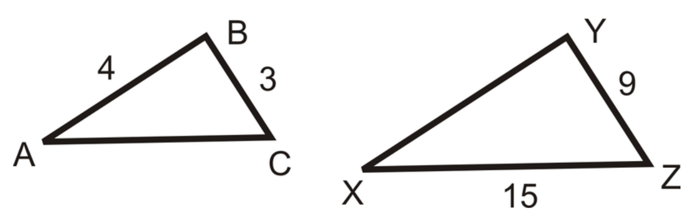
If you cross-multiply all four of these proportions, you would get $70 = 70$ for each one.

Corollary 7-4: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Corollary 7-5: If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Example 7: In the picture, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

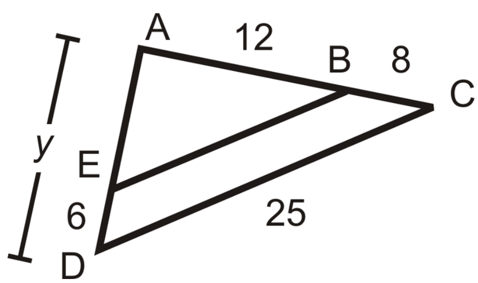
Find the measures of AC and XY.



Solution: This is an example of an *extended* proportion. Substituting in the numbers for the sides we know, we have $\frac{4}{XY} = \frac{3}{9} = \frac{AC}{15}$. Separate this into two different proportions and solve for XY and AC.

$$\begin{aligned} \frac{4}{XY} &= \frac{3}{9} \\ 36 &= 3(XY) \\ XY &= 12 \end{aligned} \qquad\qquad \begin{aligned} \frac{3}{9} &= \frac{AC}{15} \\ 9(AC) &= 45 \\ AC &= 5 \end{aligned}$$

Example 8: In the picture, $\frac{ED}{AD} = \frac{BC}{AC}$. Find y.



Solution: Substituting in the numbers for the sides we know, we have

$$\frac{6}{y} = \frac{8}{12+8}. \longrightarrow 8y = 6(20)$$
$$y = 15$$

Example 9: If $\frac{AB}{BE} = \frac{AC}{CD}$ in the picture above, find BE.

Solution:

$$\frac{12}{BE} = \frac{20}{25} \longrightarrow 20(BE) = 12(25)$$
$$BE = 15$$

Know What? Revisited Everything needs to be scaled down by a factor of $\frac{1}{18}$ $(144\ in \div 8\ in)$. Change everything into inches and then multiply by the scale factor.

Bed: 36 in by 75 in $\longrightarrow$ 2 in by 4.167 in

Desk: 48 in by 24 in $\longrightarrow$ 2.67 in by 1.33 in

Chair: 36 in by 36 in $\longrightarrow$ 2 in by 2 in

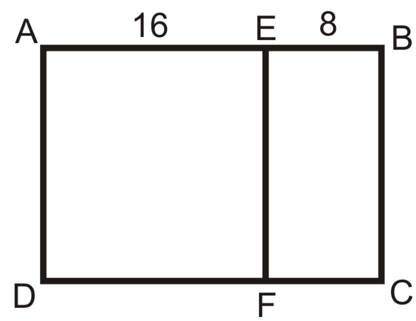
There are several layout options for these three pieces of furniture. Draw an 8 in by 8 in square and then the appropriate rectangles for the furniture. Then, cut out the rectangles and place inside the square.

Review Questions

1. The votes for president in a club election were: Smith : 24 Munoz : 32 Park : 20 Find the following ratios and write in simplest form.

a. Votes for Munoz to Smith
b. Votes for Park to Munoz
c. Votes for Smith to total votes
d. Votes for Smith to Munoz to Park

Use the picture to write the following ratios for questions 2-6.



$AEFD$ is a square

$ABCD$ is a rectangle

2. $AE : EF$
3. $EB : AB$
4. $DF : FC$
5. $EF : BC$
6. Perimeter $ABCD$: Perimeter $AEFD$: Perimeter $EBCF$
7. The measures of the angles of a triangle are have the ratio 3:3:4. What are the measures of the angles?
8. The lengths of the sides in a triangle are in a 3:4:5 ratio. The perimeter of the triangle is 36. What are the lengths of the sides?
9. The length and width of a rectangle are in a 3:5 ratio. The perimeter of the rectangle is 64. What are the length and width?
10. The length and width of a rectangle are in a 4:7 ratio. The perimeter of the rectangle is 352. What are the length and width?
11. The ratio of the short side to the long side in a parallelogram is 5:9. The perimeter of the parallelogram is 112. What are the lengths of the sides?
12. The length and width of a rectangle are in a 3:11 ratio. The area of the rectangle is 528. What are the length and width of the rectangle?
13. ***Writing*** Explain why $\frac{a+b}{b} = \frac{c+d}{d}$ is a valid proportion. HINT: Cross-multiply and see if it equals $ad = bc$.
14. ***Writing*** Explain why $\frac{a-b}{b} = \frac{c-d}{d}$ is a valid proportion. HINT: Cross-multiply and see if it equals $ad = bc$.

Solve each proportion.

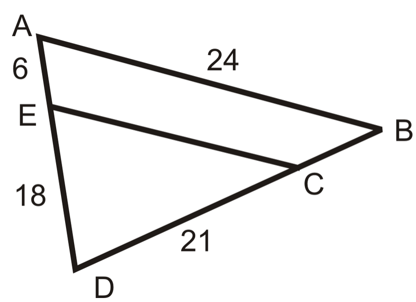
15. $\frac{x}{10} = \frac{42}{35}$
16. $\frac{x}{x-2} = \frac{5}{7}$
17. $\frac{6}{9} = \frac{y}{24}$
18. $\frac{x}{9} = \frac{16}{x}$
19. $\frac{y-3}{8} = \frac{y+6}{5}$
20. $\frac{20}{z+5} = \frac{16}{7}$
21. Shawna drove 245 miles and used 8.2 gallons of gas. At the same rate, if she drove 416 miles, how many gallons of gas will she need? Round to the nearest tenth.

22. The president, vice-president, and financial officer of a company divide the profits is a 4:3:2 ratio. If the company made $1,800,000 last year, how much did each person receive?
23. Many recipes describe ratios between ingredients. For example, one recipe for paper mache paste suggests 3 parts flour to 5 parts water. If we have one cup of flour, how much water should we add to make the paste?
24. A recipe for krispy rice treats calls for 6 cups of rice cereal and 40 large marshmallows. You want to make a larger batch of goodies and have 9 cups of rice cereal. How many large marshmallows do you need? However, you only have the miniature marshmallows at your house. You find a list of substitution quantities on the internet that suggests 10 large marshmallows are equivalent to 1 cup miniatures. How many cups of miniatures do you need?

Given the true proportion, $\frac{10}{6} = \frac{15}{d} = \frac{x}{y}$ and d, x, and y are nonzero, determine if the following proportions are also true.

25. $\frac{10}{y} = \frac{x}{6}$
26. $\frac{15}{10} = \frac{d}{6}$
27. $\frac{6+10}{10} = \frac{y+x}{x}$
28. $\frac{15}{x} = \frac{y}{d}$

For questions 24-27, $\frac{AE}{ED} = \frac{BC}{CD}$ and $\frac{ED}{AD} = \frac{CD}{DB} = \frac{EC}{AB}$.



29. Find DB.
30. Find EC.
31. Find CB.
32. Find AD.

Review Queue Answers

a. Yes, they are congruent by SAS.
b. $GI = 5$ by CPCTC
c. 12 in = 1 ft, 36 in = 3 ft, 108 in = 3 yrds
d. 2c = 1 pt, 4c = 1 qt, 16 c = 4 qt = 1 gal, 28c = 7 qt

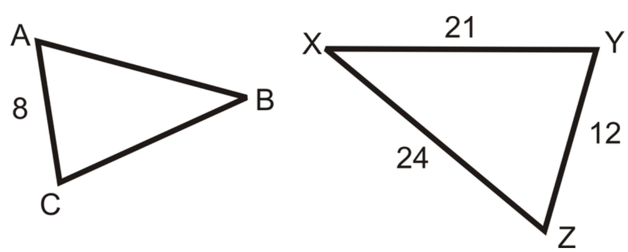
7.2 Similar Polygons

Learning Objectives

- Recognize similar polygons.
- Identify corresponding angles and sides of similar polygons from a similarity statement.
- Calculate and apply scale factors.

Review Queue

a. Solve the proportions.
   a. $\frac{6}{x} = \frac{10}{15}$
   b. $\frac{4}{7} = \frac{2x+1}{42}$
   c. $\frac{5}{8} = \frac{x-2}{2x}$
b. In the picture, $\frac{AB}{XZ} = \frac{BC}{XY} = \frac{AC}{YZ}$.
   a. Find AB.
   b. Find BC.

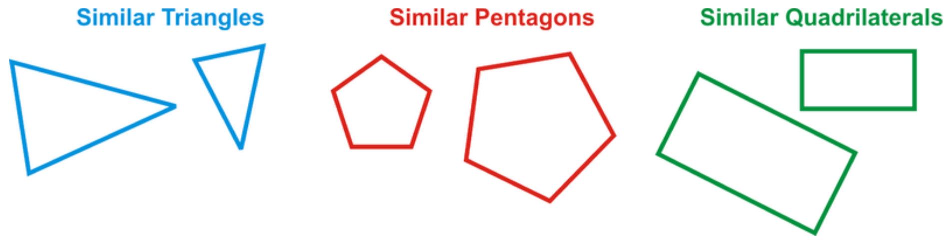


Know What? A baseball diamond is a square with 90 foot sides. A softball diamond is a square with 60 foot sides. Are the two diamonds similar? If so, what is the scale factor? Explain your answer.

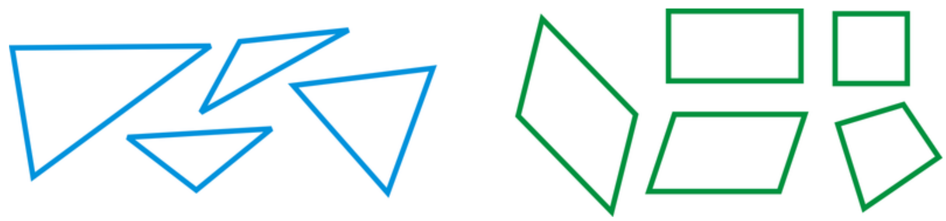
Similar Polygons

Similar Polygons: Two polygons with the same shape, but not the same size.

Think about similar polygons as an enlargement or shrinking of the same shape. So, more specifically, similar polygons have to have the same number of sides, the corresponding angles are congruent, and the corresponding sides are proportional. The symbol $\sim$ is used to represent similar. Here are some examples:



These polygons are not similar:

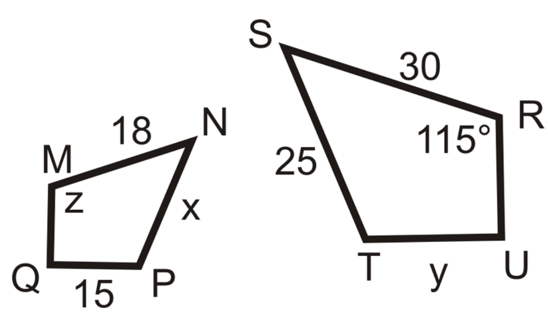


Example 1: Suppose $\triangle ABC \sim \triangle JKL$. Based on the similarity statement, which angles are congruent and which sides are proportional?

Solution: Just like a congruence statement, the congruent angles line up within the statement. So, $\angle A \cong \angle J$, $\angle B \cong \angle K$, and $\angle C \cong \angle L$. The same is true of the proportional sides. We write the sides in a proportion, $\frac{AB}{JK} = \frac{BC}{KL} = \frac{AC}{JL}$.

Because of the corollaries we learned in the last section, the proportions in Example 1 could be written several different ways. For example, $\frac{AB}{BC} = \frac{JK}{KL}$. Make sure to line up the corresponding proportional sides.

Example 2: $MNPQ \sim RSTU$. What are the values of x, y and z?

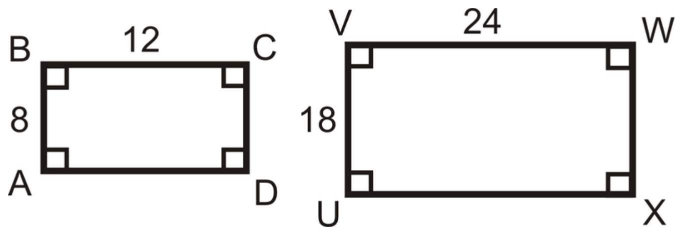


Solution: In the similarity statement, $\angle M \cong \angle R$, so $z = 115^\circ$. For x and y, set up a proportion.

$$\frac{18}{30} = \frac{x}{25} \qquad \frac{18}{30} = \frac{15}{y}$$
$$450 = 30x \qquad 450 = 18y$$
$$x = 15 \qquad y = 25$$

Specific types of triangles, quadrilaterals, and polygons will always be similar. For example, because all the angles and sides are congruent, ***all equilateral triangles are similar.*** For the same reason, ***all squares are similar.*** We can take this one step further and say that all regular polygons (with the same number of sides) are similar.

Example 3: $ABCD$ is a rectangle with length 12 and width 8. $UVWX$ is a rectangle with length 24 and width 18. Are these two rectangles similar?



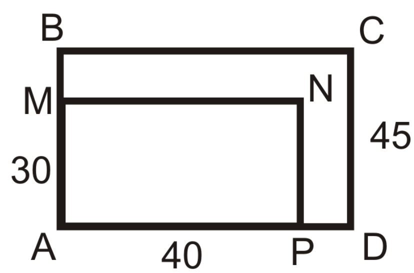
Solution: Draw a picture. First, all the corresponding angles need to be congruent. In rectangles, all the angles are congruent, so this condition is satisfied. Now, let's see if the sides are proportional. $\frac{8}{12} = \frac{2}{3}, \frac{18}{24} = \frac{3}{4}$. $\frac{2}{3} \neq \frac{3}{4}$. This tells us that the sides are not in the same proportion, so the rectangles are not similar. We could have also set up the proportion as $\frac{12}{24} = \frac{1}{2}$ and $\frac{8}{18} = \frac{4}{9}$. $\frac{1}{2} \neq \frac{4}{9}$, so you would end up with the same conclusion.

Scale Factors

If two polygons are similar, we know the lengths of corresponding sides are proportional. If k is the length of a side in one polygon, and m is the length of the corresponding side in the other polygon, then the ratio $\frac{k}{m}$ is the ***scale factor*** relating the first polygon to the second.

Scale Factor: In similar polygons, the ratio of one side of a polygon to the corresponding side of the other.

Example 5: $ABCD \sim AMNP$. Find the scale factor and the length of BC.



Solution: Line up the corresponding proportional sides. $AB : AM$, so the scale factor is $\frac{30}{45} = \frac{2}{3}$ or $\frac{3}{2}$. Because BC is in the bigger rectangle, we will multiply 40 by $\frac{3}{2}$ because it is greater than 1. $BC = \frac{3}{2}(40) = 60$.

Example 6: Find the perimeters of $ABCD$ and $AMNP$. Then find the ratio of the perimeters.

Solution: Perimeter of $ABCD = 60 + 45 + 60 + 45 = 210$

Perimeter of $AMNP = 40 + 30 + 40 + 30 = 140$

The ratio of the perimeters is 140:210, which reduces to 2:3.

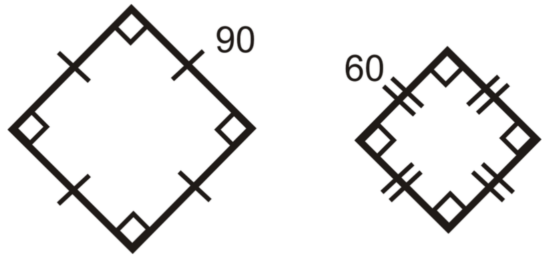
Theorem 7-2: The ratio of the perimeters of two similar polygons is the same as the ratio of the sides.

In addition the perimeter being in the same ratio as the sides, all parts of a polygon are in the same ratio as the sides. This includes diagonals, medians, midsegments, altitudes, and others.

Example 7: $\triangle ABC \sim \triangle MNP$. The perimeter of $\triangle ABC$ is 150 and $AB = 32$ and $MN = 48$. Find the perimeter of $\triangle MNP$.

Solution: From the similarity statement, AB and MN are corresponding sides. So, the scale factor is $\frac{32}{48} = \frac{2}{3}$ or $\frac{3}{2}$. The perimeter of $\triangle MNP$ is $\frac{2}{3}(150) = 100$.

Know What? Revisited All of the sides in the baseball diamond are 90 feet long and 60 feet long in the softball diamond. This means all the sides are in a $\frac{90}{60} = \frac{3}{2}$ ratio. All the angles in a square are congruent, all the angles in both diamonds are congruent. The two squares are similar and the scale factor is $\frac{3}{2}$.

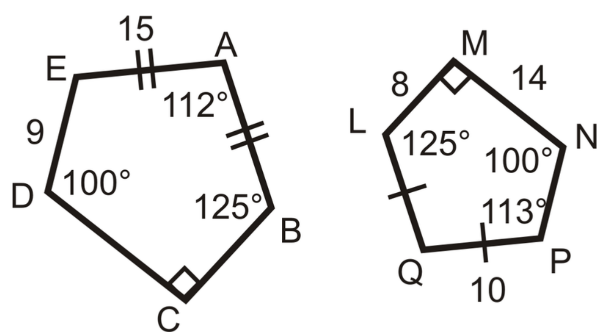


Review Questions

Determine if the following statements are true or false.

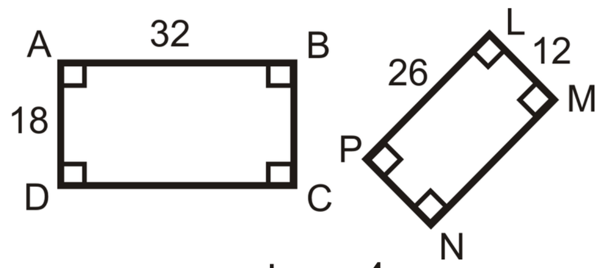
1. All equilateral triangles are similar.
2. All isosceles triangles are similar.
3. All rectangles are similar.
4. All rhombuses are similar.
5. All squares are similar.
6. All congruent polygons are similar.
7. All similar polygons are congruent.
8. All regular pentagons are similar.
9. $\triangle BIG \sim \triangle HAT$. List the congruent angles and proportions for the sides.
10. If $BI = 9$ and $HA = 15$, find the scale factor.
11. If $BG = 21$, find HT.
12. If $AT = 45$, find IG.
13. Find the perimeter of $\triangle BIG$ and $\triangle HAT$. What is the ratio of the perimeters?

Use the picture to the right to answer questions 14-18.



14. Find $m\angle E$ and $m\angle Q$.
15. $ABCDE \sim QLMNP$, find the scale factor.
16. Find BC.
17. Find CD.
18. Find NP.

Determine if the following triangles and quadrilaterals are similar. If they are, write the similarity statement.

19.


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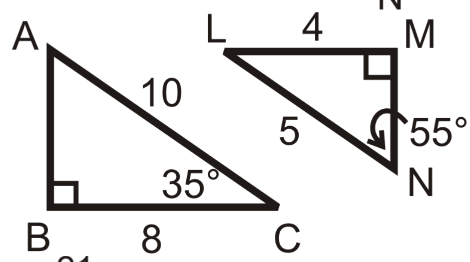

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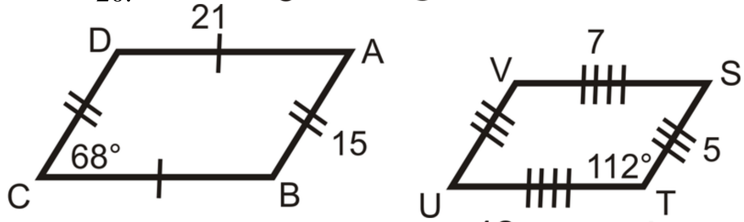

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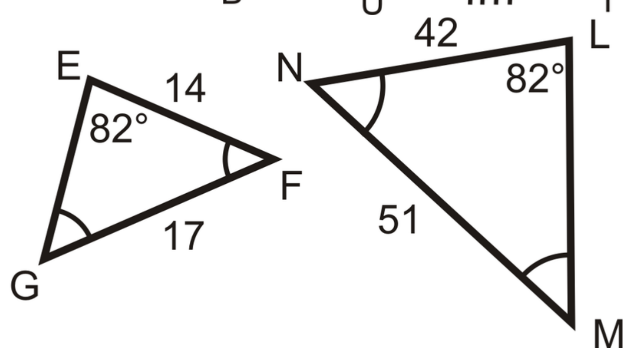
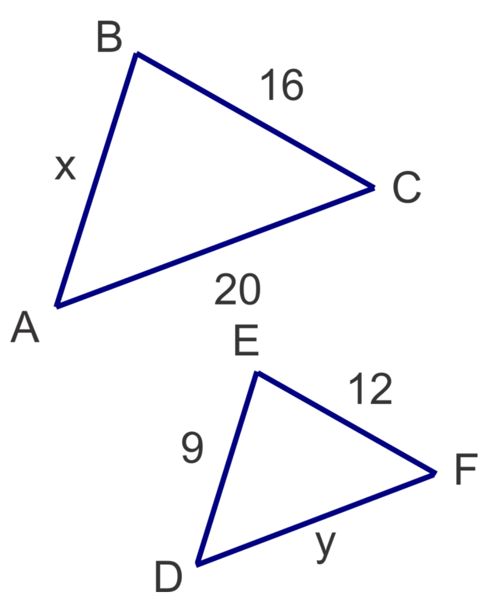
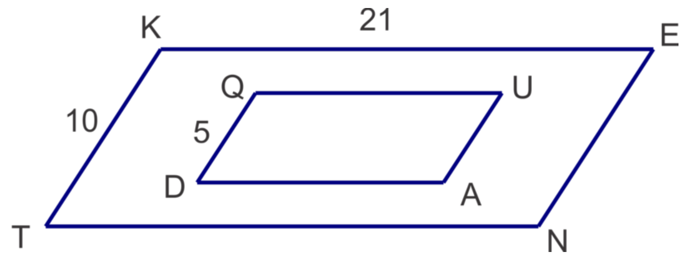


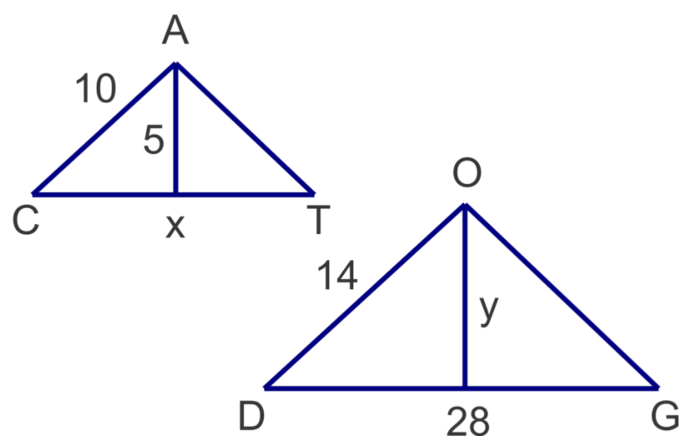
23. $\triangle ABC \sim \triangle DEF$ Solve for x and y.



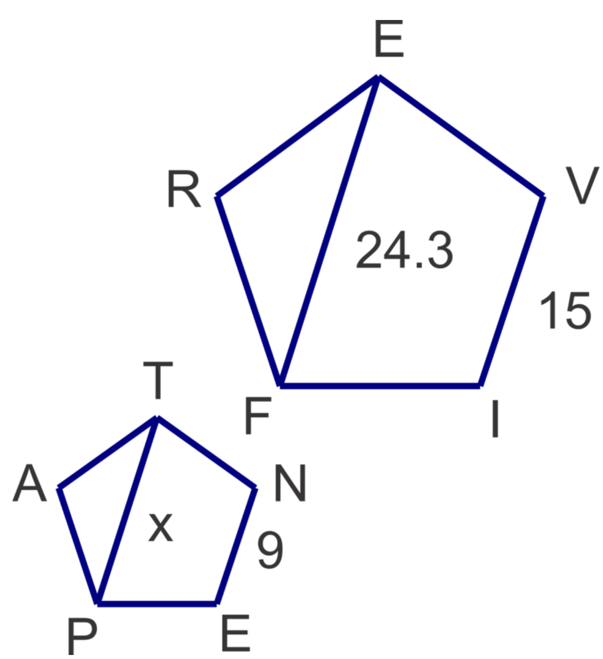
24. $QUAD \sim KENT$ Find the perimeter of $QUAD$.



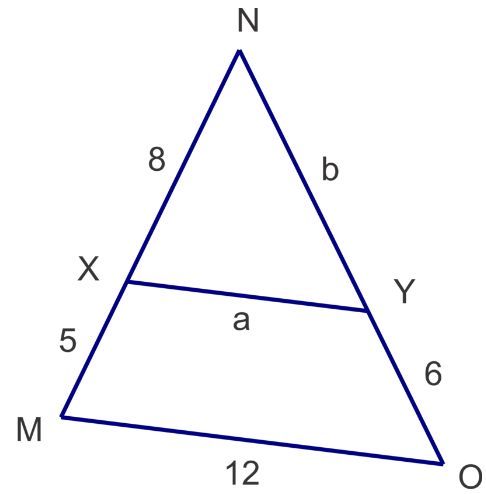
25. $\triangle CAT \sim \triangle DOG$ Solve for x and y.



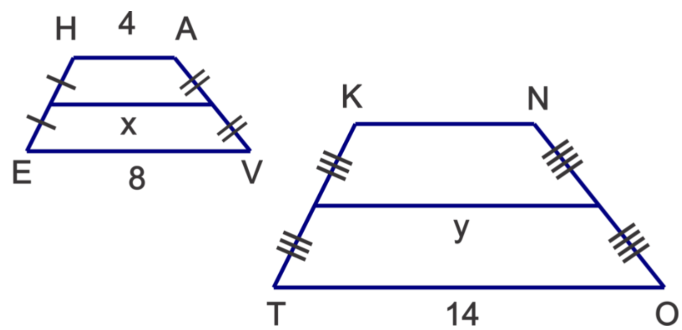
26. $PENTA \sim FIVER$ Solve for x.



27. $\triangle MNO \sim \triangle XNY$ Solve for a and b.



28. Trapezoids $HAVE \sim KNOT$ Solve for x and y.



29. Two similar octagons have a scale factor of $\frac{9}{11}$. If the perimeter of the smaller octagon is 99 meters, what is the perimeter of the larger octagon?
30. Two right triangles are similar. The legs of one of the triangles are 5 and 12. The second right triangle has a hypotenuse of length 39. What is the scale factor between the two triangles?
31. What is the area of the smaller triangle in problem 30? What is the area of the larger triangle in problem 30? What is the ratio of the areas? How does it compare to the ratio of the lengths (or scale factor)? Recall that the area of a triangle is $A = \frac{1}{2}bh$.

Review Queue Answers

a. $x = 9$
b. $x = 11.5$
c. $x = 8$

a. $AB = 16$
b. $BC = 14$

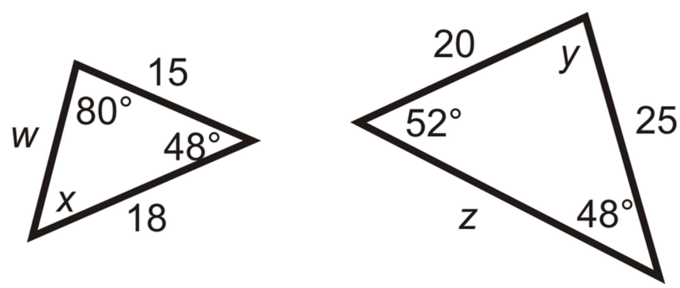
7.3 Similarity by AA

Learning Objectives

- Determine whether triangles are similar.
- Understand AA for similar triangles.
- Solve problems involving similar triangles.

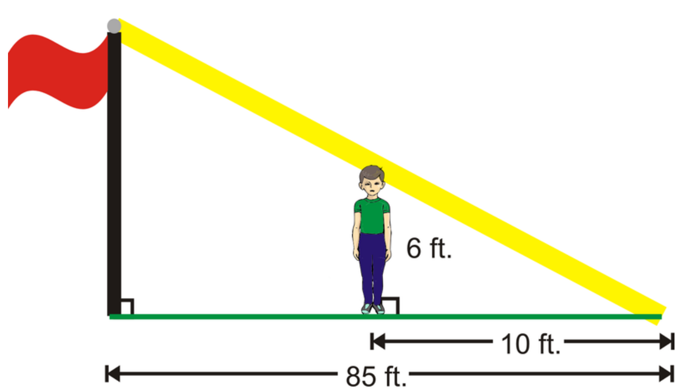
Review Queue

a.
   a. Find the measures of x and y.
   b. The two triangles are similar. Find w and z.



b. Use the true proportion $\frac{6}{8} = \frac{x}{28} = \frac{27}{y}$ to answer the following questions.
   a. Find x and y.
   b. Write another true proportion.
   c. Is $\frac{28}{8} = \frac{6+x}{12}$ true? If you solve for x, is it the same as in part a?

Know What? George wants to measure the height of a flagpole. He is 6 feet tall and his shadow is 10 feet long. At the same time, the shadow of the flagpole was 85 feet long. How tall is the flagpole?



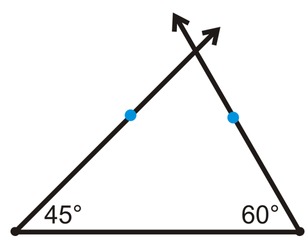
Angles in Similar Triangles

The Third Angle Theorem states if two angles are congruent to two angles in another triangle, the third angles are congruent too. Because a triangle has 180°, the third angle in any triangle is 180° minus the other two angle measures. Let's investigate what happens when two different triangles have the same angle measures. We will use Investigation 4-4 (Constructing a Triangle using ASA) to help us with this.

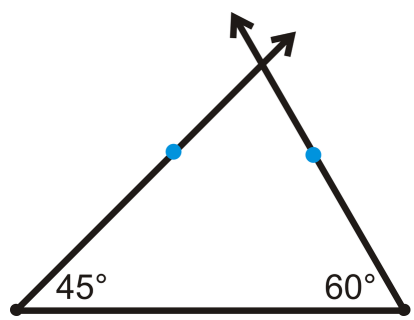
Investigation 7-1: Constructing Similar Triangles

Tools Needed: pencil, paper, protractor, ruler

a. Draw a 45° angle. Extend the horizontal side and then draw a 60° angle on the other side of this side. Extend the other side of the 45° angle and the 60° angle so that they intersect to form a triangle. What is the measure of the third angle? Measure the length of each side.



b. Repeat Step 1 and make the horizontal side between the 45° and 60° angle at least 1 inch longer than in Step 1. This will make the entire triangle larger. Find the measure of the third angle and measure the length of each side.

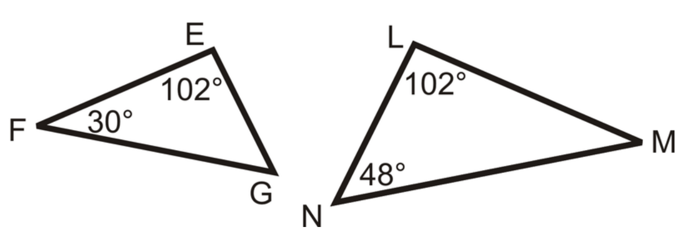


c. Find the ratio of the sides. Put the sides opposite the 45° angles over each other, the sides opposite the 60° angles over each other, and the sides opposite the third angles over each other. What happens?

AA Similarity Postulate: If two angles in one triangle are congruent to two angles in another triangle, the two triangles are similar.

The AA Similarity Postulate is a shortcut for showing that two ***triangles*** are similar. If you know that two angles in one triangle are congruent to two angles in another, which is now enough information to show that the two triangles are similar. Then, you can use the similarity to find the lengths of the sides.

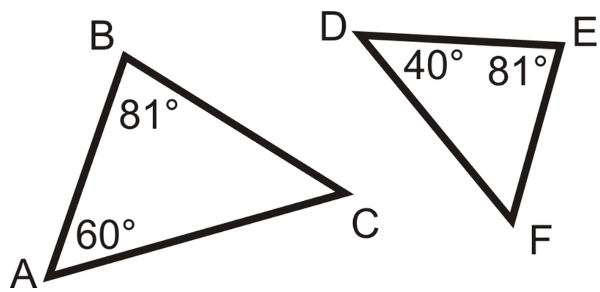
Example 1: Determine if the following two triangles are similar. If so, write the similarity statement.



Solution: Find the measure of the third angle in each triangle. $m\angle G = 48^\circ$ and $m\angle M = 30^\circ$ by the Triangle Sum Theorem. Therefore, all three angles are congruent, so the two triangles are similar. $\triangle FEG \sim \triangle MLN$.

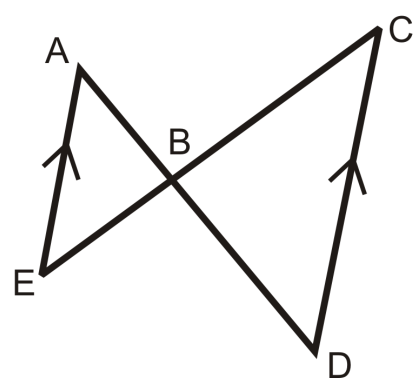
Example 2: Determine if the following two triangles are similar. If so, write the similarity statement.

Solution: $m\angle C = 39^\circ$ and $m\angle F = 59^\circ$. The angles are not equal, $\triangle ABC$ and $\triangle DEF$ are not similar.



Example 3: Are the following triangles similar? If so, write the similarity statement.

Solution: Because $\overline{AE} \parallel \overline{CD}$, $\angle A \cong \angle D$ and $\angle C \cong \angle E$ by the Alternate Interior Angles Theorem. Therefore, by the AA Similarity Postulate, $\triangle ABE \sim \triangle DBC$.

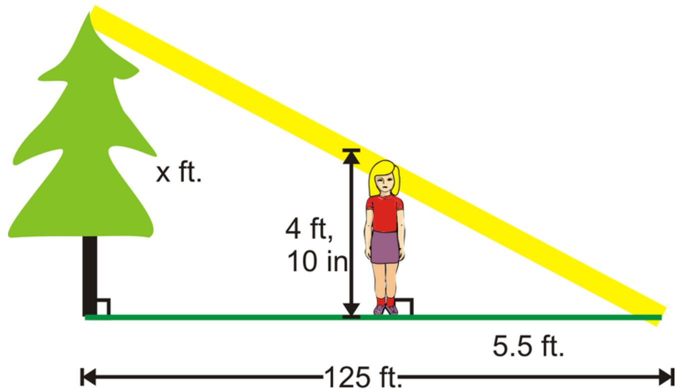


Indirect Measurement

An application of similar triangles is to measure lengths *indirectly*. The length to be measured would be some feature that was not easily accessible to a person, such as: the width of a river or canyon and the height of a tall object. To measure something indirectly, you would need to set up a pair of similar triangles.

Example 4: A tree outside Ellie's building casts a 125 foot shadow. At the same time of day, Ellie casts a 5.5 foot shadow. If Ellie is 4 feet 10 inches tall, how tall is the tree?

Solution: Draw a picture. From the picture to the right, we see that the tree and Ellie are parallel, therefore the two triangles are similar to each other. Write a proportion.



$$\frac{4ft,10in}{xft} = \frac{5.5ft}{125ft}$$

Notice that our measurements are not all in the same units. Change both numerators to inches and then we can cross multiply.

$$\frac{58in}{xft} = \frac{66in}{125ft} \longrightarrow 58(125) = 66(x)$$
$$7250 = 66x$$
$$x \approx 109.85\ ft$$

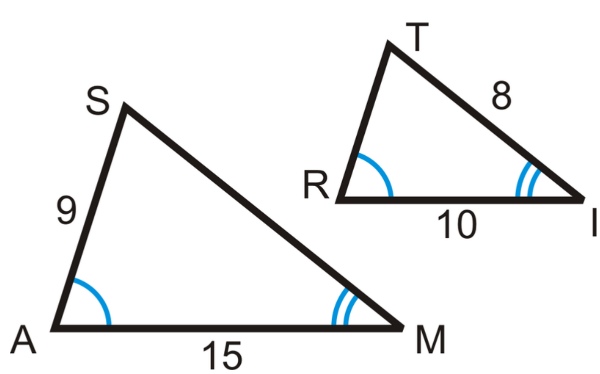
Know What? Revisited It is safe to assume that George and the flagpole stand vertically, making right angles with the ground. Also, the angle where the sun's rays hit the ground is the same for both. The two trianglesare similar. Set up a proportion.

$$\frac{10}{85} = \frac{6}{x} \longrightarrow 10x = 510$$
$$x = 51\ ft.$$

The height of the flagpole is 51 feet.

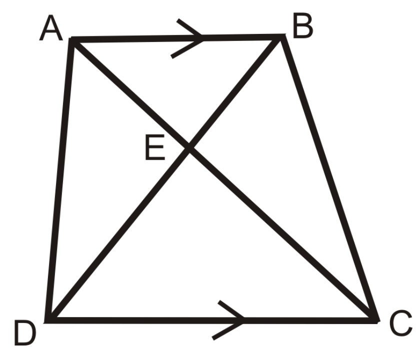
Review Questions

Use the diagram to complete each statement.



1. $\triangle SAM \sim \triangle$____
2. $\frac{SA}{?} = \frac{SM}{?} = \frac{?}{RI}$
3. $SM =$ ____
4. $TR =$ ____
5. $\frac{9}{?} = \frac{?}{8}$

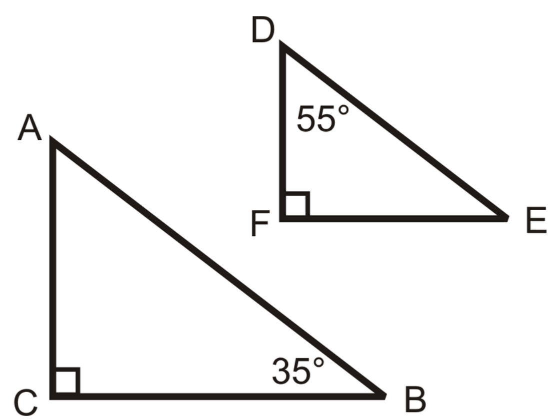
Answer questions 6-9 about trapezoid $ABCD$.



6. Name two similar triangles. How do you know they are similar?
7. Write a true proportion.
8. Name two other triangles that might *not* be similar.
9. If $AB = 10, AE = 7$, and $DC = 22$, find AC. Be careful!
10. ***Writing*** How many angles need to be congruent to show that two triangles are similar? Why?
11. ***Writing*** How do congruent triangles and similar triangles differ? How are they the same?

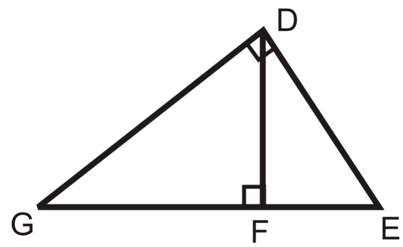
Use the triangles to the left for questions 5-9.

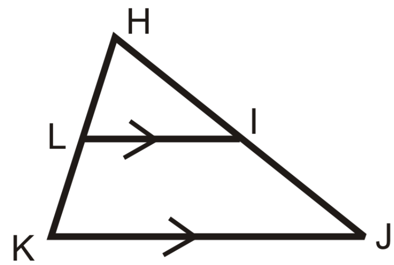
$AB = 20, DE = 15$, and $BC = k$.

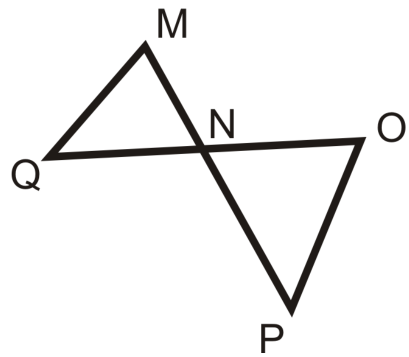


12. Are the two triangles similar? How do you know?
13. Write an expression for FE in terms of k.
14. If $FE = 12$, what is k?
15. Fill in the blanks: If an acute angle of a _______ triangle is congruent to an acute angle in another ________-triangle, then the two triangles are _______.

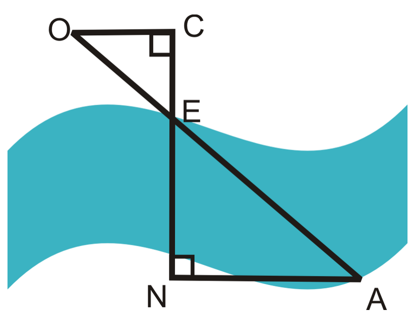
Are the following triangles similar? If so, write a similarity statement.

16. 

17. 

18. 

In order to estimate the width of a river, the following technique can be used. Use the diagram on the left.

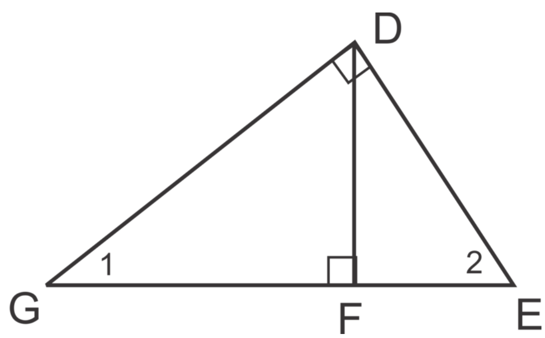


Place three markers, O, C, and E on the upper bank of the river. E is on the edge of the river and $\overline{OC} \perp \overline{CE}$. Go across the river and place a marker, N so that it is collinear with C and E. Then, walk along the lower bank of the river and place marker A, so that $\overline{CN} \perp \overline{NA}$. $OC = 50\ feet, CE = 30\ feet, NA = 80\ feet$.

19. Is $\overline{OC} \parallel \overline{NA}$? How do you know?
20. Is $\triangle OCE \sim \triangle ANE$? How do you know?
21. What is the width of the river? Find EN.
22. Can we find EA? If so, find it. If not, explain.
23. Janet wants to measure the height of her apartment building. She places a pocket mirror on the ground 20 ft from the building and steps backwards until she can see the top of the build in the mirror. She is 18 in from the mirror and her eyes are 5 ft 3 in above the ground. The angle formed by her line of sight and the ground is congruent to the angle formed by the reflection of the building and the ground. You may wish to draw a diagram to illustrate this problem. How tall is the building?

24. Sebastian is curious to know how tall the announcer's box is on his school's football field. On a sunny day he measures the shadow of the box to be 45 ft and his own shadow is 9 ft. Sebastian is 5 ft 10 in tall. Find the height of the box.
25. Juanita wonders how tall the mast of a ship she spots in the harbor is. The deck of the ship is the same height as the pier on which she is standing. The shadow of the mast is on the pier and she measures it to be 18 ft long. Juanita is 5 ft 4 in tall and her shadow is 4 ft long. How tall is the ship's mast?
26. Use shadows or a mirror to measure the height of an object in your yard or on the school grounds. Draw a picture to illustrate your method.

Use the diagram below to answer questions 27-31.



27. Draw the three separate triangles in the diagram.
28. Explain why $\triangle GDE \cong \triangle DFE \cong \triangle GFD$.

Complete the following proportionality statements.

29. $\frac{GF}{DF} = \frac{?}{FE}$
30. $\frac{GF}{GD} = \frac{?}{GE}$
31. $\frac{GE}{DE} = \frac{DE}{?}$

Review Queue Answers

a. $x = 52^\circ, y = 80^\circ$
b. $\frac{w}{20} = \frac{15}{25}$ $\qquad \frac{15}{25} = \frac{18}{z}$
$25w = 15(20)$ $\qquad 25(18) = 15z$
$25w = 300$ $\qquad 450 = 15z$
$w = 12$ $\qquad 30 = z$

a. $168 = 8x$ $\qquad 6y = 216$
$x = 21$ $\qquad y = 36$
b. Answers will vary. One possibility: $\frac{28}{8} = \frac{21}{6}$
c. $28(12) = 8(6+x)$
$336 = 48 + 8x$
$288 = 8x$
$36 = x$ Because $x \neq 21$, like in part a, this is not a true proportion.

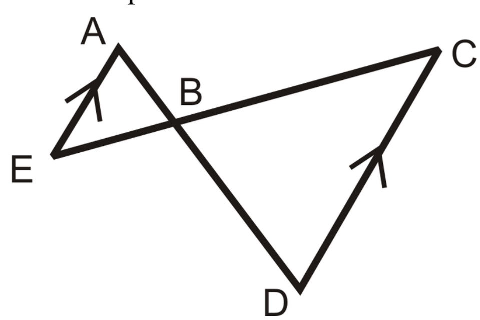
7.4 Similarity by SSS and SAS

Learning Objectives

- Use SSS and SAS to determine whether triangles are similar.
- Apply SSS and SAS to solve problems about similar triangles.

Review Queue

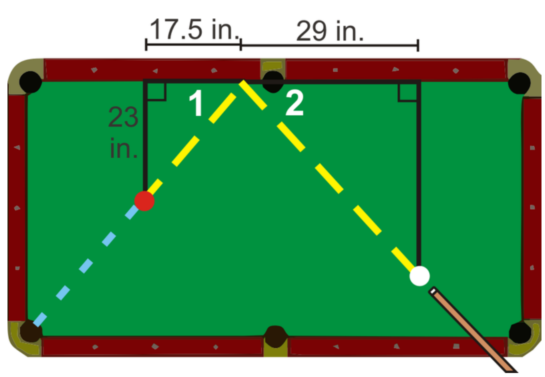
a. a. What are the congruent angles? List each pair.



   b. Write the similarity statement.
   c. If $AB = 8, BD = 20$, and $BC = 25$, find BE.

b. Solve the following proportions.
   a. $\frac{6}{8} = \frac{21}{x}$
   b. $\frac{x+2}{6} = \frac{2x-1}{15}$
   c. $\frac{x-3}{9} = \frac{4}{x+2}$

Know What? Recall from Chapter 2, that the game of pool relies heavily on angles. In Section 2.5, we discovered that $m\angle 1 = m\angle 2$.



The dimensions of a pool table are 92 inches by 46 inches. You decide to hit the cue ball so it follows the yellow path to the right. The horizontal and vertical distances are in the picture. Are the two triangles similar? Why? How far did the cue ball travel to get to the red ball?

Link for an interactive game of pool: http://www.coolmath-games.com/0-poolgeometry/index.html

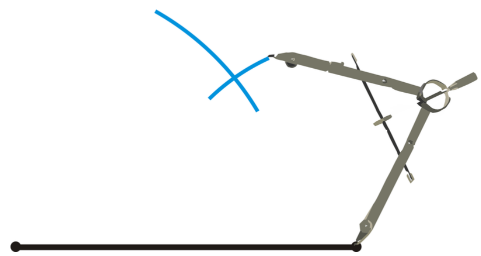
SSS for Similar Triangles

If you do not know any angle measures, can you say two triangles are similar? Let's investigate this to see. You will need to recall Investigation 4-2, Constructing a Triangle, given Three Sides.

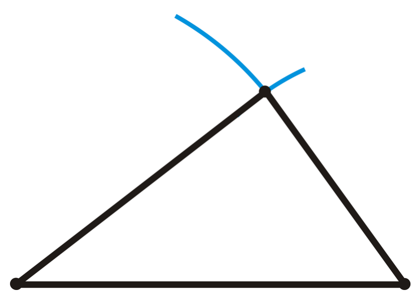
Investigation 7-2: SSS Similarity

Tools Needed: ruler, compass, protractor, paper, pencil

a. Using Investigation 4-2, construct a triangle with sides 6 cm, 8 cm, and 10 cm.



b. Construct a second triangle with sides 9 cm, 12 cm, and 15 cm.
c. Using your protractor, measure the angles in both triangles. What do you notice?
d. Line up the corresponding sides. Write down the ratios of these sides. What happens?

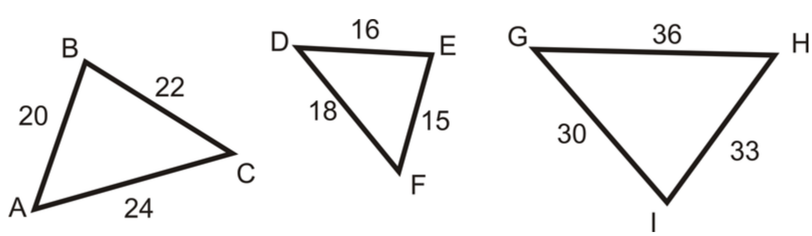


To see an animated construction of this, click: http://www.mathsisfun.com/geometry/construct-ruler-compass-1.htm l

From #3, you should notice that the angles in the two triangles are equal. Second, when the corresponding sides are lined up, the sides are all in the same proportion, $\frac{6}{9} = \frac{8}{12} = \frac{10}{15}$. If you were to repeat this activity, for a 3-4-5 or 12-16-20 triangle, you will notice that they are all similar. That is because, each of these triangles are multiples of 3-4-5. If we generalize what we found in this investigation, we have the SSS Similarity Theorem.

SSS Similarity Theorem: If the corresponding sides of two triangles are proportional, then the two triangles are similar.

Example 1: Determine if any of the triangles below are similar.



Solution: Compare two triangles at a time. In the proportions, place the shortest sides over each other, the longest sides over each other, and the middle sides over each other. Then, determine if the proportions are equal.

$\triangle ABC$ and $\triangle DEF$: $\frac{20}{15} = \frac{22}{16} = \frac{24}{18}$

Reduce each fraction to see if they are equal. $\frac{20}{15} = \frac{4}{3}$, $\frac{22}{16} = \frac{11}{8}$, and $\frac{24}{18} = \frac{4}{3}$. Because $\frac{4}{3} \neq \frac{11}{8}$, $\triangle ABC$ and $\triangle DEF$ are **not** similar.

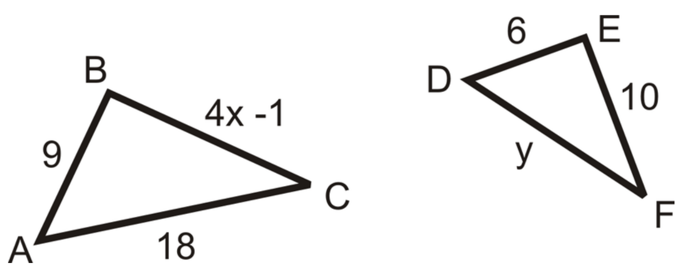
$\triangle DEF$ and $\triangle GHI$: $\frac{15}{30} = \frac{16}{33} = \frac{18}{36}$

$\frac{15}{30} = \frac{1}{2}$, $\frac{16}{33} = \frac{16}{33}$, and $\frac{18}{36} = \frac{1}{2}$. Because $\frac{1}{2} \neq \frac{16}{33}$, $\triangle DEF$ is not similar to $\triangle GHI$.

$\triangle ABC$ and $\triangle GHI$: $\frac{20}{30} = \frac{22}{33} = \frac{24}{36}$

$\frac{20}{30} = \frac{2}{3}$, $\frac{22}{33} = \frac{2}{3}$, and $\frac{24}{36} = \frac{2}{3}$. Because all three ratios reduce to $\frac{2}{3}$, $\triangle ABC \sim \triangle GIH$.

Example 2: ***Algebra Connection*** Find x and y, such that $\triangle ABC \sim \triangle DEF$.



Solution: According to the similarity statement, the corresponding sides are: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. Substituting in what we know, we have:

$$\frac{9}{6} = \frac{4x-1}{10} = \frac{18}{y}$$

$$\begin{aligned} \frac{9}{6} &= \frac{4x-1}{10} \\ 9(10) &= 6(4x-1) \\ 90 &= 24x-6 \\ 96 &= 24x \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} \frac{9}{6} &= \frac{18}{y} \\ 9y &= 18(6) \\ 9y &= 108 \\ y &= 12 \end{aligned}$$

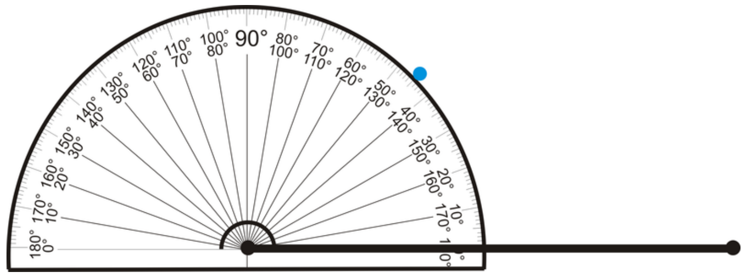
SAS for Similar Triangles

SAS is the last way to show two triangles are similar. If we know that two sides are proportional AND the included angles are congruent, then the two triangles are similar. For the following investigation, you will need to use Investigation 4-3, Constructing a Triangle with SAS.

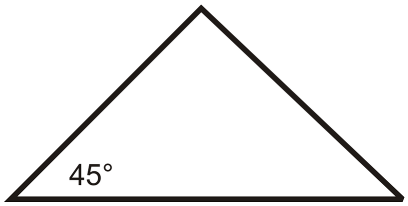
Investigation 7-3: SAS Similarity

Tools Needed: paper, pencil, ruler, protractor, compass

a. Using Investigation 4-3, construct a triangle with sides 6 cm and 4 cm and the *included* angle is 45°.



b. Repeat Step 1 and construct another triangle with sides 12 cm and 8 cm and the included angle is 45°.
c. Measure the other two angles in both triangles. What do you notice?

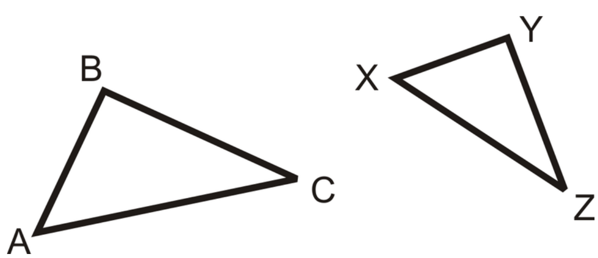


d. Measure the third side in each triangle. Make a ratio. Is this ratio the same as the ratios of the sides you were given?

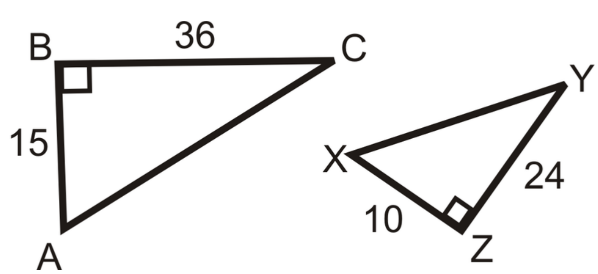
SAS Similarity Theorem: If two sides in one triangle are proportional to two sides in another triangle and the included angle in the first triangle is congruent to the included angle in the second, then the two triangles are similar.

In other words,

If $\frac{AB}{XY} = \frac{AC}{XZ}$ and $\angle A \cong \angle X$, then $\triangle ABC \sim \triangle XYZ$.

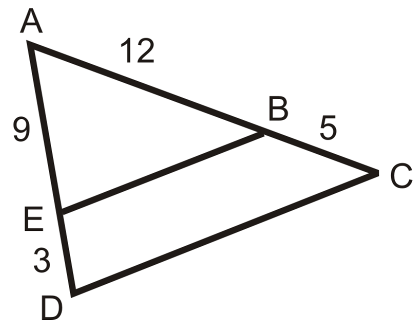


Example 3: Are the two triangles similar? How do you know?



Solution: $\angle B \cong \angle Z$ because they are both right angles. Second, $\frac{10}{15} = \frac{24}{36}$ because they both reduce to $\frac{2}{3}$. Therefore, $\frac{AB}{XZ} = \frac{BC}{ZY}$ and $\triangle ABC \sim \triangle XZY$.

Notice with this example that we can find the third sides of each triangle using the Pythagorean Theorem. If we were to find the third sides, $AC = 39$ and $XY = 26$. The ratio of these sides is $\frac{26}{39} = \frac{2}{3}$.



Example 4: Are there any similar triangles? How do you know?

Solution: $\angle A$ is shared by $\triangle EAB$ and $\triangle DAC$, so it is congruent to itself. If $\frac{AE}{AD} = \frac{AB}{AC}$ then, by SAS Similarity, the two triangles would be similar.

$$\frac{9}{9+3} = \frac{12}{12+5}$$
$$\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$$

Because the proportion is not equal, the two triangles are not similar.

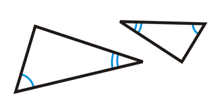
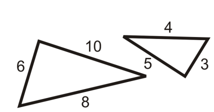
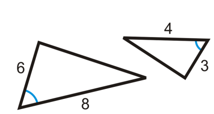
Example 5: From Example 4, what should BC equal for $\triangle EAB \sim \triangle DAC$?

Solution: The proportion we ended up with was $\frac{9}{12} = \frac{3}{4} \neq \frac{12}{17}$. AC needs to equal 16, so that $\frac{12}{16} = \frac{3}{4}$. Therefore, $AC = AB + BC$ and $16 = 12 + BC$. BC should equal 4 in order for $\triangle EAB \sim \triangle DAC$.

Similar Triangles Summary

Let's summarize what we've found that guarantees two triangles are similar.Two triangles are **similar** if and only if:

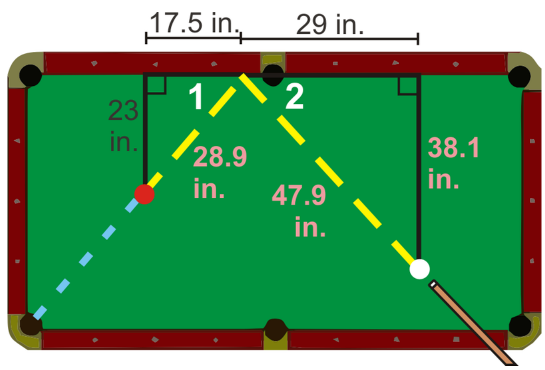
TABLE 7.2:

Name	*Description*	*Picture*
AA	Two angles in one triangle are congruent to two angles in another triangle.	
SSS for Similar Triangles	All three sides in one triangle are proportional to three sides in another triangle.	
SAS for Similar Triangles	Two sides in one triangle are proportional with two sides in another triangle AND the included angles are congruent.	

Know What? Revisited First, we need to find the vertical length of the larger triangle. The two triangles are similar by AA, two right angles and $\angle 1 \cong \angle 2$. Set up a proportion.

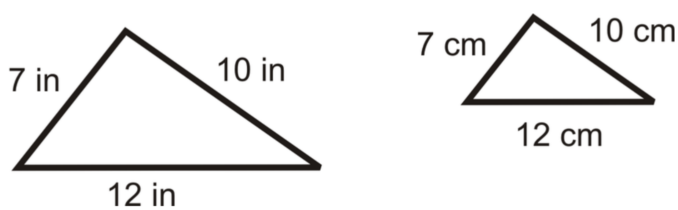
$$\frac{17.5}{23} = \frac{29}{v}$$

Doing cross-multiplication, $v = 38.1$. Second, to find the distance that the cue ball travels, use the Pythagorean Theorem. $17.5^2 + 23^2 = d_1^2$ and $38.1^2 + 29^2 = d_2^2$, the lengths 28.9 and 47.9, and the total length is 76.8 inches.



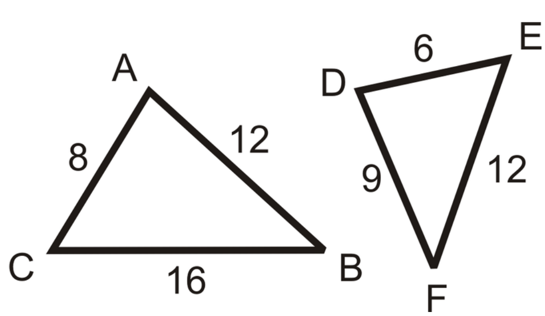
Review Questions

Use the following diagram for questions 1-3. *The diagram is to scale.*



1. Are the two triangles similar? Explain your answer.
2. Are the two triangles congruent? Explain your answer.
3. What is the scale factor for the two triangles?
4. ***Writing*** How come there is no ASA Similarity Theorem?

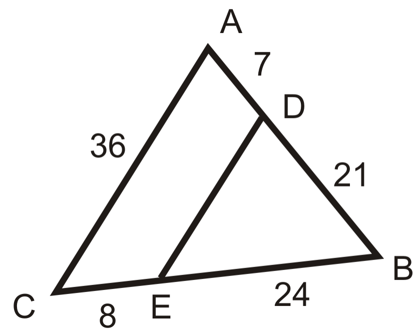
Fill in the blanks in the statements below. Use the diagram to the left.



5. $\triangle ABC \sim \triangle$____

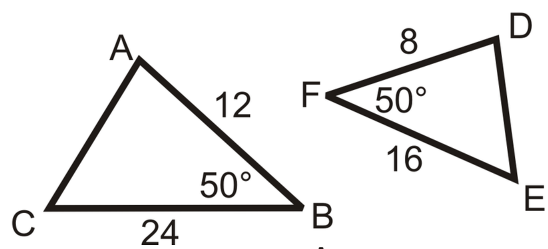
6. $\frac{AB}{?} = \frac{BC}{?} = \frac{AC}{?}$
7. If $\triangle ABC$ had an altitude, $AG = 10$, what would be the length of altitude $\overline{DH}$?

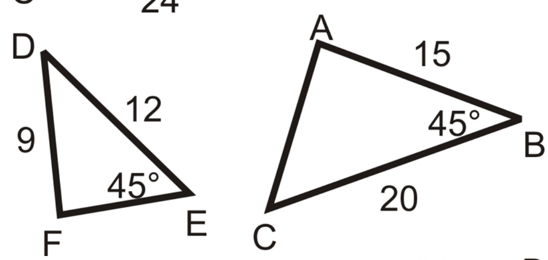
Use the diagram to the right for questions 8-12.

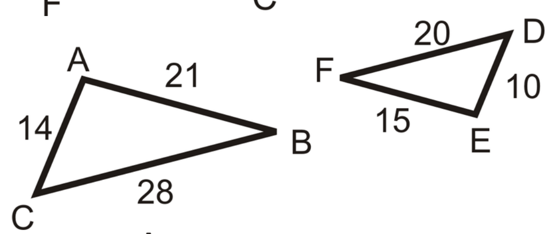


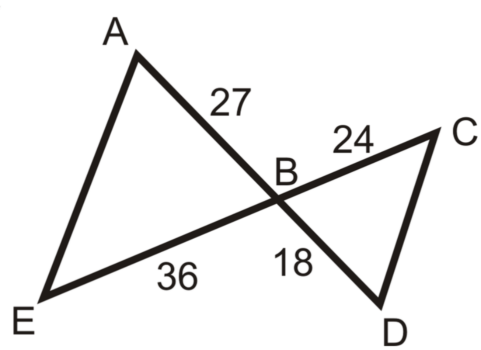
8. $\triangle ABC \sim \triangle$____
9. Why are the two triangles similar?
10. Find ED.
11. $\frac{BD}{?} = \frac{?}{BC} = \frac{DE}{?}$
12. Is $\frac{AD}{DB} = \frac{CE}{EB}$ a valid proportion? How do you know?

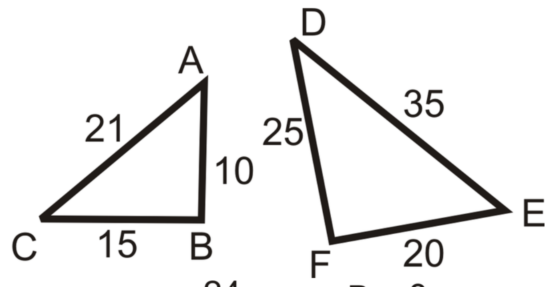
Determine if the following triangles are similar. If so, write the similarity theorem and statement.

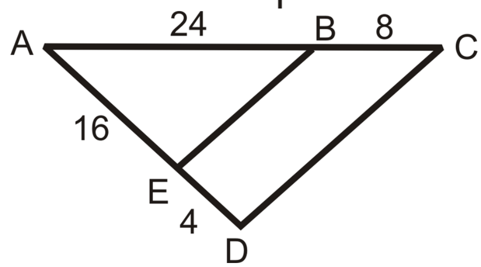
13. 

14. 

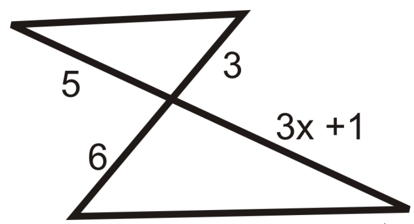
15. 

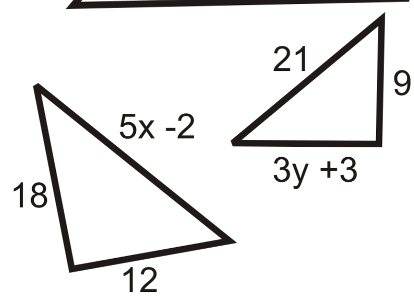
16. 

17. 

18. 

Algebra Connection Find the value of the missing variable(s) that makes the two triangles similar.

19. 

20. 

21. At a certain time of day, a building casts a 25 ft shadow. At the same time of day, a 6 ft tall stop sign casts a 15 ft shadow. How tall is the building?
22. A child who is 42 inches tall is standing next to the stop sign in #21. How long is her shadow?
23. An architect wants to build 3 similar right triangles such that the ratio of the middle triangle to the small triangle is the same as the ratio of the largest triangle to the middle triangle. The smallest one has side lengths 5, 12 and 13. The largest triangle has side lengths 45, 108 and 117. What are the lengths of the sides of the middle triangle?
24. Jaime wants to find the height of a radio tower in his neighborhood. He places a mirror on the ground 30 ft from the tower and walks backwards 3 ft until he can see the top of the tower in the mirror. Jaime is 5 ft 6 in tall. How tall is the radio tower?

For questions 25-27, use $\triangle ABC$ with $A(-3,0), B(-1.5,3)$ and $C(0,0)$ and $\triangle DEF$ with $D(0,2), E(1,4)$ and $F(2,2)$.

25. Find AB, BC, AC, DE, EF and DF.
26. Use these values to find the following proportions: $\frac{AB}{DE}, \frac{BC}{EF}$ and $\frac{AC}{DF}$.
27. Are these triangles similar? Justify your answer.

For questions 28-31, use $\triangle CAR$ with $C(-3,3), A(-3,-1)$ and $R(0,-1)$ and $\triangle LOT$ with $L(5,-2), O(5,6)$ and $T(-1,6)$.

28. Find the slopes of $\overline{CA}, \overline{AR}, \overline{LO}$ and $\overline{OT}$.
29. What are the measures of $\angle A$ and $\angle O$? Explain.
30. Find LO, OT, CA and AR. Use these values to write the ratios $LO : CA$ and $OT : AR$.
31. Are the triangles similar? Justify your answer.

Review Queue Answer

a.
   a. $\angle A \cong \angle D, \angle E \cong \angle C$
   b. $\triangle ABE \sim \triangle DBC$
   c. $BE = 10$

   a. $\frac{6}{8} = \frac{21}{x}, x = 28$
   b. $$\begin{aligned} 15(x+2) &= 6(2x-1) \\ 15x+30 &= 12x-6 \\ 3x &= -36 \\ x &= -12 \end{aligned}$$

b. $$\begin{aligned} (x-3)(x+2) &= 36 \\ x^2 - x - 6 &= 36 \\ x^2 - x - 42 &= 0 \\ (x-7)(x+6) &= 0 \\ x &= 7, -6 \end{aligned}$$

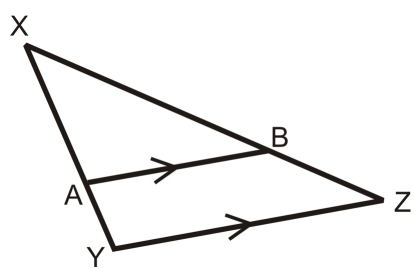
7.5 Proportionality Relationships

Learning Objectives

- Identify proportional segments when two sides of a triangle are cut by a segment parallel to the third side.
- Extend triangle proportionality to parallel lines.

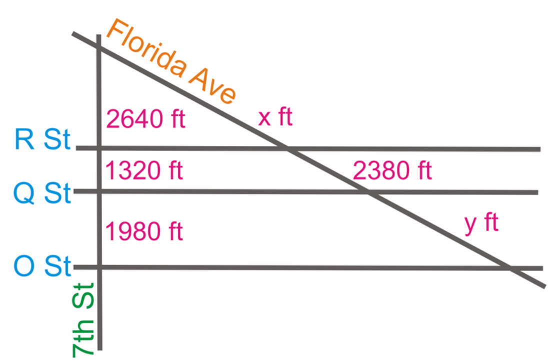
Review Queue

a. Write a similarity statement for the two triangles in the diagram. Why are they similar?



b. If $XA = 16, XY = 18, XB = 32$, find XZ.
c. If $YZ = 27$, find AB.
d. Find AY and BZ.
e. Is $\frac{AY}{AX} = \frac{BZ}{BX}$?

Know What? To the right is a street map of part of Washington DC. R Street, Q Street, and O Street are parallel and 7^{th} Street is perpendicular to all three. R and Q are one "city block" (usually $\frac{1}{4}$ mile or 1320 feet) apart. The other given measurements are on the map. What are x and y?



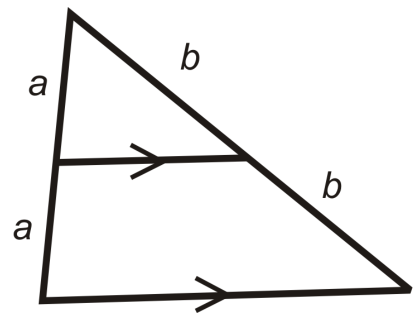
What is the distance from:

- R and 7^{th} to R and Florida?
- Q and 7^{th} to Q and Florida?
- O and 7^{th} to O and Florida?

Triangle Proportionality

Think about a midsegment of a triangle. A midsegment is parallel to one side of a triangle and divides the other two sides into congruent halves. The midsegment divides those two sides ***proportionally.***

Example 1: A triangle with its midsegment is drawn below. What is the ratio that the midsegment divides the sides into?



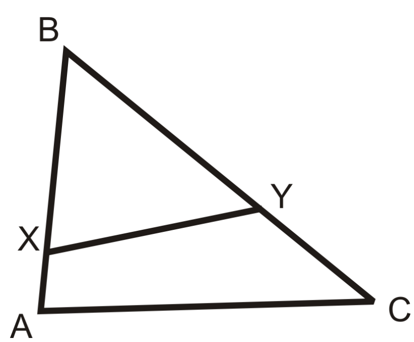
Solution: The midsegment's endpoints are the midpoints of the two sides it connects. The midpoints split the sides evenly. Therefore, the ratio would be $a : a$ or $b : b$. Both of these reduce to 1:1.

The midsegment divides the two sides of the triangle proportionally, but what about other segments?

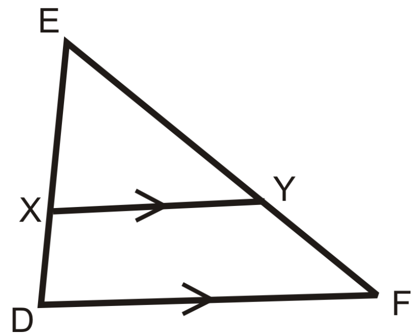
Investigation 7-4: Triangle Proportionality

Tools Needed: pencil, paper, ruler

a. Draw $\triangle ABC$. Label the vertices.
b. Draw $\overline{XY}$ so that X is on $\overline{AB}$ and Y is on $\overline{BC}$. X and Y can be ***anywhere*** on these sides.



c. Is $\triangle XBY \sim \triangle ABC$? Why or why not? Measure AX, XB, BY, and YC. Then set up the ratios $\frac{AX}{XB}$ and $\frac{YC}{YB}$. Are they equal?
d. Draw a second triangle, $\triangle DEF$. Label the vertices.
e. Draw $\overline{XY}$ so that X is on $\overline{DE}$ and Y is on $\overline{EF}$ AND $\overline{XY} \,||\, \overline{DF}$.
f. Is $\triangle XEY \sim \triangle DEF$? Why or why not? Measure DX, XE, EY, and YF. Then set up the ratios $\frac{DX}{XE}$ and $\frac{FY}{YE}$. Are they equal?

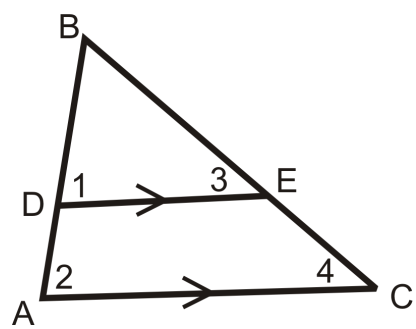


From this investigation, it is clear that if the line segments are parallel, then $\overline{XY}$ divides the sides proportionally.

Triangle Proportionality Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Triangle Proportionality Theorem Converse: If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Proof of the Triangle Proportionality Theorem



Given: $\triangle ABC$ with $\overline{DE} \parallel \overline{AC}$

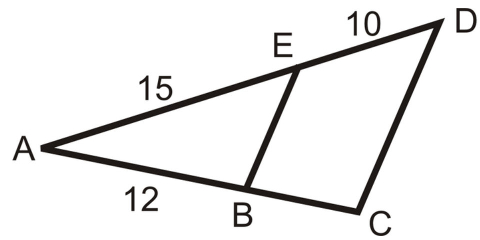
Prove: $\frac{AD}{DB} = \frac{CE}{EB}$

TABLE 7.3:

Statement	***Reason***
1. $\overline{DE} \parallel \overline{AC}$	Given
2. $\angle 1 \cong \angle 2, \angle 3 \cong \angle 4$	Corresponding Angles Postulate
3. $\triangle ABC \sim \triangle DBE$	AA Similarity Postulate
4. $AD + DB = AB$	
EC + EB = BC	Segment Addition Postulate
5. $\frac{AB}{BD} = \frac{BC}{BE}$	Corresponding sides in similar triangles are proportional
6. $\frac{AD+DB}{BD} = \frac{EC+EB}{BE}$	Substitution PoE
7. $\frac{AD}{BD} + \frac{DB}{DB} = \frac{EC}{BE} + \frac{BE}{BE}$	Separate the fractions
8. $\frac{AD}{BD} + 1 = \frac{EC}{BE} + 1$	Substitution PoE (something over itself always equals 1)
9. $\frac{AD}{BD} = \frac{EC}{BE}$	Subtraction PoE

We will not prove the converse, it is essentially this proof but in the reverse order. Using the corollaries from earlier in this chapter, $\frac{BD}{DA} = \frac{BE}{EC}$ is also a true proportion.

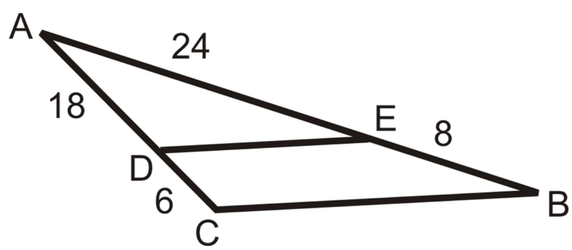
Example 2: In the diagram below, $\overline{EB} \parallel \overline{BD}$. Find BC.



Solution: Use the Triangle Proportionality Theorem.

$$\frac{10}{15} = \frac{BC}{12} \longrightarrow 15(BC) = 120$$
$$BC = 8$$

Example 3: Is $\overline{DE} \parallel \overline{CB}$?



Solution: Use the Triangle Proportionality Converse. If the ratios are equal, then the lines are parallel.

$\frac{6}{18} = \frac{1}{3}$ and $\frac{8}{24} = \frac{1}{3}$

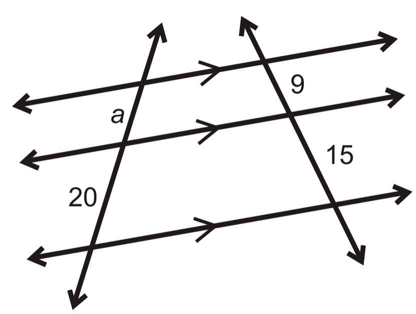
Because the ratios are equal, $\overline{DE} \parallel \overline{CB}$.

Parallel Lines and Transversals

We can extend the Triangle Proportionality Theorem to multiple parallel lines.

Theorem 7-7: If three parallel lines are cut by two transversals, then they divide the transversals proportionally.

Example 4: Find a.

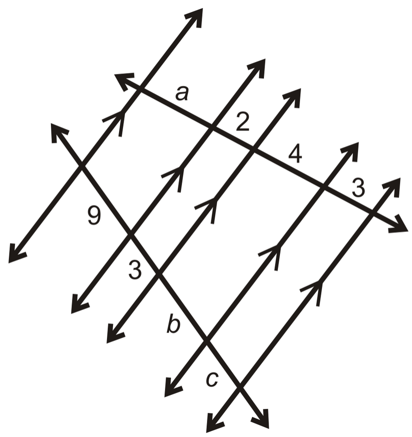


Solution: The three lines are marked parallel, so you can set up a proportion.

$$\frac{a}{20} = \frac{9}{15}$$
$$180 = 15a$$
$$a = 12$$

Theorem 7-7 can be expanded to ***any*** number of parallel lines with ***any*** number of transversals. When this happens all corresponding segments of the transversals are proportional.

Example 5: Find a, b, and c.

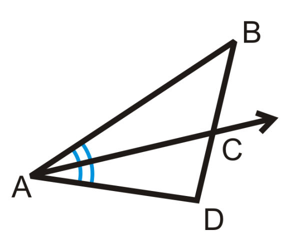


Solution: Look at the corresponding segments. Only the segment marked "2" is opposite a number, all the other segments are opposite variables. That means we will be using this ratio, 2:3 in all of our proportions.

$\frac{a}{2} = \frac{9}{3}$	$\frac{2}{4} = \frac{3}{b}$	$\frac{2}{3} = \frac{3}{c}$
$3a = 18$	$2b = 12$	$2c = 9$
$a = 6$	$b = 6$	$c = 4.5$

There are several ratios you can use to solve this example. To solve for b, you could have used the proportion $\frac{6}{4} = \frac{9}{b}$, which will still give you the same answer.

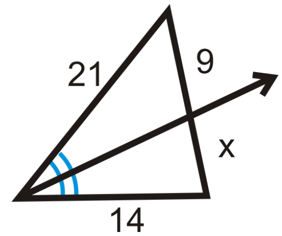
Proportions with Angle Bisectors



The last proportional relationship we will explore is how an angle bisector intersects the opposite side of a triangle. By definition, $\overrightarrow{AC}$ divides $\angle BAD$ equally, so $\angle BAC \cong \angle CAD$. The proportional relationship is $\frac{BC}{CD} = \frac{AB}{AD}$. The proof is in the review exercises.

Theorem 7-8: If a ray bisects an angle of a triangle, then it divides the opposite side into segments that are proportional to the lengths of the other two sides.

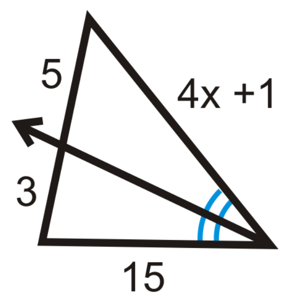
Example 6: Find x.



Solution: Because the ray is the angle bisector it splits the opposite side in the same ratio as the sides. So, the proportion is:

$$\frac{9}{x} = \frac{21}{14}$$
$$21x = 126$$
$$x = 6$$

Example 7: ***Algebra Connection*** Determine the value of x that would make the proportion true.



Solution: You can set up this proportion just like the previous example.

$$\frac{5}{3} = \frac{4x+1}{15}$$
$$75 = 3(4x+1)$$
$$75 = 12x+3$$
$$72 = 12x$$
$$6 = x$$

Know What? Revisited To find x and y, you need to set up a proportion using parallel the parallel lines.

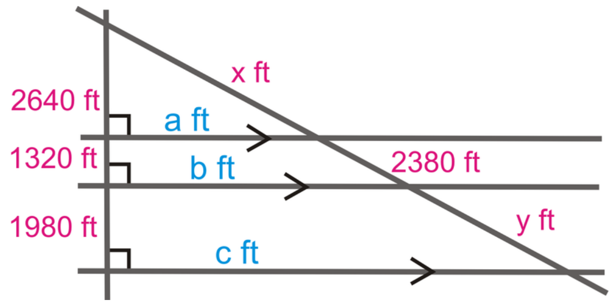
$$\frac{2640}{x} = \frac{1320}{2380} = \frac{1980}{y}$$

From this, $x = 4760\ ft$ and $y = 3570\ ft$.

To find a, b, and c, use the Pythagorean Theorem.

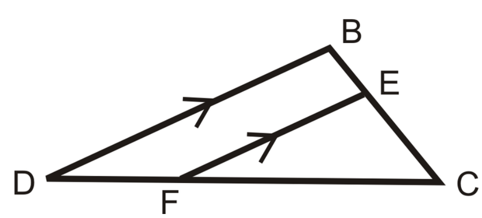
$$2640^2 + a^2 = 4760^2$$
$$3960^2 + b^2 = 7140^2$$
$$5940^2 + c^2 = 10710^2$$

$$a = 3960.81, \ b = 5941.21, \ c = 8911.82$$



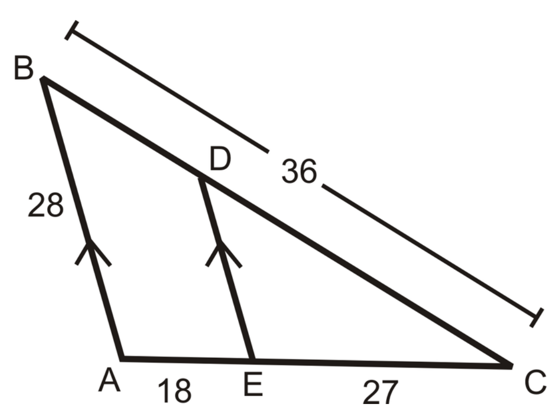
Review Questions

Use the diagram to answers questions 1-5. $\overline{DB} \,||\, \overline{FE}$.



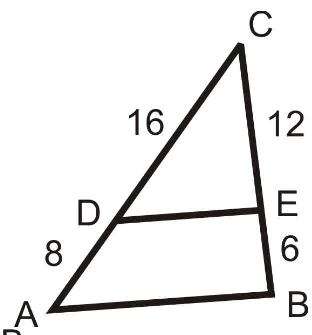
1. Name the similar triangles. Write the similarity statement.
2. $\frac{BE}{EC} = \frac{?}{FC}$
3. $\frac{EC}{CB} = \frac{CF}{?}$
4. $\frac{DB}{?} = \frac{BC}{EC}$
5. $\frac{FC+?}{FC} = \frac{?}{FE}$

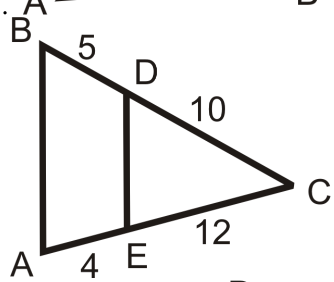
Use the diagram to answer questions 6-10. $\overline{AB} \,||\, \overline{DE}$.

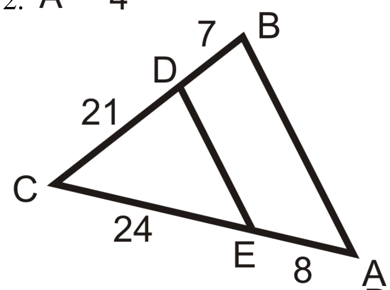


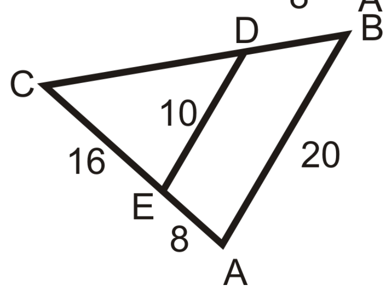
6. Find BD.
7. Find DC.
8. Find DE.
9. Find AC.
10. We know that $\frac{BD}{DC} = \frac{AE}{EC}$ and $\frac{BA}{DE} = \frac{BC}{DC}$. Why is $\frac{BA}{DE} \neq \frac{BD}{DC}$?

Use the given lengths to determine if $\overline{AB} \parallel \overline{DE}$.

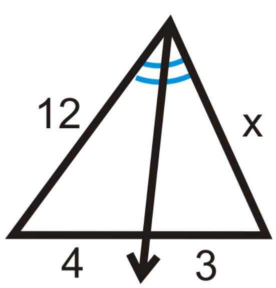
11. 

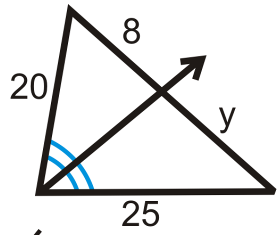
12. 

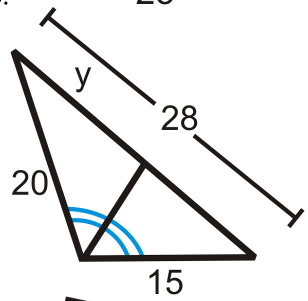
13. 

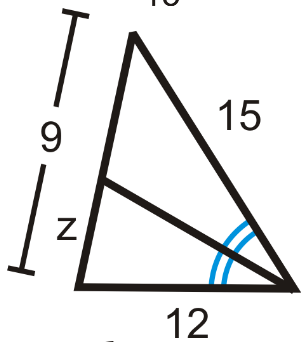
14. 

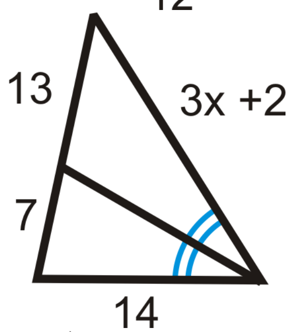
Algebra Connection Find the value of the missing variable(s).

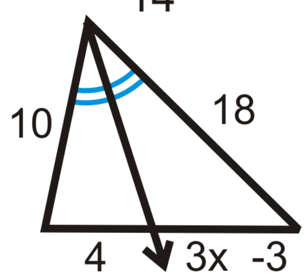
15. 

16. 

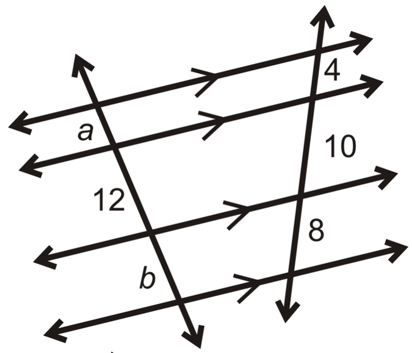
17. 

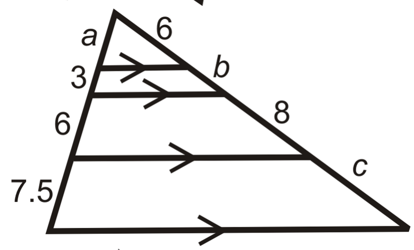
18. 

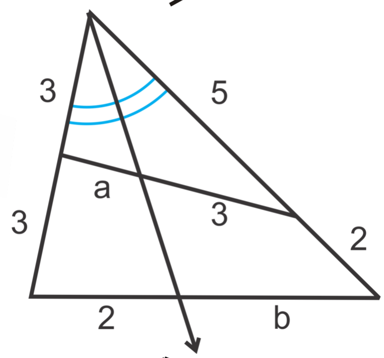
19. 

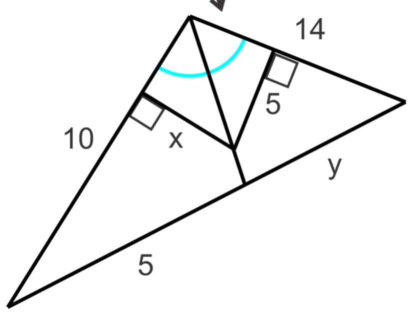
20. 

Find the value of each variable in the pictures below.

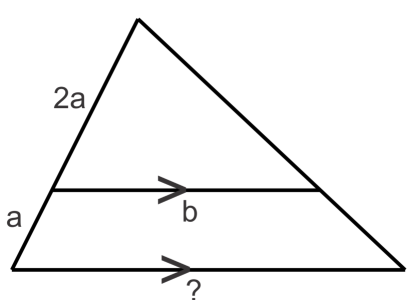
21. 

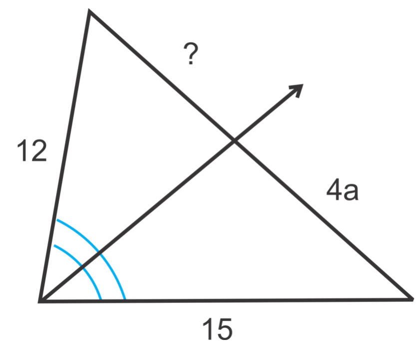
22. 

23. 

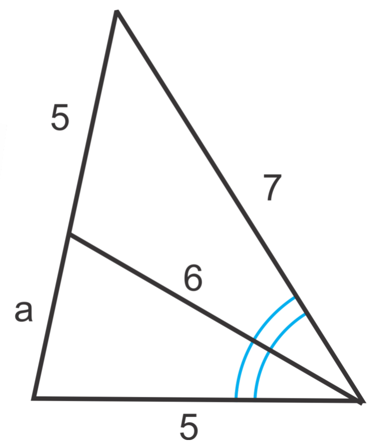
24. 

Find the unknown lengths.

25. 

26. 

27. ***Error Analysis***

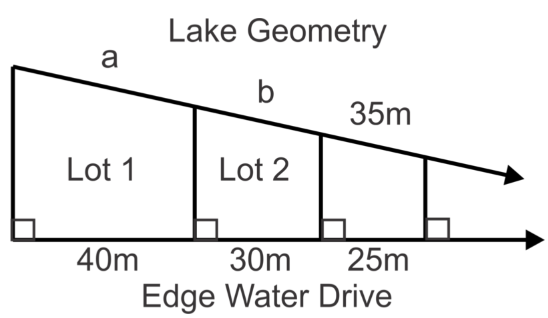


Casey attempts to solve for a in the diagram using the proportion

$$\frac{5}{a} = \frac{6}{5}$$

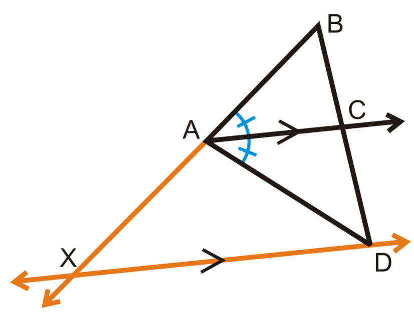
What did Casey do wrong? Write the correct proportion and solve for a.

28. Michael has a triangular shaped garden with sides of length 3, 5 and 6 meters. He wishes to make a path along the perpendicular bisector of the angle between the sides of length 3 m and 5 m. Where will the path intersect the third side?

29. 

This is a map of lake front properties. Find a and b, the length of the edge of Lot 1 and Lot 2 that is adjacent to the lake.

30. Fill in the blanks of the proof of Theorem 7-8.



Given: $\triangle BAD$ with $\overrightarrow{AC}$ is the angle bisector of $\angle BAD$ Auxiliary lines $\overrightarrow{AX}$ and $\overleftrightarrow{XD}$, such that X, A, B are collinear and $\overrightarrow{AC} \parallel \overleftrightarrow{XD}$. Prove: $\frac{BC}{CD} = \frac{BA}{AD}$

TABLE 7.4:

Statement	Reason
1. $\overrightarrow{AC}$ is the angle bisector of $\angle BADX, A, B$ are collinear and $\overrightarrow{AC} \parallel \overleftrightarrow{XD}$	
2. $\angle BAC \cong \angle CAD$	
3.	Corresponding Angles Postulate
4. $\angle CAD \cong \angle ADX$	
5. $\angle X \cong \angle ADX$	
6. $\triangle XAD$ is isosceles	
7.	Definition of an Isosceles Triangle
8.	Congruent segments are also equal
9.	Theorem 7-7
10.	

Review Queue Answers

a. $\triangle AXB \sim \triangle YXZ$ by AA Similarity Postulate
b. $\frac{16}{18} = \frac{32}{XZ}, XZ = 36$
c. $\frac{16}{18} = \frac{AB}{27}, AB = 24$
d. $AY = 18 - 16 = 2, BZ = 36 - 32 = 4$
e. $\frac{2}{16} = \frac{4}{32}$. Yes, this is a true proportion.

7.6 Extension: Self-Similarity

Learning Objectives

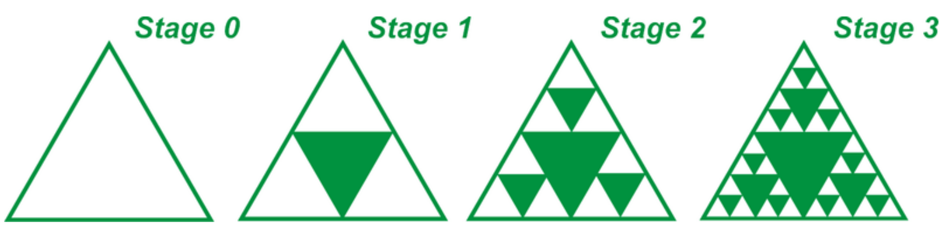
- Draw sets of the Sierpinski Triangle.
- Understand basic fractals.

Self-Similar: When one part of an object can be enlarged (or shrunk) to look like the whole object.

To explore self-similarity, we will go through a couple of examples. Typically, each step of repetition is called an iteration or level. The first level is called the Start Level or Stage 0.

Sierpinski Triangle

The Sierpinski triangle iterates an equilateral triangle (but, any triangle can be used) by connecting the midpoints of the sides and shading the central triangle (Stage 1). Repeat this process for the unshaded triangles in Stage 1 to get Stage 2. This series was part of the **Know What?** in Section 5.1.



Example 1: Determine the number of shaded and unshaded triangles in each stage of the Sierpinkski triangles. Determine if there is a pattern.

Solution:

TABLE 7.5:

	Stage 0	*Stage 1*	*Stage 2*	*Stage 3*
Unshaded	1	3	9	27
Shaded	0	1	4	13

The unshaded triangles seem to be powers of $3, 3^0, 3^1, 3^2, 3^3, \ldots$ The shaded triangles are add the previous number of unshaded triangles to the total. For Example, Stage 4 would equal $9+13$ shaded trangles.

Fractals

A fractal is another self-similar object that is repeated at successively smaller scales. Below are the first three stages of the Koch snowflake.

Example 2: Determine the number of edges and the perimeter of each snowflake.

TABLE 7.6:

	Stage 0	*Stage 1*	*Stage 2*
Number of Edges	3	12	48
Edge Length	1	$\frac{1}{3}$	$\frac{1}{9}$
Perimeter	3	4	$\frac{48}{9} = 5.\overline{3}$

The Cantor Set

The Cantor set is another fractal that consists of dividing a segment into thirds and then erasing the middle third.

Review Questions

1. Draw Stage 4 of the Cantor set.
2. Use the Cantor Set to fill in the table below.

TABLE 7.7:

	Number of Segments	*Length of each Segment*	*Total Length of the Segments*
Stage 0	1	1	1
Stage 1	2	$\frac{1}{3}$	$\frac{2}{3}$
Stage 2	4	$\frac{1}{9}$	$\frac{4}{9}$
Stage 3			
Stage 4			

TABLE 7.7: (continued)

	Number of Segments	Length of each Segment	Total Length of the Segments
Stage 5			

3. How many segments are in Stage n?
4. What is the length of each segment in Stage n?
5. Draw Stage 3 of the Koch snowflake.
6. Fill in the table from Example 2 for Stage 3 of the Koch snowflake.
7. A variation on the Sierpinski triangle is the Sierpinski carpet, which splits a square into 9 equal squares, coloring the middle one only. Then, split the uncolored squares to get the next stage. Draw the first 3 stages of this fractal.
8. How many shaded vs. unshaded squares are in each stage?
9. Fractals are very common in nature. For example, a fern leaf is a fractal. As the leaves get closer to the end, they get smaller and smaller. Find three other examples of fractals in nature.

10. Use the internet to explore fractals further. Write a paragraph about another example of a fractal in music, art or another field that interests you.

7.7 Chapter 7 Review

Keywords and Theorems

Ratio
A way to compare two numbers.

Proportion
When two ratios are set equal to each other.

Means
Mean (also called the arithmetic mean): The numerical balancing point of the data set. Calculated by adding all the data values and dividing the sum by the total number of data points.

Extremes
the product of the means must equal the product of the extremes.

Cross-Multiplication Theorem
the product of the means must equal the product of the extremes

Corollary
A theorem that follows quickly, easily, and directly from another theorem.

Corollary 7-1
If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$.

Corollary 7-2
Corollary 7-2 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{d}{b} = \frac{c}{a}$.

Corollary 7-3
Corollary 7-3 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$.

Corollary 7-4
Corollary 7-4 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{b} = \frac{c+d}{d}$.

Corollary 7-5
Corollary 7-5 If a, b, c, and d are nonzero and $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

Similar Polygons
Two polygons with the same shape, but not the same size.

Scale Factor
In similar polygons, the ratio of one side of a polygon to the corresponding side of the other.

Theorem 7-2
The ratio of the perimeters of two similar polygons is the same as the ratio of the sides.

AA Similarity Postulate
If two angles in one triangle are congruent to two angles in another triangle, the two triangles are similar.

Indirect Measurement
An application of similar triangles is to measure lengths *indirectly*.

SSS Similarity Theorem
If the corresponding sides of two triangles are proportional, then the two triangles are similar.

SAS Similarity Theorem
If two sides in one triangle are proportional to two sides in another triangle and the included angle in the first triangle is congruent to the included angle in the second, then the two triangles are similar.

Triangle Proportionality Theorem
If a line parallel to one side of a triangle intersects the other two sides, then it divides those sides proportionally.

Triangle Proportionality Theorem Converse
If a line divides two sides of a triangle proportionally, then it is parallel to the third side.

Theorem 7-7
If three parallel lines are cut by two transversals, then they divide the transversals proportionally.

Theorem 7-8
If a ray bisects an angle of a triangle, then it divides the opposite side into segments that are proportional to the lengths of the other two sides.

Transformation
An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation
Transformations that preserve size are ***rigid***

Non-rigid Transformation
Transformations that preserve size are ***rigid*** and ones that do not are ***non-rigid.***

Dilation
A non-rigid transformation that preserves shape but not size.

Self-Similar
When one part of an object can be enlarged (or shrunk) to look like the whole object.

Fractal
A fractal is another self-similar object that is repeated at successively smaller scales.

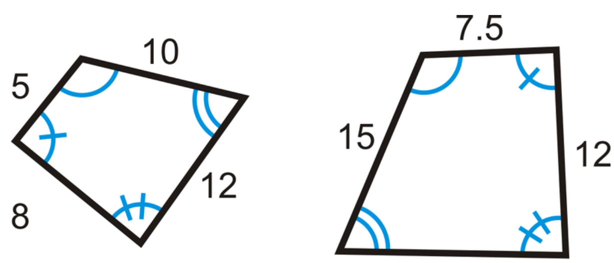
Review Questions

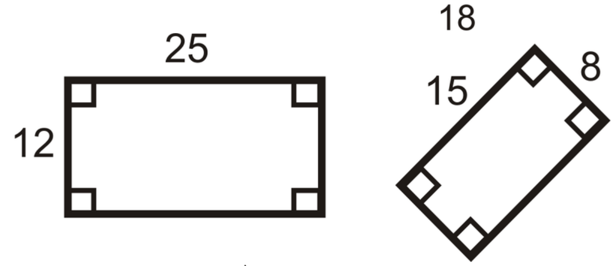
1. Solve the following proportions.

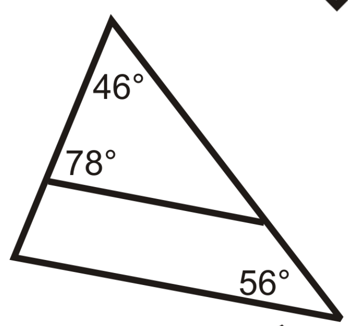
   a. $\frac{x+3}{3} = \frac{10}{2}$
   b. $\frac{8}{5} = \frac{2x-1}{x+3}$

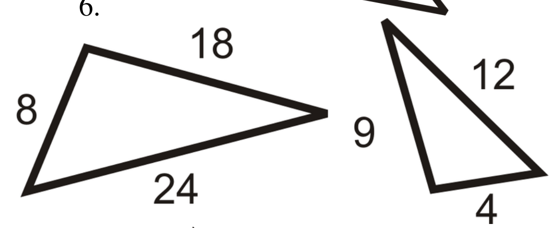
2. The extended ratio of the angle in a triangle are 5:6:7. What is the measure of each angle?
3. Rewrite 15 quarts in terms of gallons.

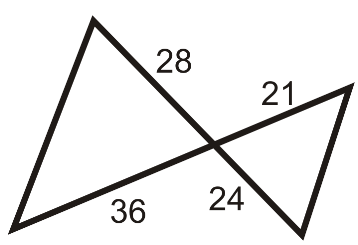
Determine if the following pairs of polygons are similar. If it is two triangles, write *why* they are similar.

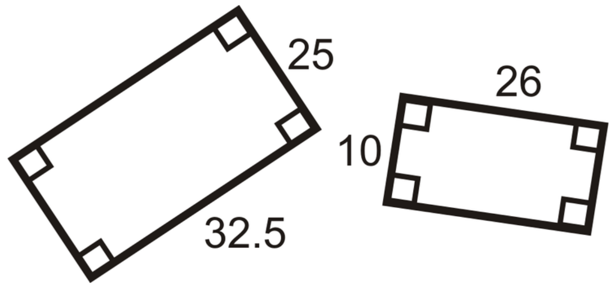
4. 

5. 

6. 

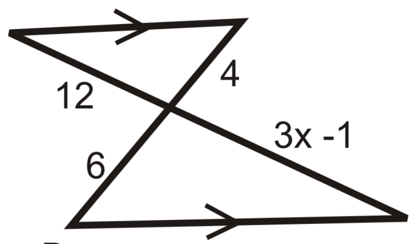
7. 

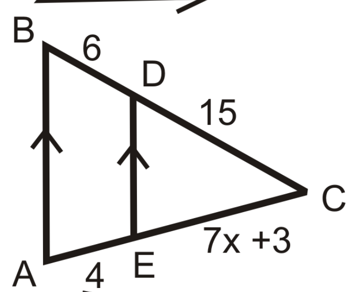
8. 

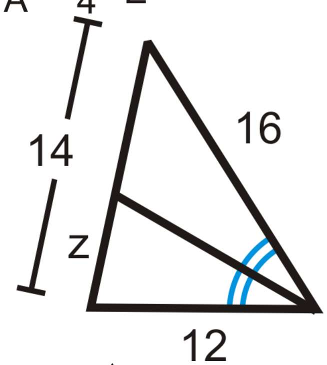
9. 

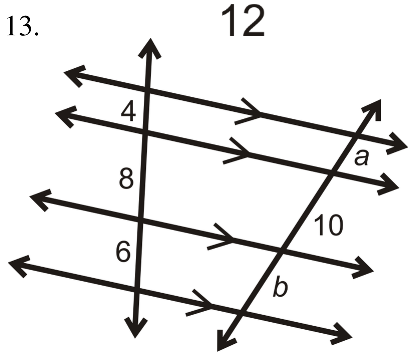
10. Draw a dilation of $A(7,2), B(4,9)$, and $C(-1,4)$ with $k = \frac{3}{2}$.

Algebra Connection Find the value of the missing variable(s).

11. 

12. 

13. 

14. 

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9692 .

CHAPTER 8 Right Triangle Trigonometry

Chapter Outline

Chapter 8 explores right triangles in far more depth than Chapters 4 and 5. Recall that a right triangle is a triangle with exactly one right angle. In this chapter, we will first prove the Pythagorean Theorem and its converse, followed by analyzing the sides of certain types of triangles. Then, we will introduce trigonometry, which starts with the tangent, sine and cosine ratios. Finally, we will extend sine and cosine to any triangle, through the Law of Sines and the Law of Cosines.

8.1 The Pythagorean Theorem

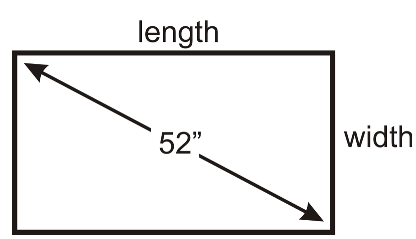
Learning Objectives

- Review simplifying and reducing radicals.
- Prove and use the Pythagorean Theorem.
- Use the Pythagorean Theorem to derive the distance formula.

Review Queue

1. Draw a right scalene triangle.
2. Draw an isosceles right triangle.
3. List all the factors of 75.
4. Write the prime factorization of 75.

Know What? For a 52" TV, 52" is the length of the diagonal. High Definition Televisions (HDTVs) have sides in a ratio of 16:9. What are the length and width of a 52" HDTV?



Simplifying and Reducing Radicals

In algebra, you learned how to simplify radicals. Let's review it here.

Example 1: Simplify the radical.

a) $\sqrt{50}$

b) $\sqrt{27}$

c) $\sqrt{272}$

Solution: For each radical, find the square number(s) that are factors.

a) $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$

b) $\sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}$

c) $\sqrt{272} = \sqrt{16 \cdot 17} = 4\sqrt{17}$

When adding radicals, you can only combine radicals with the same number underneath it. For example, $2\sqrt{5}+3\sqrt{6}$ cannot be combined, because 5 and 6 are not the same number.

Example 2: Simplify the radicals.

a) $2\sqrt{10}+\sqrt{160}$

b) $5\sqrt{6}\cdot 4\sqrt{18}$

c) $\sqrt{8}\cdot 12\sqrt{2}$

d) $\left(5\sqrt{2}\right)^2$

Solution:

a) Simplify $\sqrt{160}$ before adding: $2\sqrt{10}+\sqrt{160}=2\sqrt{10}+\sqrt{16\cdot 10}=2\sqrt{10}+4\sqrt{10}=6\sqrt{10}$

b) To multiply two radicals, multiply what is under the radicals and what is in front.

$5\sqrt{6}\cdot 4\sqrt{18}=5\cdot 4\sqrt{6\cdot 18}=20\sqrt{108}=20\sqrt{36\cdot 3}=20\cdot 6\sqrt{3}=120\sqrt{3}$

c) $\sqrt{8}\cdot 12\sqrt{2}=12\sqrt{8\cdot 2}=12\sqrt{16}=12\cdot 4=48$

d) $\left(5\sqrt{2}\right)^2=5^2\left(\sqrt{2}\right)^2=25\cdot 2=50\rightarrow$ the $\sqrt{\ }$ and the ^{2} cancel each other out

Lastly, to divide radicals, you need to simplify the denominator, which means multiplying the top and bottom of the fraction by the radical in the denominator.

Example 3: Divide and simplify the radicals.

a) $4\sqrt{6}\div\sqrt{3}$

b) $\frac{\sqrt{30}}{\sqrt{8}}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}}$

Solution: Rewrite all division problems like a fraction.

a)

$$4\sqrt{6}\div\sqrt{3}=\frac{4\sqrt{6}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{4\sqrt{18}}{\sqrt{9}}=\frac{4\sqrt{9\cdot 2}}{3}=\frac{4\cdot 3\sqrt{2}}{3}=4\sqrt{2}$$

like multiplying by 1, $\frac{\sqrt{3}}{\sqrt{3}}$ does not change the value of the fraction

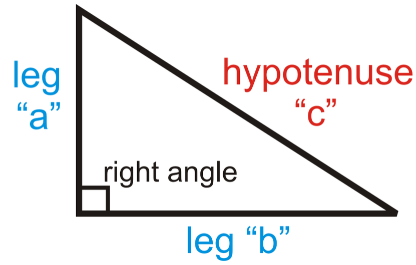
b) $\frac{\sqrt{30}}{\sqrt{8}}\cdot\frac{\sqrt{8}}{\sqrt{8}}=\frac{\sqrt{240}}{\sqrt{64}}=\frac{\sqrt{16\cdot 15}}{8}=\frac{4\sqrt{15}}{8}=\frac{\sqrt{15}}{2}$

c) $\frac{8\sqrt{2}}{6\sqrt{7}}\cdot\frac{\sqrt{7}}{\sqrt{7}}=\frac{8\sqrt{14}}{6\cdot 7}=\frac{4\sqrt{14}}{3\cdot 7}=\frac{4\sqrt{14}}{21}$

Notice, we do not really "divide" radicals, but get them out of the denominator of a fraction.

The Pythagorean Theorem

We have used the Pythagorean Theorem already in this text, but have not proved it. Recall that the sides of a right triangle are the **legs** (the sides of the right angle) and the **hypotenuse** (the side opposite the right angle). For the Pythagorean Theorem, the legs are "a" and "b" and the hypotenuse is "c".

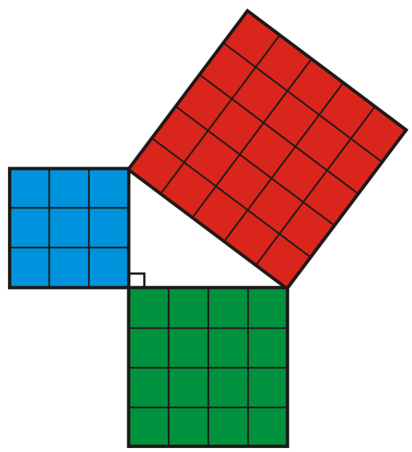


Pythagorean Theorem: Given a right triangle with legs of lengths a and b and a hypotenuse of length c, then $a^2 + b^2 = c^2$.

Investigation 8-1: Proof of the Pythagorean Theorem

Tools Needed: pencil, 2 pieces of graph paper, ruler, scissors, colored pencils (optional)

1. On the graph paper, draw a 3 in. square, a 4 in. square, a 5 in. square and a right triangle with legs of 3 in. and 4 in.
2. Cut out the triangle and square and arrange them like the picture on the right.



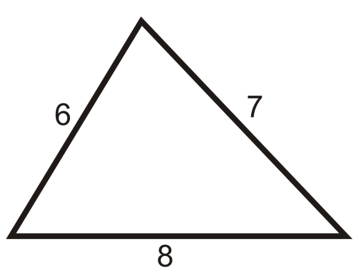
3. This theorem relies on area. Recall that the area of a square is $side^2$. In this case, we have three squares with sides 3 in., 4 in., and 5 in. What is the area of each square?
4. Now, we know that $9 + 16 = 25$, or $3^2 + 4^2 = 5^2$. Cut the smaller squares to fit into the larger square, thus proving the areas are equal.

For two more proofs, go to: http://www.mathsisfun.com/pythagoras.html and scroll down to "And You Can Prove the Theorem Yourself."

Using the Pythagorean Theorem

Here are several examples of the Pythagorean Theorem in action.

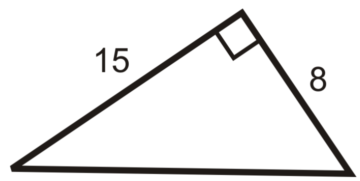
Example 4: Do 6, 7, and 8 make the sides of a right triangle?



Solution: Plug in the three numbers to the Pythagorean Theorem. ***The largest length will always be the hypotenuse.*** If $6^2 + 7^2 = 8^2$, then they are the sides of a right triangle.

$$6^2 + 7^2 = 36 + 49 = 85$$
$$8^2 = 64 \qquad 85 \neq 64, \text{ so the lengths are not the sides of a right triangle.}$$

Example 5: Find the length of the hypotenuse.

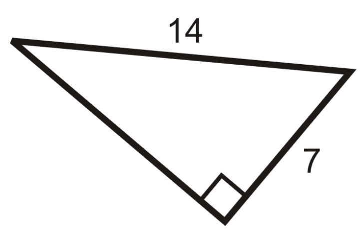


Solution: Use the Pythagorean Theorem. Set $a = 8$ and $b = 15$. Solve for c.

$$\begin{aligned} 8^2 + 15^2 &= c^2 \\ 64 + 225 &= c^2 \\ 289 &= c^2 \qquad \textit{Take the square root of both sides.} \\ 17 &= c \end{aligned}$$

When you take the square root of an equation, the answer is 17 or -17. ***Length is never negative,*** which makes 17 the answer.

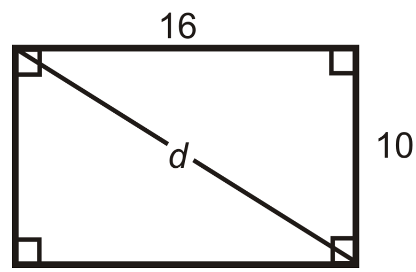
Example 6: Find the missing side of the right triangle below.



Solution: Here, we are given the hypotenuse and a leg. Let's solve for b.

$$\begin{aligned} 7^2 + b^2 &= 14^2 \\ 49 + b^2 &= 196 \\ b^2 &= 147 \\ b &= \sqrt{147} = \sqrt{49 \cdot 3} = 7\sqrt{3} \end{aligned}$$

Example 7: What is the diagonal of a rectangle with sides 10 and 16?



Solution: For any square and rectangle, you can use the Pythagorean Theorem to find the length of a diagonal. Plug in the sides to find d.

$$\begin{aligned} 10^2+16^2 &= d^2 \\ 100+256 &= d^2 \\ 356 &= d^2 \\ d &= \sqrt{356} = 2\sqrt{89} \approx 18.87 \end{aligned}$$

Pythagorean Triples

In Example 5, the sides of the triangle were 8, 15, and 17. This combination of numbers is called a ***Pythagorean triple.***

Pythagorean Triple: A set of three whole numbers that makes the Pythagorean Theorem true.

$$3,4,5 \qquad 5,12,13 \qquad 7,24,25 \qquad 8,15,17 \qquad 9,12,15 \qquad 10,24,26$$

Any multiple of a Pythagorean triple is also considered a triple because it would still be three whole numbers. Multiplying 3, 4, 5 by 2 gives 6, 8, 10, which is another triple. To see if a set of numbers makes a triple, plug them into the Pythagorean Theorem.

Example 8: Is 20, 21, 29 a Pythagorean triple?

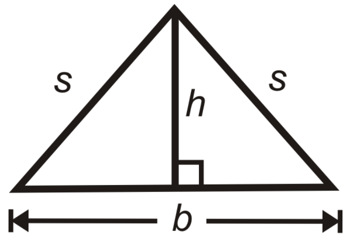
Solution: If $20^2+21^2=29^2$, then the set is a Pythagorean triple.

$$\begin{aligned} 20^2+21^2 &= 400+441=841 \\ 29^2 &= 841 \end{aligned}$$

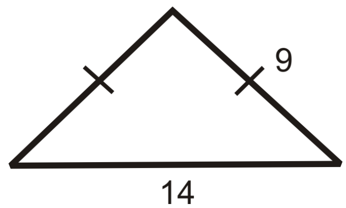
Therefore, 20, 21, and 29 is a Pythagorean triple.

Height of an Isosceles Triangle

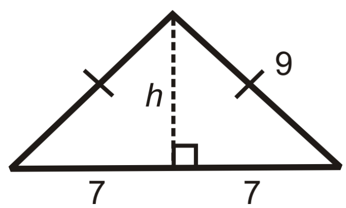
One way to use The Pythagorean Theorem is to find the height of an isosceles triangle.



Example 9: What is the height of the isosceles triangle?



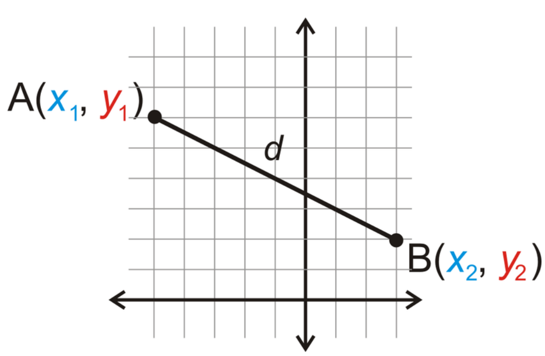
Solution: Draw the altitude from the vertex between the congruent sides, which bisect the base.



$$\begin{aligned}7^2+h^2&=9^2\\49+h^2&=81\\h^2&=32\\h&=\sqrt{32}=\sqrt{16\cdot 2}=4\sqrt{2}\end{aligned}$$

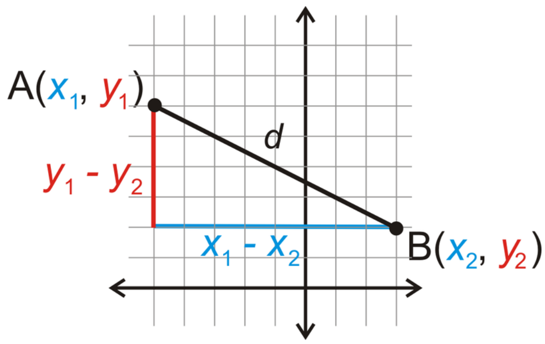
The Distance Formula

Another application of the Pythagorean Theorem is the Distance Formula. We will prove it here.



Let's start with point $A(x_1,y_1)$ and point $B(x_2,y_2)$, to the left. We will call the distance between A and B, d.

Draw the vertical and horizontal lengths to make a right triangle.



Now that we have a right triangle, we can use the Pythagorean Theorem to find the hypotenuse, d.

$$d^2 = (x_1-x_2)^2+(y_1-y_2)^2$$
$$d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Distance Formula: The distance $A(x_1,y_1)$ and $B(x_2,y_2)$ is $d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Example 10: Find the distance between (1, 5) and (5, 2).

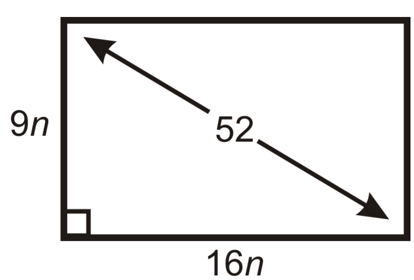
Solution: Make $A(1,5)$ and $B(5,2)$. Plug into the distance formula.

$$\begin{aligned} d &= \sqrt{(1-5)^2+(5-2)^2} \\ &= \sqrt{(-4)^2+(3)^2} \\ &= \sqrt{16+9} = \sqrt{25} = 5 \end{aligned}$$

Just like the lengths of the sides of a triangle, distances are always positive.

Know What? Revisited To find the length and width of a 52" HDTV, plug in the ratios and 52 into the Pythagorean Theorem. We know that the sides are going to be a multiple of 16 and 9, which we will call n.

$$\begin{aligned} (16n)^2+(9n)^2 &= 52^2 \\ 256n^2+81n^2 &= 2704 \\ 337n^2 &= 2704 \\ n^2 &= 8.024 \\ n &= 2.83 \end{aligned}$$



The dimensions of the TV are $16(2.83'') \times 9(2.83'')$, or $45.3'' \times 25.5''$.

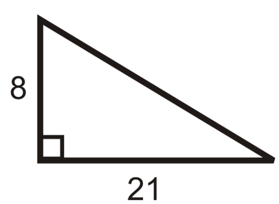
Review Questions

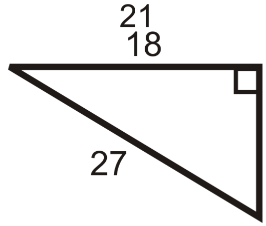
- Questions 1-9 are similar to Examples 1-3.
- Questions 10-15 are similar to Example 5 and 6.
- Questions 16-19 are similar to Example 7.
- Questions 20-25 are similar to Example 8.
- Questions 26-28 are similar to Example 9.
- Questions 29-31 are similar to Example 10.
- Questions 32 and 33 are similar to the Know What?
- Question 34 and 35 are a challenge and similar to Example 9.

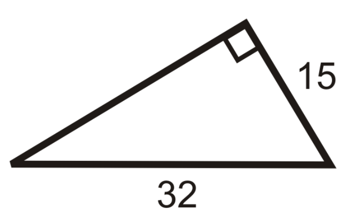
Simplify the radicals.

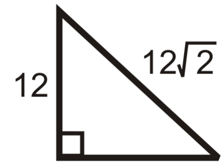
1. $2\sqrt{5} + \sqrt{20}$
2. $\sqrt{24}$
3. $\left(6\sqrt{3}\right)^2$
4. $8\sqrt{8} \cdot \sqrt{10}$
5. $\left(2\sqrt{30}\right)^2$
6. $\sqrt{320}$
7. $\frac{4\sqrt{5}}{\sqrt{6}}$
8. $\frac{12}{\sqrt{10}}$
9. $\frac{21\sqrt{5}}{9\sqrt{15}}$

Find the length of the missing side. Simplify all radicals.

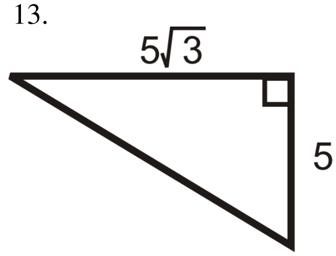
10. 

11. 

12. 



13.



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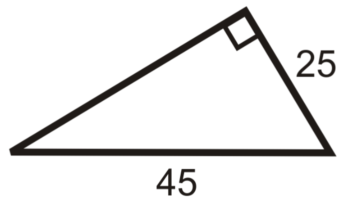


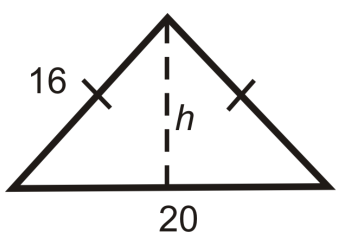
15.

16. If the legs of a right triangle are 10 and 24, then the hypotenuse is __________.
17. If the sides of a rectangle are 12 and 15, then the diagonal is _____________.
18. If the sides of a square are 16, then the diagonal is ____________.
19. If the sides of a square are 9, then the diagonal is _____________.

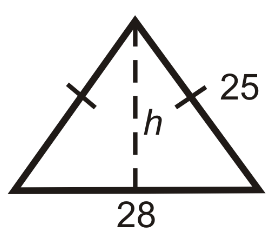
Determine if the following sets of numbers are Pythagorean Triples.

20. 12, 35, 37
21. 9, 17, 18
22. 10, 15, 21
23. 11, 60, 61
24. 15, 20, 25
25. 18, 73, 75

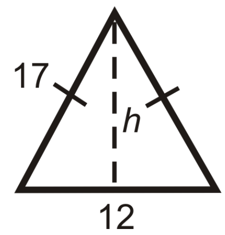
Find the height of each isosceles triangle below. Simplify all radicals.



26.



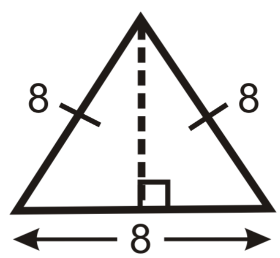
27.



28.

Find the length between each pair of points.

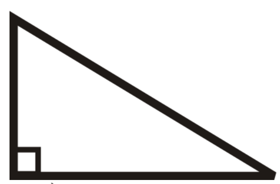
29. (-1, 6) and (7, 2)
30. (10, -3) and (-12, -6)
31. (1, 3) and (-8, 16)
32. What are the length and width of a 42" HDTV? Round your answer to the nearest tenth.
33. Standard definition TVs have a length and width ratio of 4:3. What are the length and width of a 42" Standard definition TV? Round your answer to the nearest tenth.
34. ***Challenge*** An equilateral triangle is an isosceles triangle. If all the sides of an equilateral triangle are 8, find the height. Leave your answer in simplest radical form.



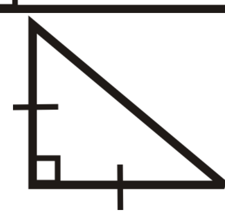
35. If the sides are length s, what would the height be?

Review Queue Answers

1.



2.



3. Factors of 75: 1, 3, 5, 15, 25, 75
4. Prime Factorization of 75: $3 \cdot 5 \cdot 5$

8.2 Simplifying Square Roots

Here you'll simplify, add, subtract and multiply square roots.

The length of the two legs of a right triangle are $2\sqrt{5}$ and $3\sqrt{4}$. What is the length of the triangle's hypotenuse?

Guidance

Before we can solve a quadratic equation using square roots, we need to review how to simplify, add, subtract, and multiply them. Recall that the **square root** is a number that, when multiplied by itself, produces another number. 4 is the square root of 16, for example. -4 is also the square root of 16 because $(-4)^2 = 16$. The symbol for square root is the **radical** sign, or $\sqrt{\ }$. The number under the radical is called the **radicand**. If the square root of a number is not an integer, it is an irrational number.

Example A

Find $\sqrt{50}$ using:

a) A calculator.

b) By simplifying the square root.

Solution:

a) To plug the square root into your graphing calculator, typically there is a $\sqrt{\ }$ or SQRT button. Depending on your model, you may have to enter 50 before or after the square root button. Either way, your answer should be $\sqrt{50} = 7.071067811865\ldots$ In general, we will round to the hundredths place, so 7.07 is sufficient.

b) To simplify the square root, the square numbers must be "pulled out." Look for factors of 50 that are square numbers: 4, 9, 16, 25... 25 is a factor of 50, so break the factors apart.

$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. *This is the most accurate answer.*

Radical Rules

1. $\sqrt{ab} = \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ Any two radicals can be multiplied together.

2. $x\sqrt{a} \pm y\sqrt{a} = x \pm y\sqrt{a}$ The radicands must be the same in order to add or subtract.

3. $\left(\sqrt{a}\right)^2 = \sqrt{a}^2 = a$ The square and square root cancel each other out.

Example B

Simplify $\sqrt{45} + \sqrt{80} - 2\sqrt{5}$.

Solution: At first glance, it does not look like we can simplify this. But, we can simplify each radical by pulling out the perfect squares.

$$\sqrt{45} = \sqrt{9 \cdot 5} = 3\sqrt{5}$$
$$\sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}$$

Rewriting our expression, we have: $3\sqrt{5}+4\sqrt{5}-2\sqrt{5}$ and all the radicands are the same. Using the Order of Operations, our answer is $5\sqrt{5}$.

Example C

Simplify $2\sqrt{35}\cdot 4\sqrt{7}$.

Solution: Multiply across.

$$2\sqrt{35}\cdot 4\sqrt{7}=2\cdot 4\sqrt{35\cdot 7}=8\sqrt{245}$$

Now, simplify the radical. $8\sqrt{245}=8\sqrt{49\cdot 5}=8\cdot 7\sqrt{5}=56\sqrt{5}$

Intro Problem Revisit We must use the Pythagorean Theorem, which states that the square of one leg of a right triangle plus the square of the other leg equals the square of the hypotenuse.

So we are looking for c such that $(2\sqrt{5})^2+(3\sqrt{4})^2=c^2$.

Simplifying, we get $4\cdot 5+9\cdot 4=c^2$, or $20+36=c^2$.

Therefore, $c^2=56$, so to find c, we must take the square root of 56.

$\sqrt{56}=\sqrt{4\cdot 14}=2\sqrt{14}$.

Therefore, $c=2\sqrt{14}$.

Guided Practice

Simplify the following radicals.

1. $\sqrt{150}$
2. $2\sqrt{3}-\sqrt{6}+\sqrt{96}$
3. $\sqrt{8}\cdot\sqrt{20}$

Answers

1. Pull out all the square numbers.

$$\sqrt{150}=\sqrt{25\cdot 6}=5\sqrt{6}$$

Alternate Method: Write out the prime factorization of 150.

$$\sqrt{150}=\sqrt{2\cdot 3\cdot 5\cdot 5}$$

Now, pull out any number that has a pair. Write it *once* in front of the radical and multiply together what is left over under the radical.

$$\sqrt{150}=\sqrt{2\cdot 3\cdot 5\cdot 5}=5\sqrt{6}$$

2. Simplify $\sqrt{96}$ to see if anything can be combined. We will use the alternate method above.

$$\sqrt{96} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = 2 \cdot 2\sqrt{6} = 4\sqrt{6}$$

Rewrite the expression: $2\sqrt{3} - \sqrt{6} + 4\sqrt{6} = 2\sqrt{3} + 3\sqrt{6}$. This is fully simplified. $\sqrt{3}$ and $\sqrt{6}$ cannot be combined because they do not have the same value under the radical.

3. This problem can be done two different ways.

First Method: Multiply radicals, then simplify the answer.

$$\sqrt{8} \cdot \sqrt{20} = \sqrt{160} = \sqrt{16 \cdot 10} = 4\sqrt{10}$$

Second Method: Simplify radicals, then multiply.

$$\sqrt{8} \cdot \sqrt{20} = \left(\sqrt{4 \cdot 2}\right) \cdot \left(\sqrt{4 \cdot 5}\right) = 2\sqrt{2} \cdot 2\sqrt{5} = 2 \cdot 2\sqrt{2 \cdot 5} = 4\sqrt{10}$$

Depending on the complexity of the problem, either method will work. Pick whichever method you prefer.

Vocabulary

Square Root
A number, that when multiplied by itself, produces another number.

Perfect Square
A number that has an integer for a square root.

Radical
The $\sqrt{}$, or square root, sign.

Radicand
The number under the radical.

Practice

Find the square root of each number by using the calculator. Round your answer to the nearest hundredth.

1. 56
2. 12
3. 92

Simplify the following radicals. If it cannot be simplified further, write "*cannot be simplified*".

4. $\sqrt{18}$
5. $\sqrt{75}$
6. $\sqrt{605}$

7. $\sqrt{48}$
8. $\sqrt{50} \cdot \sqrt{2}$
9. $4\sqrt{3} \cdot \sqrt{21}$
10. $\sqrt{6} \cdot \sqrt{20}$
11. $\left(4\sqrt{5}\right)^2$
12. $\sqrt{24} \cdot \sqrt{27}$
13. $\sqrt{16} + 2\sqrt{8}$
14. $\sqrt{28} + \sqrt{7}$
15. $-8\sqrt{3} - \sqrt{12}$
16. $\sqrt{72} - \sqrt{50}$
17. $\sqrt{6} + 7\sqrt{6} - \sqrt{54}$
18. $8\sqrt{10} - \sqrt{90} + 7\sqrt{5}$

8.3 Converse of the Pythagorean Theorem

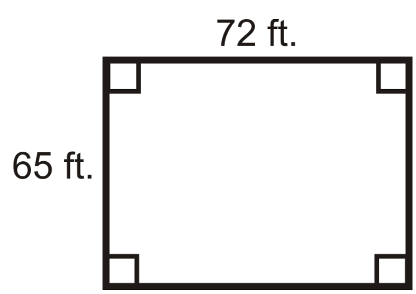
Learning Objectives

- Understand the converse of the Pythagorean Theorem.
- Identify acute and obtuse triangles from side measures.

Review Queue

a. Determine if the following sets of numbers are Pythagorean triples.
   a. 14, 48, 50
   b. 9, 40, 41
   c. 12, 43, 44
b. Do the following lengths make a right triangle? How do you know?
   a. $\sqrt{5}, 3, \sqrt{14}$
   b. $6, 2\sqrt{3}, 8$
   c. $3\sqrt{2}, 4\sqrt{2}, 5\sqrt{2}$

Know What? A friend of yours is designing a building and wantsit to be rectangular. One wall 65 ft. long and the other is 72 ft. long. How can he ensure the walls are going to be perpendicular?



Converse of the Pythagorean Theorem

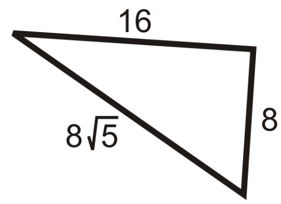
In the last lesson, you learned about the Pythagorean Theorem and how it can be used. The converse of the Pythagorean Theorem is also true. We touched on this in the last section with Example 1.

Pythagorean Theorem Converse: If the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

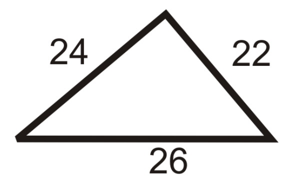
With this converse, you can use the Pythagorean Theorem to prove that a triangle is a right triangle, even if you do not know any of the triangle's angle measurements.

Example 1: Determine if the triangles below are right triangles.

a)



b)



Solution: Check to see if the three lengths satisfy the Pythagorean Theorem. Let the longest sides represent c, in the equation.

a) $a^2+b^2=c^2$
$8^2+16^2 \stackrel{?}{=} \left(8\sqrt{5}\right)^2$
$64+256 \stackrel{?}{=} 64\cdot 5$
$320=320$

The triangle is a right triangle.

b) $a^2+b^2=c^2$
$22^2+24^2 \stackrel{?}{=} 26^2$
$484+576=676$
$1060 \neq 676$

The triangle is not a right triangle.

Identifying Acute and Obtuse Triangles

We can extend the converse of the Pythagorean Theorem to determine if a triangle has an obtuse angle or is acute. We know that if the sum of the squares of the two smaller sides equals the square of the larger side, then the triangle is right. We can also interpret the outcome if the sum of the squares of the smaller sides does not equal the square of the third.

Theorem 8-3: If the sum of the squares of the two shorter sides in a right triangle is ***greater*** than the square of the longest side, then the triangle is ***acute***.

Theorem 8-4: If the sum of the squares of the two shorter sides in a right triangle is ***less*** than the square of the longest side, then the triangle is ***obtuse***.

In other words: The sides of a triangle are a, b, and c and $c>b$ and $c>a$.

If $a^2+b^2>c^2$, then the triangle is acute.

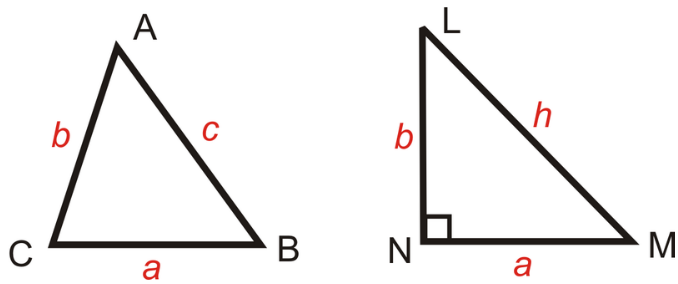
If $a^2+b^2=c^2$, then the triangle is right.

If $a^2+b^2<c^2$, then the triangle is obtuse.

Proof of Theorem 8-3

Given: In $\triangle ABC, a^2+b^2>c^2$, where c is the longest side.

In $\triangle LMN, \angle N$ is a right angle.



Prove: $\triangle ABC$ is an acute triangle. (all angles are less than 90°)

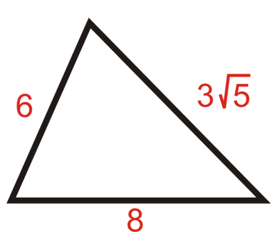
TABLE 8.1:

Statement	***Reason***
1. In $\triangle ABC, a^2+b^2>c^2$, and c is the longest side. In $\triangle LMN, \angle N$ is a right angle.	Given
2. $a^2+b^2=h^2$	Pythagorean Theorem
3. $c^2<h^2$	Transitive PoE
4. $c<h$	Take the square root of both sides
5. $\angle C$ is the largest angle in $\triangle ABC$.	The largest angle is opposite the longest side.
6. $m\angle N=90^\circ$	Definition of a right angle
7. $m\angle C<m\angle N$	SSS Inequality Theorem
8. $m\angle C<90^\circ$	Transitive PoE
9. $\angle C$ is an acute angle.	Definition of an acute angle
10. $\triangle ABC$ is an acute triangle.	If the largest angle is less than 90°, then all the angles are less than 90°.

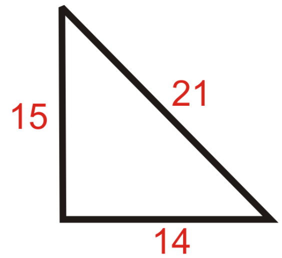
The proof of Theorem 8-4 is very similar and is in the review questions.

Example 2: Determine if the following triangles are acute, right or obtuse.

a)



b)



Solution: Set the shorter sides in each triangle equal to a and b and the longest side equal to c.

a) $6^2 + (3\sqrt{5})^2 \ ? \ 8^2$
$36 + 45 \ ? \ 64$
$81 > 64$

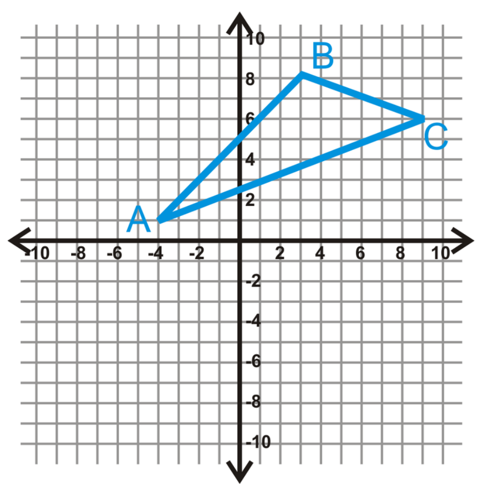
The triangle is acute.

b) $15^2 + 14^2 \ ? \ 21^2$
$225 + 196 \ ? \ 441$
$421 < 441$

The triangle is obtuse.

Example 3: Graph $A(-4,1), B(3,8)$, and $C(9,6)$. Determine if $\triangle ABC$ is acute, obtuse, or right.

Solution: This looks like an obtuse triangle, but we need proof to draw the correct conclusion. Use the distance formula to find the length of each side.



$$AB = \sqrt{(-4-3)^2 + (1-8)^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$$
$$BC = \sqrt{(3-9)^2 + (8-6)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$
$$AC = \sqrt{(-4-9)^2 + (1-6)^2} = \sqrt{169+25} = \sqrt{194}$$

Now, let's plug these lengths into the Pythagorean Theorem.

$$\left(\sqrt{98}\right)^2 + \left(\sqrt{40}\right)^2 \ ? \ \left(\sqrt{194}\right)^2$$
$$98 + 40 \ ? \ 194$$
$$138 < 194$$

$\triangle ABC$ is an obtuse triangle.

Know What? Revisited To make the walls perpendicular, find the length of the diagonal.

$$\begin{aligned} 65^2+72^2 &= c^2 \\ 4225+5184 &= c^2 \\ 9409 &= c^2 \\ 97 &= c \end{aligned}$$

In order to make the building rectangular, both diagonals must be 97 feet.

Review Questions

1. The two *shorter* sides of a triangle are 9 and 12.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
2. The two *longer* sides of a triangle are 24 and 25.
   a. What would be the length of the third side to make the triangle a right triangle?
   b. What is a possible length of the third side to make the triangle acute?
   c. What is a possible length of the third side to make the triangle obtuse?
3. The lengths of the sides of a triangle are $8x, 15x$, and $17x$. Determine if the triangle is acute, right, or obtuse.

Determine if the following lengths make a right triangle.

4. 15, 20, 25
5. 20, 25, 30
6. $8\sqrt{3}, 6, 2\sqrt{39}$

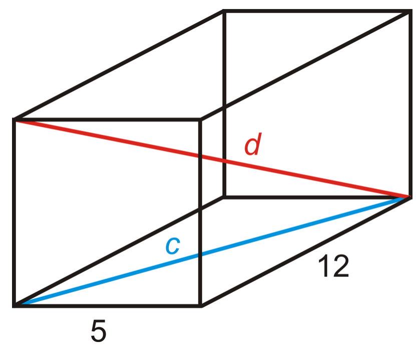
Determine if the following triangles are acute, right or obtuse.

7. 7, 8, 9
8. 14, 48, 50
9. 5, 12, 15
10. 13, 84, 85
11. 20, 20, 24
12. 35, 40, 51
13. 39, 80, 89
14. 20, 21, 38
15. 48, 55, 76

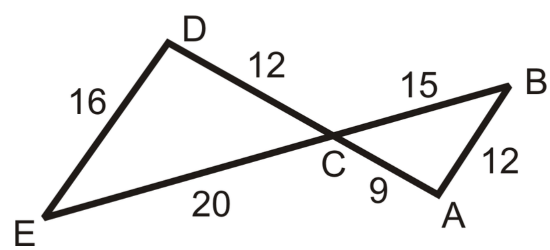
Graph each set of points and determine if $\triangle ABC$ is acute, right, or obtuse.

16. $A(3,-5), B(-5,-8), C(-2,7)$
17. $A(5,3), B(2,-7), C(-1,5)$
18. ***Writing*** Explain the two different ways you can show that a triangle in the coordinate plane is a right triangle.

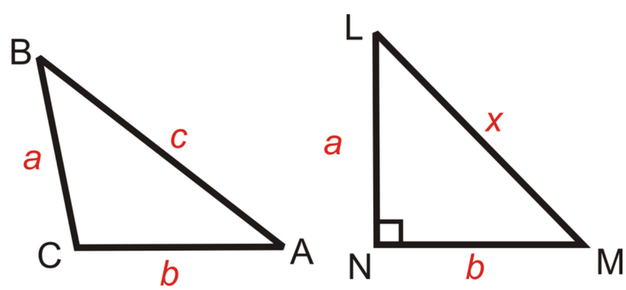
The figure to the right is a rectangular prism. All sides (or faces) are either squares (the front and back) or rectangles (the four around the middle). All sides are perpendicular.



19. Find c.
20. Find d.



21. ***Writing*** Explain why $m\angle A = 90^\circ$.
22. Fill in the blanks for the proof of Theorem 8-4.



Given: In $\triangle ABC, a^2+b^2 < c^2$, where c is the longest side. In $\triangle LMN, \angle N$ is a right angle. Prove: $\triangle ABC$ is an obtuse triangle. (one angle is greater than 90°)

TABLE 8.2:

Statement	***Reason***
1. In $\triangle ABC, a^2+b^2 < c^2$, and c is the longest side. In $\triangle LMN, \angle N$ is a right angle.	
2. $a^2+b^2 = h^2$	
3. $c^2 > h^2$	
4.	
5. $\angle C$ is the largest angle in $\triangle ABC$.	
6. $m\angle N = 90^\circ$	
7. $m\angle C > m\angle N$	
8.	Transitive PoE
9. $\angle C$ is an obtuse angle.	
10. $\triangle ABC$ is an obtuse triangle.	

Given $\overline{AB}$, with $A(3,3)$ and $B(2,-3)$ determine whether the given point C in problems 23-25 makes an acute, right or obtuse triangle.

23. $C(3,-3)$
24. $C(4,-1)$
25. $C(5,-2)$

Given $\overline{AB}$, with $A(-2,5)$ and $B(1,-3)$ find at least two possible points, C, such that $\triangle ABC$ is

26. right, with right $\angle C$.
27. acute, with acute $\angle C$.
28. obtuse, with obtuse $\angle C$.
29. ***Construction***
    a. Draw $\overline{AB}$, such that $AB = 3$ *in*.
    b. Draw $\overrightarrow{AD}$ such that $\angle BAD < 90^\circ$.
    c. Construct a line through B which is perpendicular to $\overrightarrow{AD}$, label the intersection C.
    d. $\triangle ABC$ is a right triangle with right $\angle C$.
30. Is the triangle you made unique? In other words, could you have multiple different outcomes with the same AB? Why or why not? You may wish to experiment to find out.
31. Why do the instructions specifically require that $\angle BAD < 90^\circ$?
32. Describe how this construction could be changed so that $\angle B$ is the right angle in the triangle.

Review Queue Answers

a. Yes
b. Yes
c. No

a. Yes
b. No
c. Yes

8.4 Using Similar Right Triangles

Learning Objectives

- Identify similar triangles inscribed in a larger triangle.
- Evaluate the geometric mean.
- Find the length of an altitude or leg using the geometric mean.

Review Queue

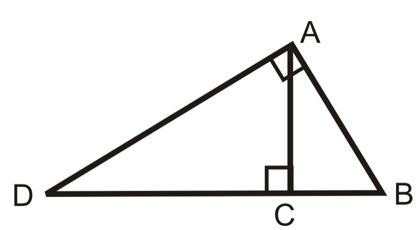
a. If two triangles are right triangles, does that mean they are similar? Explain.
b. If two triangles are isosceles right triangles, does that mean they are similar? Explain.
c. Solve the ratio: $\frac{3}{x} = \frac{x}{27}$.
d. If the legs of an isosceles right triangle are 4, find the length of the hypotenuse. Draw a picture and simplify the radical.

Know What? In California, the average home price increased 21.3% in 2004 and another 16.0% in 2005. What is the average rate of increase for these two years?

Inscribed Similar Triangles

You may recall that if two objects are similar, corresponding angles are congruent and their sides are proportional in length. Let's look at a right triangle, with an altitude drawn from the right angle.

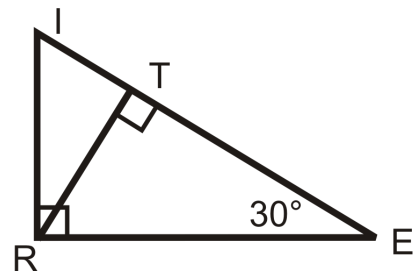
There are three right triangles in this picture, $\triangle ADB, \triangle CDA$, and $\triangle CAB$. Both of the two smaller triangles are similar to the larger triangle because they each share an angle with $\triangle ADB$. That means all three triangles are similar to each other.



Theorem 8-5: If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.

The proof of Theorem 8-5 is in the review questions.

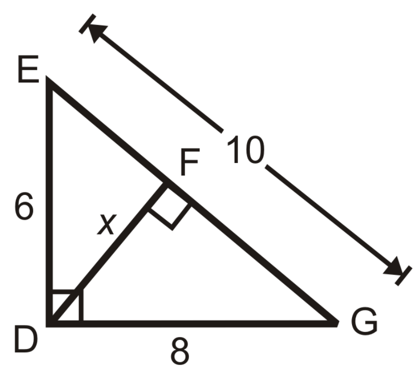
Example 1: Write the similarity statement for the triangles below.



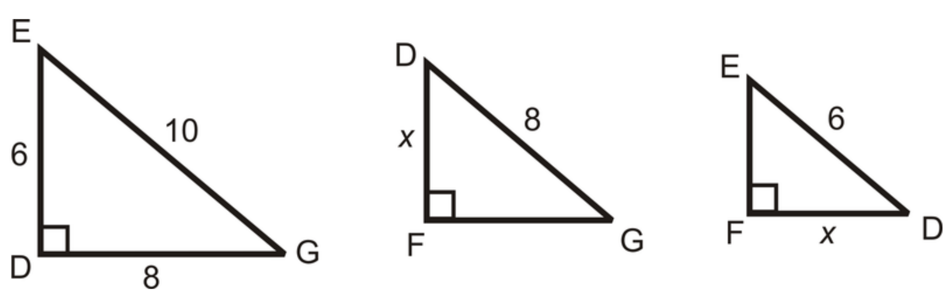
Solution: If $m\angle E = 30^\circ$, then $m\angle I = 60^\circ$ and $m\angle TRE = 60^\circ$. $m\angle IRT = 30^\circ$ because it is complementary to $\angle TRE$. Line up the congruent angles in the similarity statement. $\triangle IRE \sim \triangle ITR \sim \triangle RTE$

We can also use the side proportions to find the length of the altitude.

Example 2: Find the value of x.



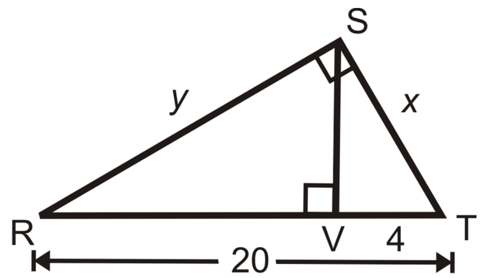
Solution: First, let's separate the triangles to find the corresponding sides.



Now we can set up a proportion.

$$\frac{\text{shorter leg in } \triangle EDG}{\text{shorter leg in } \triangle DFG} = \frac{\text{hypotenuse in } \triangle EDG}{\text{hypotenuse in } \triangle DFG}$$
$$\frac{6}{x} = \frac{10}{8}$$
$$48 = 10x$$
$$4.8 = x$$

Example 3: Find the value of x.



Solution: Let's set up a proportion.

$$\begin{aligned}
\frac{\text{shorter leg in } \triangle SVT}{\text{shorter leg in } \triangle RST} &= \frac{\text{hypotenuse in } \triangle SVT}{\text{hypotenuse in } \triangle RST} \\
\frac{4}{x} &= \frac{x}{20} \\
x^2 &= 80 \\
x &= \sqrt{80} = 4\sqrt{5}
\end{aligned}$$

Example 4: Find the value of y in $\triangle RST$ above.

Solution: Use the Pythagorean Theorem.

$$\begin{aligned}
y^2 + \left(4\sqrt{5}\right)^2 &= 20^2 \\
y^2 + 80 &= 400 \\
y^2 &= 320 \\
y &= \sqrt{320} = 8\sqrt{5}
\end{aligned}$$

The Geometric Mean

You are probably familiar with the arithmetic mean, which ***divides the sum*** of n numbers by n. This is commonly used to determine the average test score for a group of students.

The geometric mean is a different sort of average, which takes the n^{th} ***root of the product*** of n numbers. In this text, we will primarily compare two numbers, so we would be taking the square root of the product of two numbers. This mean is commonly used with rates of increase or decrease.

Geometric Mean: The geometric mean of two positive numbers a and b is the number x, such that $\frac{a}{x} = \frac{x}{b}$ or $x^2 = ab$ and $x = \sqrt{ab}$.

Example 5: Find the geometric mean of 24 and 36.

Solution: $x = \sqrt{24 \cdot 36} = \sqrt{12 \cdot 2 \cdot 12 \cdot 3} = 12\sqrt{6}$

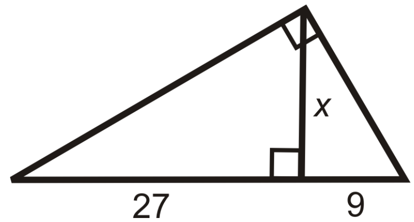
Example 6: Find the geometric mean of 18 and 54.

Solution: $x = \sqrt{18 \cdot 54} = \sqrt{18 \cdot 18 \cdot 3} = 18\sqrt{3}$

Notice that in both of these examples, we did not actually multiply the two numbers together, but kept them separate. This made it easier to simplify the radical.

A practical application of the geometric mean is to find the altitude of a right triangle.

Example 7: Find the value of x.



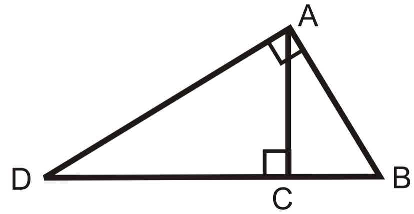
Solution: Using similar triangles, we have the proportion

$$\frac{\text{shortest leg of smallest } \triangle}{\text{shortest leg of middle } \triangle} = \frac{\text{longer leg of smallest } \triangle}{\text{longer leg of middle } \triangle}$$
$$\frac{9}{x} = \frac{x}{27}$$
$$x^2 = 243$$
$$x = \sqrt{243} = 9\sqrt{3}$$

In Example 7, $\frac{9}{x} = \frac{x}{27}$ is in the definition of the geometric mean. So, the altitude is the geometric mean of the two segments that it divides the hypotenuse into.

Theorem 8-6: In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of the altitude is the geometric mean of these two segments.

Theorem 8-7: In a right triangle, the altitude drawn from the right angle to the hypotenuse divides the hypotenuse into two segments. The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.



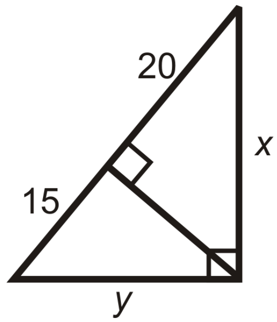
In other words

Theorem 8-6: $\frac{BC}{AC} = \frac{AC}{DC}$

Theorem 8-7: $\frac{BC}{AB} = \frac{AB}{DB}$ and $\frac{DC}{AD} = \frac{AD}{DB}$

Both of these theorems are proved using similar triangles.

Example 8: Find the value of x and y.



Solution: Use theorem 8-7 to solve for x and y.

$$\begin{aligned} \frac{20}{x} &= \frac{x}{35} \\ x^2 &= 20 \cdot 35 \\ x &= \sqrt{20 \cdot 35} \\ x &= 10\sqrt{7} \end{aligned} \qquad \begin{aligned} \frac{15}{y} &= \frac{y}{35} \\ y^2 &= 15 \cdot 35 \\ y &= \sqrt{15 \cdot 35} \\ y &= 5\sqrt{21} \end{aligned}$$

You could also use the Pythagorean Theorem to solve for y, once x has been solved for.

$$\begin{aligned} \left(10\sqrt{7}\right)^2 + y^2 &= 35^2 \\ 700 + y^2 &= 1225 \\ y &= \sqrt{525} = 5\sqrt{21} \end{aligned}$$

Either method is acceptable.

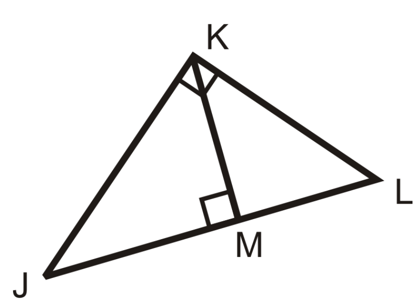
Know What? Revisited The average rate of increase can be found by using the geometric mean.

$$x = \sqrt{0.213 \cdot 0.16} = 0.1846$$

Over the two year period, housing prices increased 18.46%.

Review Questions

Use the diagram to answer questions 1-4.

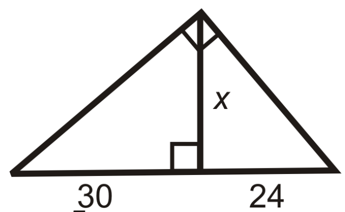


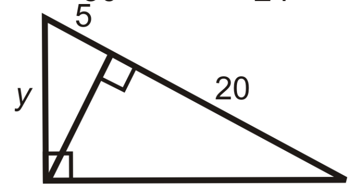
1. Write the similarity statement for the three triangles in the diagram.
2. If $JM = 12$ and $ML = 9$, find KM.
3. Find JK.
4. Find KL.

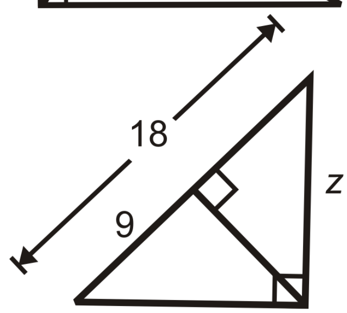
Find the geometric mean between the following two numbers. Simplify all radicals.

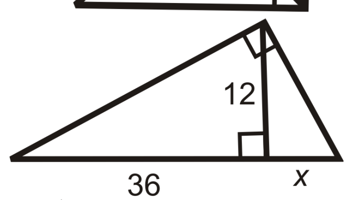
5. 16 and 32
6. 45 and 35
7. 10 and 14
8. 28 and 42
9. 40 and 100
10. 51 and 8

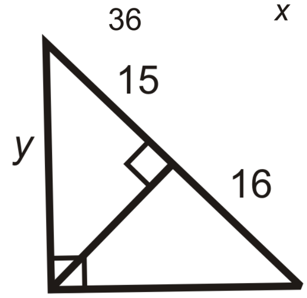
Find the length of the missing variable(s). Simplify all radicals.

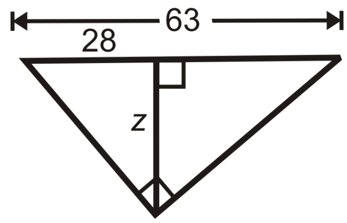
11. 

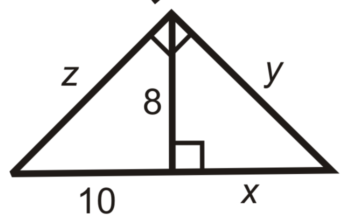
12. 

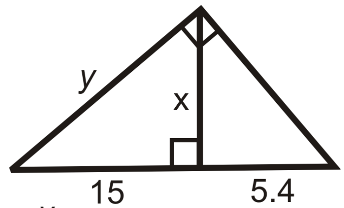
13. 

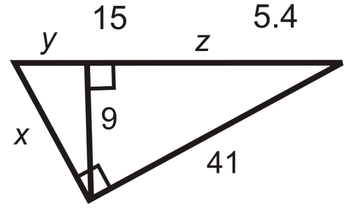
14. 

15. 

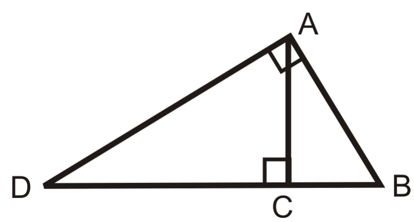
16. 

17. 

18. 

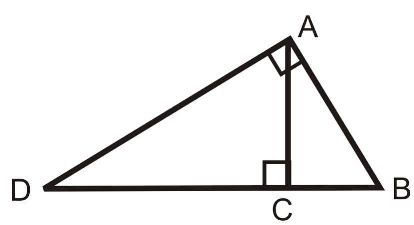
19. 

20. Write a proof for Theorem 8-5.



Given: $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle. Prove: $\triangle ABD \sim \triangle CBA \sim \triangle CAD$

21. Fill in the blanks for the proof of Theorem 8-7.



Given: $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle. Prove: $\frac{BC}{AB} = \frac{AB}{DB}$

TABLE 8.3:

Statement	***Reason***
1. $\triangle ABD$ with $\overline{AC} \perp \overline{DB}$ and $\angle DAB$ is a right angle.	
2. $\triangle ABD \sim \triangle CBA \sim \triangle CAD$	
3. $\frac{BC}{AB} = \frac{AB}{DB}$	

22. Last year Poorva's rent increased by 5% and this year her landlord wanted to raise her rent by 7.5%. What is the average rate at which her landlord has raised her rent over the course of these two years?

23. Mrs. Smith teaches AP Calculus. Between the first and second years she taught the course her students' average score improved by 12%. Between the second and third years, the scores increased by 9%. What is the average rate of improvement in her students' scores?
24. According to the US Census Bureau, http://www.census.gov/ipc/www/idb/country.php the rate of growth of the US population was 0.8% and in 2009 it was 1.0%. What was the average rate of population growth during that time period?

Algebra Connection A geometric sequence is a sequence of numbers in which each successive term is determined by multiplying the previous term by the common ratio. An example is the sequence 1, 3, 9, 27, ... Here each term is multiplied by 3 to get the next term in the sequence. Another way to look at this sequence is to compare the ratios of the consecutive terms.

25. Find the ratio of the 2^{nd} to 1^{st} terms and the ratio of the 3^{rd} to 2^{nd} terms. What do you notice? Is this true for the next set (4^{th} to 3^{rd} terms)?
26. Given the sequence 4, 8, 16,..., if we equate the ratios of the consecutive terms we get: $\frac{8}{4}=\frac{16}{8}$. This means that 8 is the ____________________ of 4 and 16. We can generalize this to say that every term in a geometric sequence is the ____________________ of the previous and subsequent terms.

Use what you discovered in problem 26 to find the middle term in the following geometric sequences.

27. 5, _____, 20
28. 4, _____, 100
29. 2, _____, $\frac{1}{2}$
30. We can use what we have learned in this section in another proof of the Pythagorean Theorem. Use the diagram to fill in the blanks in the proof below.

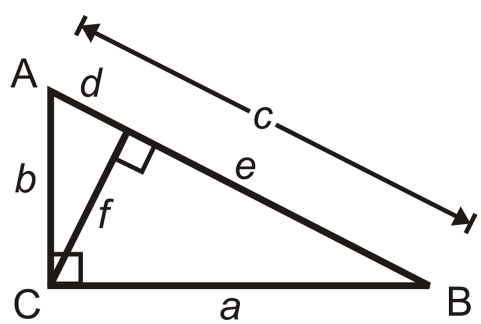


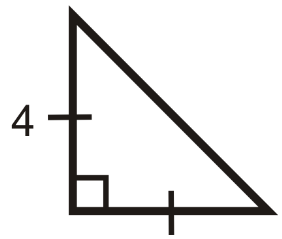
TABLE 8.4:

Statement	***Reason***
1. $\frac{e}{a}=\frac{?}{d+e}$ and $\frac{d}{b}=\frac{b}{?}$	Theorem 8-7
2. $a^2=e(d+e)$ and $b^2=d(d+e)$	?
3. $a^2+b^2=?$	Combine equations from #2.
4. ?	Distributive Property
5. $c=d+e$	?
6. ?	Substitution PoE

Review Queue Answers

a. No, another angle besides the right angles must also be congruent.

b. Yes, the three angles in an isosceles right triangle are $45^\circ, 45^\circ$, and 90°. Isosceles right triangles will always be similar.
c. $\frac{3}{x} = \frac{x}{27} \rightarrow x^2 = 81 \rightarrow x = \pm 9$
d. $4^2 + 4^2 = h^2$
$h = \sqrt{32} = 4\sqrt{2}$



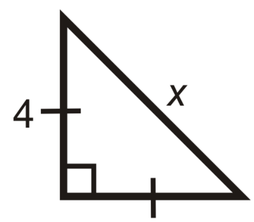
8.5 Special Right Triangles

Learning Objectives

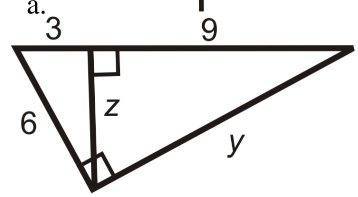
- Identify and use the ratios involved with isosceles right triangles.
- Identify and use the ratios involved with 30-60-90 triangles.

Review Queue

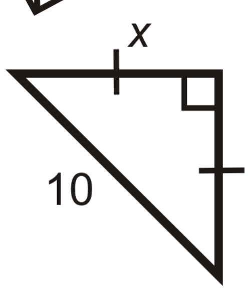
Find the value of the missing variable(s). Simplify all radicals.



a.



b.



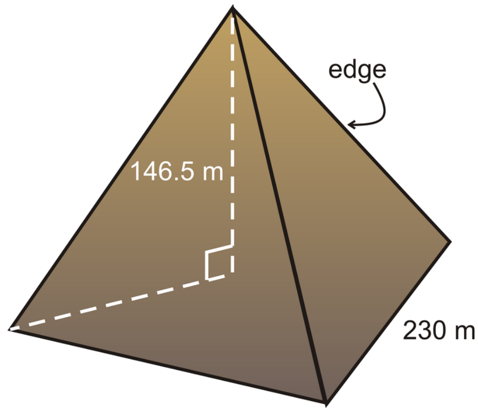
c.

d. Do the lengths 6, 6, and $6\sqrt{2}$ make a right triangle?
e. Do the lengths $3, 3\sqrt{3}$, and 6 make a right triangle?

Know What? The Great Giza Pyramid is a pyramid with a square base and four isosceles triangles that meet at a point. It is thought that the original height was 146.5 meters and the base edges were 230 meters.

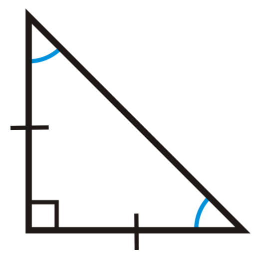
First, find the length of the edge of the isosceles triangles. Then, determine if the isosceles triangles are also equilateral triangles. Round your answers to the nearest tenth.

You can assume that the height of the pyramid is from the center of the square base and is a vertical line.



Isosceles Right Triangles

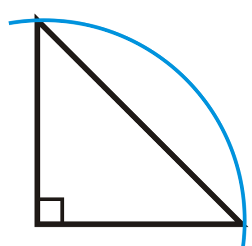
There are two types of special right triangles, based on their angle measures. The first is an isosceles right triangle. Here, the legs are congruent and, by the Base Angles Theorem, the base angles will also be congruent. Therefore, the angle measures will be $90^\circ, 45^\circ$, and 45°. You will also hear an isosceles right triangle called a 45-45-90 triangle. Because the three angles are always the same, all isosceles right triangles are similar.



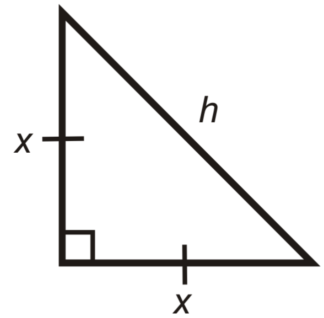
Investigation 8-2: Properties of an Isosceles Right Triangle

Tools Needed: Pencil, paper, compass, ruler, protractor

a. Construct an isosceles right triangle with 2 in legs. Use the SAS construction that you learned in Chapter 4.



b. Find the measure of the hypotenuse. What is it? Simplify the radical.
c. Now, let's say the legs are of length x and the hypotenuse is h. Use the Pythagorean Theorem to find the hypotenuse. What is it? How is this similar to your answer in #2?



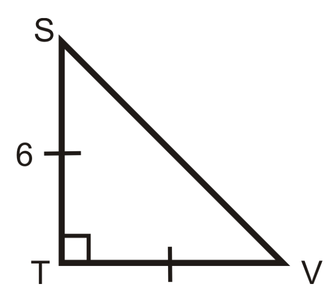
$$\begin{aligned} x^2 + x^2 &= h^2 \\ 2x^2 &= h^2 \\ x\sqrt{2} &= h \end{aligned}$$

45-45-90 Corollary: If a triangle is an isosceles right triangle, then its sides are in the extended ratio $x : x : x\sqrt{2}$.

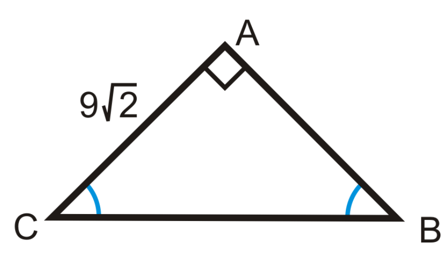
Step 3 in the above investigation proves the 45-45-90 Triangle Theorem. So, anytime you have a right triangle with congruent legs or congruent angles, then the sides will always be in the ratio $x : x : x\sqrt{2}$. The hypotenuse is always $x\sqrt{2}$ because that is the longest length. This is a specific case of the Pythagorean Theorem, so it will still work, if for some reason you forget this corollary.

Example 1: Find the length of the missing sides.

a)



b)



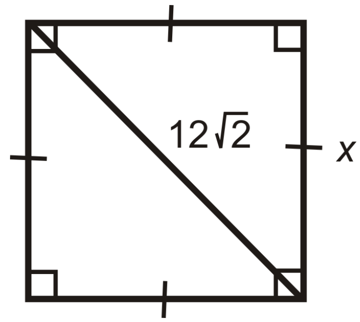
Solution: Use the $x : x : x\sqrt{2}$ ratio.

a) $TV = 6$ because it is equal to ST. So, $SV = 6\sqrt{2}$.

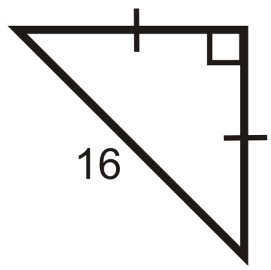
b) $AB = 9\sqrt{2}$ because it is equal to AC. So, $BC = 9\sqrt{2} \cdot \sqrt{2} = 9 \cdot 2 = 18$.

Example 2: Find the length of x.

a)



b)



Solution: Again, use the $x : x : x\sqrt{2}$ ratio, but in these two we are given the hypotenuse. We need to solve for x in the ratio.

a) $12\sqrt{2} = x\sqrt{2}$
$12 = x$

b) $x\sqrt{2} = 16$
$x = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$

In part b, we ***rationalized the denominator***. Whenever there is a radical in the denominator of a fraction, multiply the top and bottom by that radical. This will cancel out the radical from the denominator and reduce the fraction.

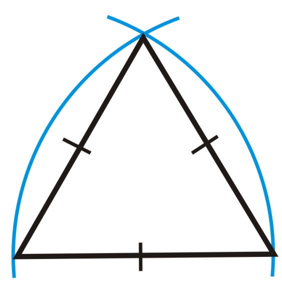
30-60-90 Triangles

The second special right triangle is called a 30-60-90 triangle, after the three angles. To construct a 30-60-90 triangle, start with an equilateral triangle.

Investigation 8-3: Properties of a 30-60-90 Triangle

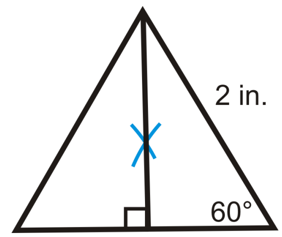
Tools Needed: Pencil, paper, ruler, compass

1. Construct an equilateral triangle with 2 in sides.



2. Draw or construct the altitude from the top vertex to the base for two congruent triangles.

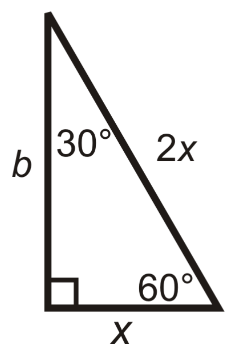
3. Find the measure of the two angles at the top vertex and the length of the shorter leg.



The top angles are each 30° *and the shorter leg is 1 in because the altitude of an equilateral triangle is also the angle and perpendicular bisector.*

4. Find the length of the longer leg, using the Pythagorean Theorem. Simplify the radical.

5. Now, let's say the shorter leg is length x and the hypotenuse is $2x$. Use the Pythagorean Theorem to find the longer leg. What is it? How is this similar to your answer in #4?



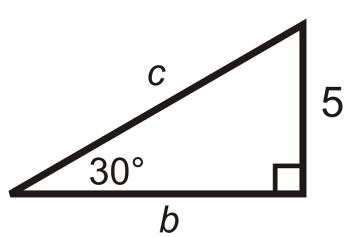
$$\begin{aligned}
x^2 + b^2 &= (2x)^2 \\
x^2 + b^2 &= 4x^2 \\
b^2 &= 3x^2 \\
b &= x\sqrt{3}
\end{aligned}$$

30-60-90 Corollary: If a triangle is a 30-60-90 triangle, then its sides are in the extended ratio $x : x\sqrt{3} : 2x$.

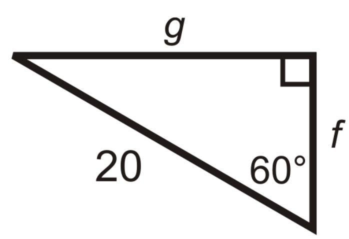
Step 5 in the above investigation proves the 30-60-90 Corollary. The shortest leg is always x, the longest leg is always $x\sqrt{3}$, and the hypotenuse is always $2x$. If you ever forget this corollary, then you can still use the Pythagorean Theorem.

Example 3: Find the length of the missing sides.

a)



b)



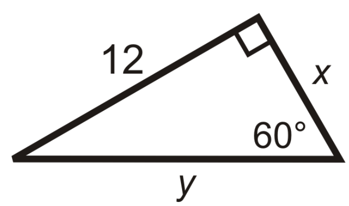
Solution: In part a, we are given the shortest leg and in part b, we are given the hypotenuse.

a) If $x = 5$, then the longer leg, $b = 5\sqrt{3}$, and the hypotenuse, $c = 2(5) = 10$.

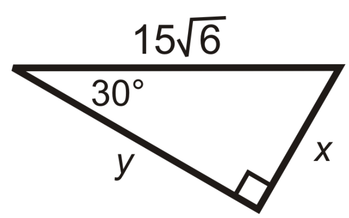
b) Now, $2x = 20$, so the shorter leg, $f = 10$, and the longer leg, $g = 10\sqrt{3}$.

Example 4: Find the value of x and y.

a)



b)

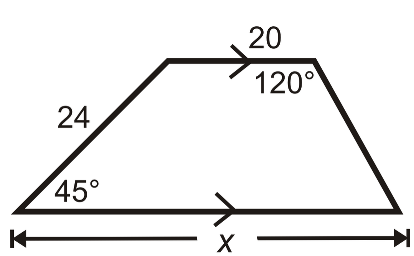


Solution: In part a, we are given the longer leg and in part b, we are given the hypotenuse.

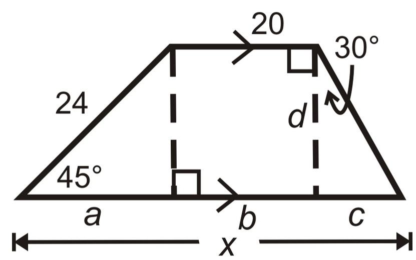
a) $x\sqrt{3} = 12$
$x = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$
Then, the hypotenuse would be
$y = 2\left(4\sqrt{3}\right) = 8\sqrt{3}$

b) $2x = 15\sqrt{6}$
$x = \frac{15\sqrt{6}}{2}$
The, the longer leg would be
$y = \left(\frac{15\sqrt{6}}{2}\right) \cdot \sqrt{3} = \frac{15\sqrt{18}}{2} = \frac{45\sqrt{2}}{2}$

Example 5: Find the measure of x.



Solution: Think of this trapezoid as a rectangle, between a 45-45-90 triangle and a 30-60-90 triangle.



From this picture, $x = a + b + c$. First, find a, which is a leg of an isosceles right triangle.

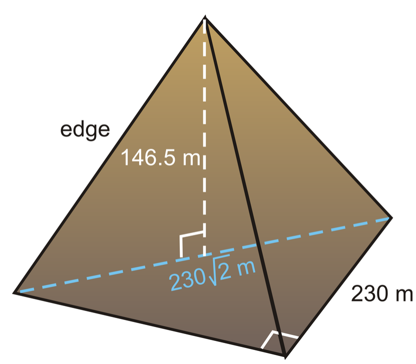
$$a = \frac{24}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{24\sqrt{2}}{2} = 12\sqrt{2}$$

$a = d$, so we can use this to find c, which is the shorter leg of a 30-60-90 triangle.

$$c = \frac{12\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{6}}{3} = 4\sqrt{6}$$

$b = 20$, so $x = 12\sqrt{2} + 20 + 4\sqrt{6}$. Nothing simplifies, so this is how we leave our answer.

Know What? Revisited The line that the vertical height is perpendicular to is the diagonal of the square base. This length (blue) is the same as the hypotenuse of an isosceles right triangle because half of a square is an isosceles right triangle. So, the diagonal is $230\sqrt{2}$. Therefore, the base of the right triangle with 146.5 as the leg is half of $230\sqrt{2}$ or $115\sqrt{2}$. Do the Pythagorean Theorem to find the edge.



$$edge = \sqrt{\left(115\sqrt{2}\right)^2 + 146.5^2} \approx 218.9\ m$$

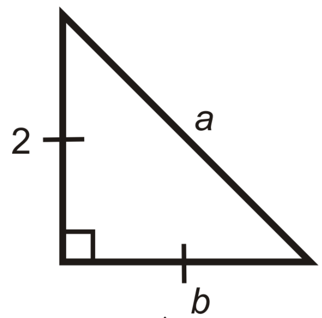
In order for the sides to be equilateral triangles, this length should be 230 meters. It is not, so the triangles are isosceles.

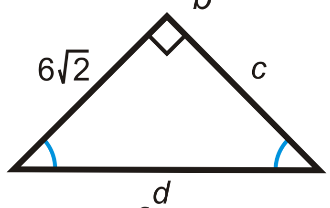
Review Questions

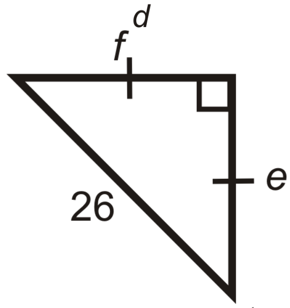
1. In an isosceles right triangle, if a leg is x, then the hypotenuse is __________.

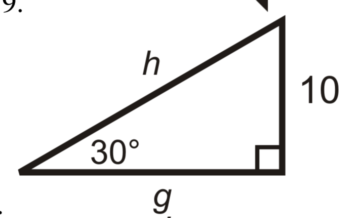
2. In a 30-60-90 triangle, if the shorter leg is x, then the longer leg is __________ and the hypotenuse is _____-______.
3. A square has sides of length 15. What is the length of the diagonal?
4. A square's diagonal is 22. What is the length of each side?
5. A rectangle has sides of length 4 and $4\sqrt{3}$. What is the length of the diagonal?
6. A baseball diamond is a square with 90 foot sides. What is the distance from home base to second base? (HINT: It's the length of the diagonal).

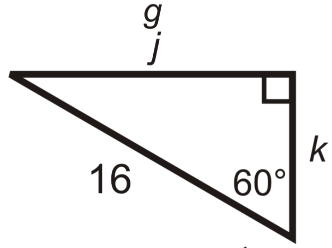
For questions 7-18, find the lengths of the missing sides.

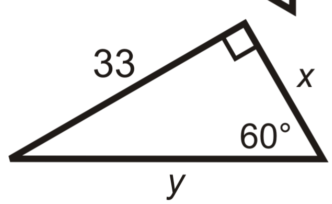
7. 

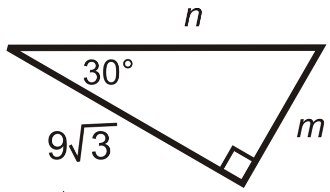
8. 

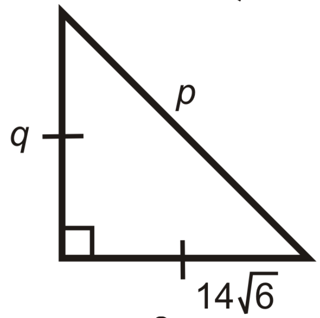
9. 

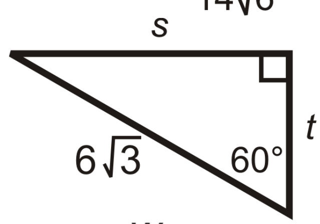
10. 

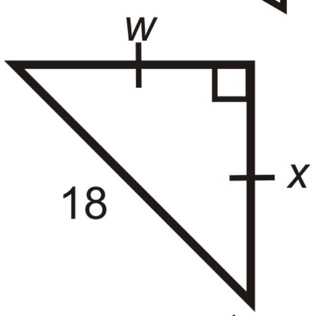
11. 

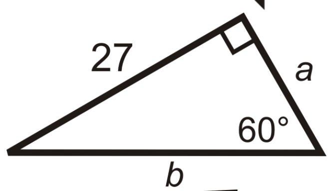
12. 

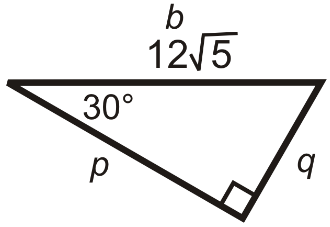
13. 

14. 

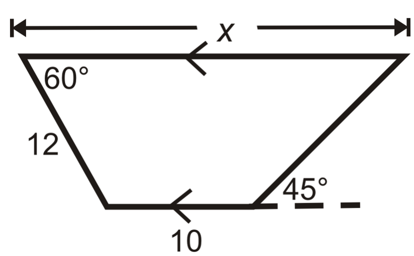
15. 

16. 

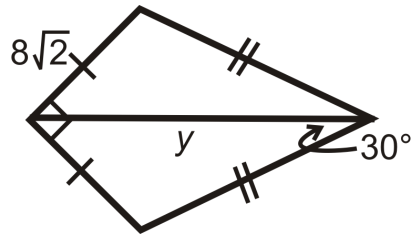
17. 

18. 

19. Do the lengths $8\sqrt{2}, 8\sqrt{6}$, *and* $16\sqrt{2}$ make a special right triangle? If so, which one?
20. Do the lengths $4\sqrt{3}, 4\sqrt{6}$ *and* $8\sqrt{3}$ make a special right triangle? If so, which one?
21. Find the measure of x.



22. Find the measure of y.



23. What is the ratio of the sides of a rectangle if the diagonal divides the rectangle into two 30-60-90 triangles?
24. What is the length of the sides of a square with diagonal 8 in?

For questions 25-28, it might be helpful to recall #25 from section 8.1.

25. What is the height of an equilateral triangle with sides of length 3 in?
26. What is the area of an equilateral triangle with sides of length 5 ft?
27. A regular hexagon has sides of length 3 in. What is the area of the hexagon? (*Hint: the hexagon is made up a 6 equilateral triangles.*)
28. The area of an equilateral triangle is $36\sqrt{3}$. What is the length of a side?
29. If a road has a grade of 30°, this means that its angle of elevation is 30°. If you travel 1.5 miles on this road, how much elevation have you gained in feet (5280 ft = 1 mile)?
30. Four isosceles triangles are formed when both diagonals are drawn in a square. If the length of each side in the square is *s*, what are the lengths of the legs of the isosceles triangles?

Review Queue Answers

a. $$\begin{aligned} 4^2+4^2 &= x^2 \\ 32 &= x^2 \\ x &= 4\sqrt{2} \end{aligned}$$

b. $$\begin{aligned} 3^2+z^2 &= 6^2 & \left(3\sqrt{3}\right)^2+9^2 &= y^2 \\ z^2 &= 27 & 108 &= y^2 \\ z &= 3\sqrt{3} & y &= 6\sqrt{3} \end{aligned}$$

c. $$\begin{aligned} x^2+x^2 &= 10^2 \\ 2x^2 &= 100 \\ x^2 &= 50 \\ x &= 5\sqrt{2} \end{aligned}$$

d. Yes, $6^2+6^2 = \left(6\sqrt{2}\right)^2 \rightarrow 36+36 = 72$

e. Yes, $3^2+\left(3\sqrt{3}\right)^2 = 6^2 \rightarrow 9+27 = 36$

8.6 The Distance Formula

Learning Objectives

- Find the distance between two points.
- Find the shortest distance between a point and a line and two parallel lines.
- Determine the equation of a perpendicular bisector of a line segment in the coordinate plane.

Review Queue

1. What is the equation of the line between (-1, 3) and (2, -9)?
2. Find the equation of the line that is perpendicular to $y = -2x + 5$ through the point (-4, -5).
3. Find the equation of the line that is parallel to $y = \frac{2}{3}x - 7$ through the point (3, 8).

Know What? The shortest distance between two points is a straight line. To the right is an example of how far apart cities are in the greater Los Angeles area. There are always several ways to get somewhere in Los Angeles. Here, we have the distances between Los Angeles and Orange. Which distance is the shortest? Which is the longest?

The Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) can be defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This formula will be derived in Chapter 9.

Example 1: Find the distance between (4, -2) and (-10, 3).

Solution: Plug in (4, -2) for (x_1, y_1) and (-10, 3) for (x_2, y_2) and simplify.

$$\begin{aligned} d &= \sqrt{(-10-4)^2+(3+2)^2} \\ &= \sqrt{(-14)^2+(5^2)} \qquad \textit{Distances are always positive!} \\ &= \sqrt{196+25} \\ &= \sqrt{221} \approx 14.87 \textit{ units} \end{aligned}$$

Example 2: The distance between two points is 4 units. One point is (1, -6). What is the second point? You may assume that the second point is made up of integers.

Solution: We will still use the distance formula for this problem, however, we know d and need to solve for $(x_2,\ y_2)$.

$$\begin{aligned} 4 &= \sqrt{(1-x_2)^2+(-6-y_2)^2} \\ 16 &= (1-x_2)^2+(-6-y_2)^2 \end{aligned}$$

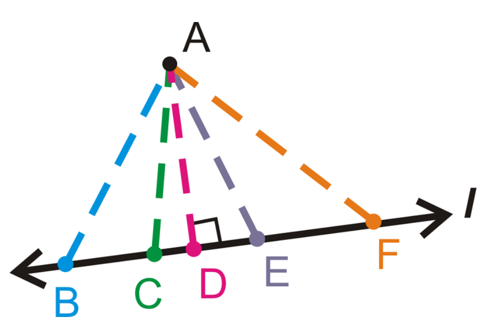
At this point, we need to figure out two square numbers that add up to 16. The only two square numbers that add up to 16 are $16+0$.

$$\begin{array}{lll} 16 = \underbrace{(1-x_2)^2}_{4^2}+\underbrace{(-6-y_2)^2}_{0^2} & \text{or} & 16 = \underbrace{(1-x_2)^2}_{0^2}+\underbrace{(-6-y_2)^2}_{4^2} \\ 1-x_2 = \pm 4 \qquad\qquad -6-y_2 = 0 & & 1-x_2 = 0 \qquad\qquad -6-y_2 = \pm 4 \\ \quad -x_2 = -5 \text{ or } 3 \quad \text{and} \quad -y_2 = 6 & \text{or} & \quad -x_2 = -1 \quad \text{and} \quad y_2 = 10 \text{ or } 2 \\ \qquad x_2 = 5 \text{ or } -3 \qquad\quad y_2 = -6 & & \qquad x_2 = 1 \qquad\qquad y_2 = -10 \text{ or } -2 \end{array}$$

Therefore, the second point could have 4 possibilities: (5, -6), (-3, -6), (1, -10), and (1, -2).

Shortest Distance between a Point and a Line

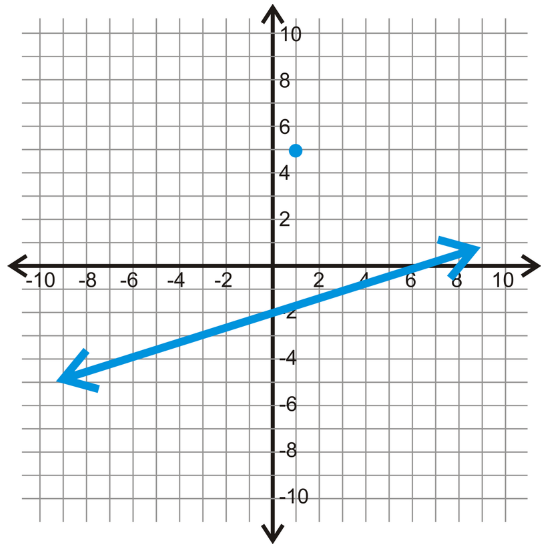
We know that the shortest distance between two points is a straight line. This distance can be calculated by using the distance formula. Let's extend this concept to the shortest distance between a point and a line.



Just by looking at a few line segments from A to line l, we can tell that the shortest distance between a point and a line is the ***perpendicular line*** between them. Therefore, AD is the shortest distance between A and line l.

Putting this onto a graph can be a little tougher.

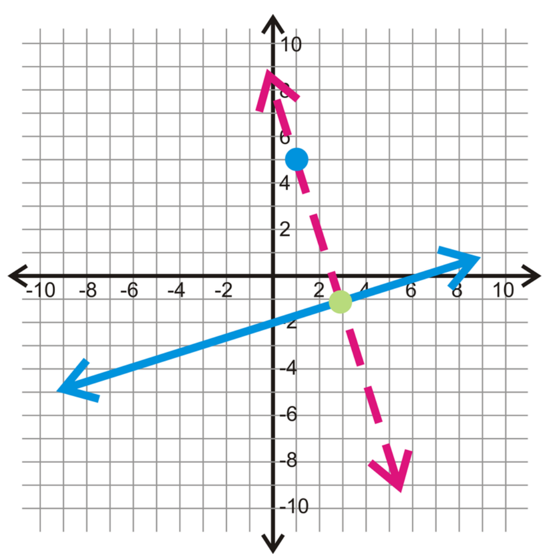
Example 3: Determine the shortest distance between the point (1, 5) and the line $y = \frac{1}{3}x - 2$.



Solution: First, graph the line and point. Second determine the equation of the perpendicular line. The opposite sign and reciprocal of $\frac{1}{3}$ is -3, so that is the slope. We know the line must go through the given point, (1, 5), so use that to find the $y-$intercept.

$$\begin{aligned} y &= -3x+b \\ 5 &= -3(1)+b \qquad \text{The equation of the line is } y=-3x+8. \\ 8 &= b \end{aligned}$$

Next, we need to find the point of intersection of these two lines. By graphing them on the same axes, we can see that the point of intersection is (3, -1), the green point.

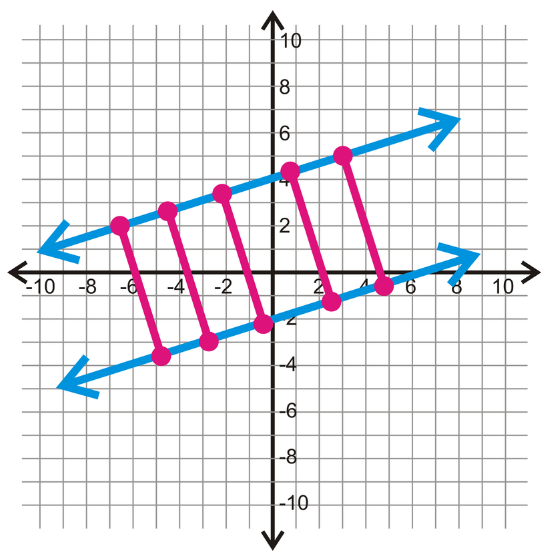


Finally, plug (1, 5) and (3,-1) into the distance formula to find the shortest distance.

$$
\begin{aligned}
d &= \sqrt{(3-1)^2+(-1-5)^2} \\
&= \sqrt{(2)^2+(-6)^2} \\
&= \sqrt{2+36} \\
&= \sqrt{38} \approx 6.16 \; units
\end{aligned}
$$

Shortest Distance between Two Parallel Lines

The shortest distance between two parallel lines is the length of the perpendicular segment between them. It doesn't matter which perpendicular line you choose, as long as the two points are on the lines. Recall that there are infinitely many perpendicular lines between two parallel lines.



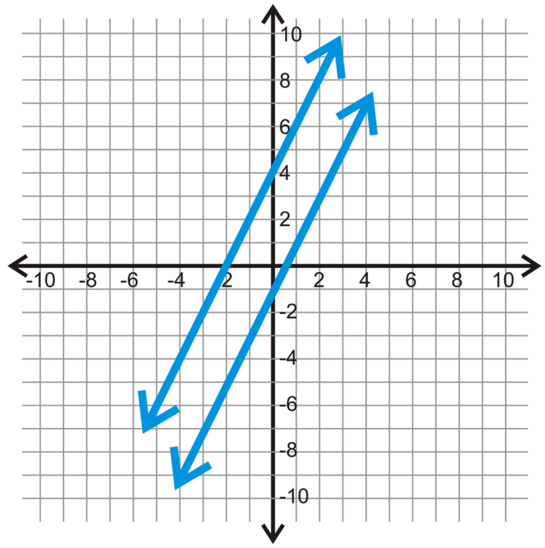
Notice that all of the pink segments are the same length. So, when picking a perpendicular segment, be sure to pick one with endpoints that are integers.

Example 3: Find the distance between $x = 3$ and $x = -5$.

Solution: Any line with $x = a$ number is a vertical line. In this case, we can just count the squares between the two lines. The two lines are $3-(-5)$ units apart, or 8 units.

You can use this same method with horizontal lines as well. For example, $y = -1$ and $y = 3$ are $3-(-1)$ units, or 4 units apart.

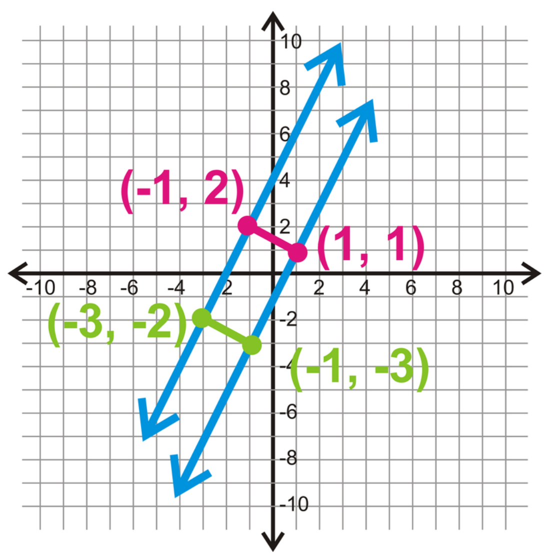
Example 4: What is the shortest distance between $y = 2x+4$ and $y = 2x-1$?



Solution: Graph the two lines and determine the perpendicular slope, which is $-\frac{1}{2}$. Find a point on $y = 2x+4$, let's say (-1, 2). From here, use the slope of the perpendicular line to find the corresponding point on $y = 2x-1$. If you move down 1 from 2 and over to the right 2 from -1, you will hit $y = 2x-1$ at (1, 1). Use these two points to determine the distance between the two lines.

$$\begin{aligned} d &= \sqrt{(1+1)^2+(1-2)^2} \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \approx 2.24 \ units \end{aligned}$$

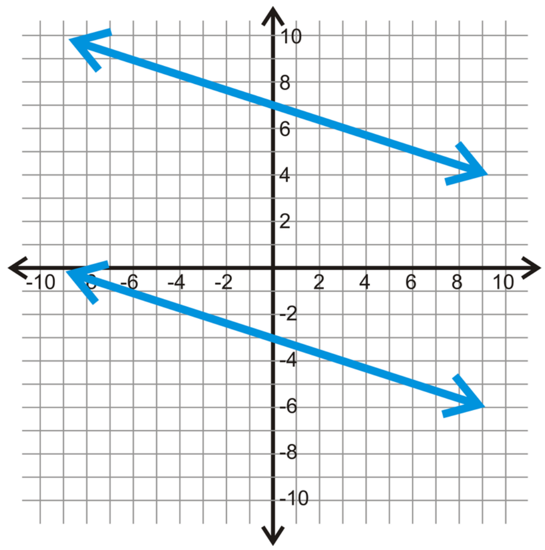
The lines are about 2.24 units apart.



Notice that you could have used any two points, as long as they are on the same perpendicular line. For example, you could have also used (-3, -2) and (-1, -3) and you still would have gotten the same answer.

$$\begin{aligned} d &= \sqrt{(-1+3)^2+(-3+2)^2} \\ &= \sqrt{2^2+(-1)^2} \\ &= \sqrt{4+1} \\ &= \sqrt{5} \approx 2.24 \ units \end{aligned}$$

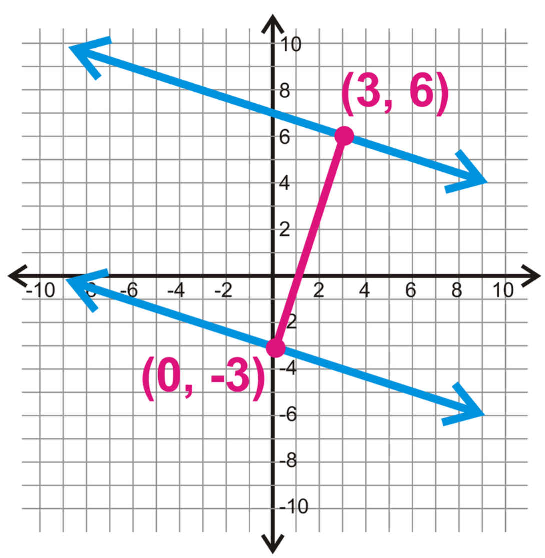
Example 5: Find the distance between the two parallel lines below.



Solution: First you need to find the slope of the two lines. Because they are parallel, they are the same slope, so if you find the slope of one, you have the slope of both.

Start at the $y-$intercept of the top line, 7. From there, you would go down 1 and over 3 to reach the line again. Therefore the slope is $-\frac{1}{3}$ and the perpendicular slope would be 3.

Next, find two points on the lines. Let's use the $y-$intercept of the bottom line, (0, -3). Then, rise 3 and go over 1 until your reach the second line. Doing this three times, you would hit the top line at (3, 6). Use these two points in the distance formula to find how far apart the lines are.

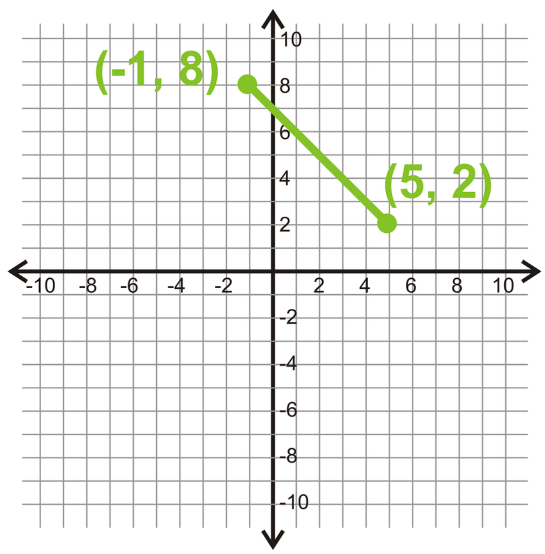


$$\begin{aligned} d &= \sqrt{(0-3)^2+(-3-6)^2} \\ &= \sqrt{(-3)^2+(-9)^2} \\ &= \sqrt{9+81} \\ &= \sqrt{90} \approx 9.49 \text{ units} \end{aligned}$$

Perpendicular Bisectors in the Coordinate Plane

Recall that the definition of a perpendicular bisector is a perpendicular line that goes through the midpoint of a line segment. Using what we have learned in this chapter and the formula for a midpoint, we can find the equation of a perpendicular bisector.

Example 6: Find the equation of the perpendicular bisector of the line segment between (-1, 8) and (5, 2).



Solution: First, find the midpoint of the line segment.

$$\left(\frac{-1+5}{2}, \frac{8+2}{2}\right) = \left(\frac{4}{2}, \frac{10}{2}\right) = (2,\ 5)$$

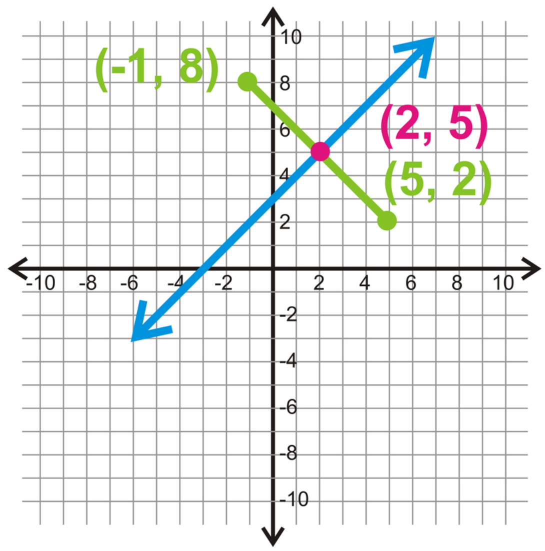
Second, find the slope between the two endpoints. This will help us figure out the perpendicular slope for the perpendicular bisector.

$$m = \frac{2-8}{5+1} = \frac{-6}{6} = -1$$

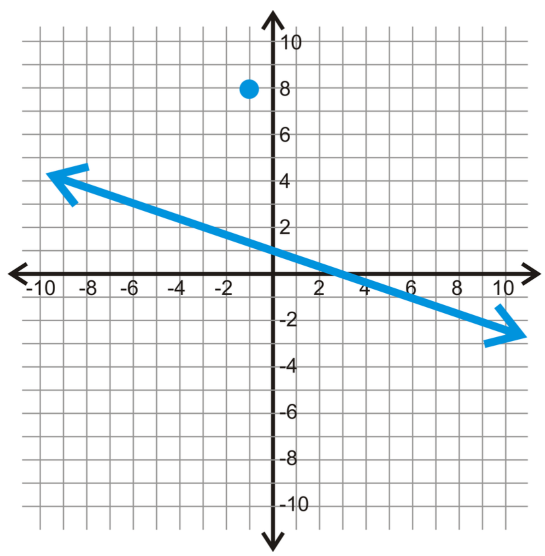
If the slope of the segment is -1, then the slope of the perpendicular bisector will be 1. The last thing to do is to find the $y-$intercept of the perpendicular bisector. We know it goes through the midpoint, (2, 5), of the segment, so substitute that in for x and y in the slope-intercept equation.

$$
\begin{aligned}
y &= mx + b \\
5 &= 1(2) + b \\
5 &= 2 + b \\
3 &= b
\end{aligned}
$$

The equation of the perpendicular bisector is $y = x + 3$.



Example 7: The perpendicular bisector of $\overline{AB}$ has the equation $y = -\frac{1}{3}x + 1$. If A is (-1, 8) what are the coordinates of B?



Solution: The easiest way to approach this problem is to graph it. Graph the perpendicular line and plot the point. See the graph to the left.

Second, determine the slope of $\overline{AB}$. If the slope of the perpendicular bisector is $-\frac{1}{3}$, then the slope of $\overline{AB}$ is 3.

Using the slope, count *down* 3 and over to the *right* 1 until you hit the perpendicular bisector. Counting down 6 and over 2, you land on the line at (-3, 2). This is the *midpoint* of $\overline{AB}$. If you count down another 6 and over to the right 2 more, you will find the coordinates of B, which are (-5, -4).

Know What? Revisited Draw two intersecting lines. Make sure they are not perpendicular. Label the 26.3 miles along hwy 5. The longest distance is found by adding the distances along the 110 and 405, or 41.8 miles.

Review Questions

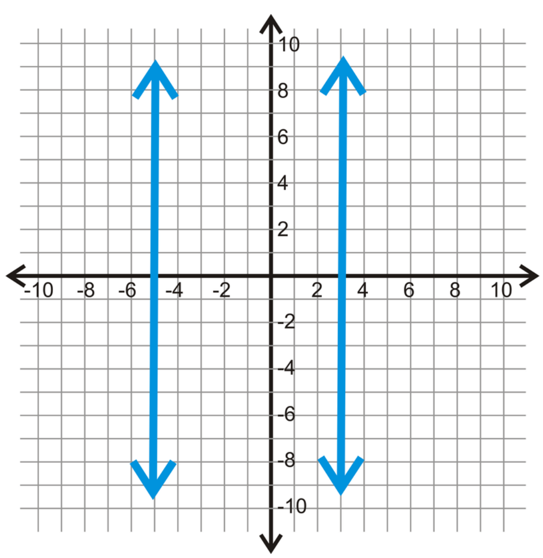
Find the distance between each pair of points. Round your answer to the nearest hundredth.

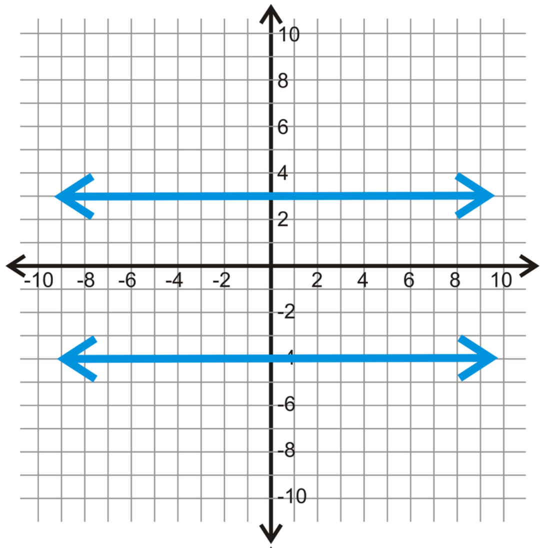
1. (4, 15) and (-2, -1)
2. (-6, 1) and (9, -11)
3. (0, 12) and (-3, 8)
4. (-8, 19) and (3, 5)
5. (3, -25) and (-10, -7)
6. (-1, 2) and (8, -9)
7. (5, -2) and (1, 3)
8. (-30, 6) and (-23, 0)

Determine the shortest distance between the given line and point. Round your answers to the nearest hundredth.

9. $y = \frac{1}{3}x + 4;\ (5,\ -1)$
10. $y = 2x - 4;\ (-7,\ -3)$
11. $y = -4x + 1;\ (4,\ 2)$
12. $y = -\frac{2}{3}x - 8;\ (7,\ 9)$

Use each graph below to determine how far apart each theparallel lines are. Round your answers to the nearest hundredth.

13. 



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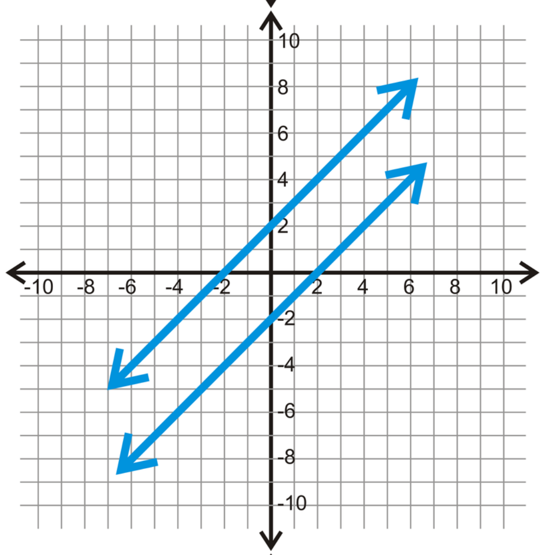

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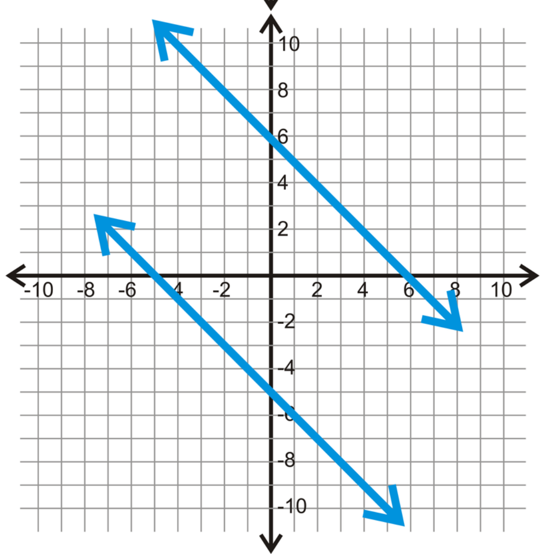


16.

Determine the shortest distance between the each pair of parallel lines. Round your answer to the nearest hundredth.

17. $x = 5,\ x = 1$
18. $y = -6,\ y = 4$
19. $y = x + 5,\ y = x - 3$

20. $y = -\frac{1}{3}x + 2,\ y = -\frac{1}{3}x - 8$
21. $y = 4x + 9,\ y = 4x - 8$
22. $y = \frac{1}{2}x,\ y = \frac{1}{2}x - 5$

Find the equation of the perpendicular bisector for pair of points.

23. (1, 5) and (7, -7)
24. (1, -8) and (7, -6)
25. (9, 2) and (-9, -10)
26. (-7, 11) and (-3, 1)
27. The perpendicular bisector of $\overline{CD}$ has the equation $y = 3x - 11$. If D is (-3, 0) what are the coordinates of C?
28. The perpendicular bisector of $\overline{LM}$ has the equation $y = -x + 5$. If L is (6, -3) what are the coordinates of M?
29. ***Construction*** Plot the points (5, -3) and (-5, -9). Draw the line segment between the points. Construct the perpendicular bisector for these two points. (Construction was in Chapter 1). Determine the equation of the perpendicular bisector and the midpoint.
30. ***Construction*** Graph the line $y = -\frac{1}{2}x - 5$ and the point (2, 5). Construct the perpendicular line, through (2, 5) and determine the equation of this line.
31. ***Challenge*** The distance between two points is 25 units. One point is (-2, 9). What is the second point? You may assume that the second point is made up of integers.
32. ***Writing*** List the steps you would take to find the distance between two parallel lines, like the two in #24.

Review Queue Answers

1. $y = -4x - 1$
2. $y = \frac{1}{2}x - 3$
3. $y = \frac{2}{3}x + 6$

8.7 Tangent, Sine and Cosine

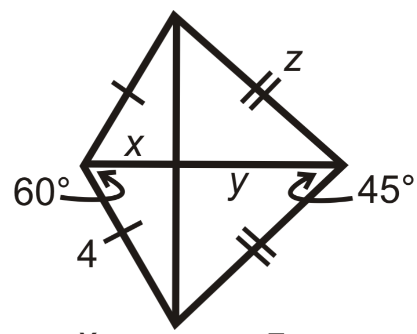
Learning Objectives

- Use the tangent, sine and cosine ratios in a right triangle.
- Understand these trigonometric ratios in special right triangles.
- Use a scientific calculator to find sine, cosine and tangent.
- Use trigonometric ratios in real-life situations.

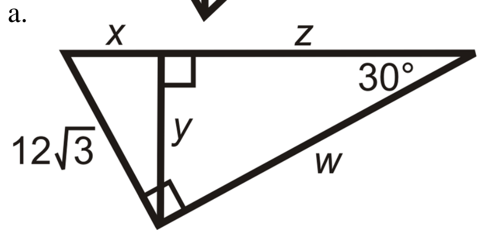
Review Queue

1. The legs of an isosceles right triangle have length 14. What is the hypotenuse?
2. Do the lengths 8, 16, 20 make a right triangle? If not, is the triangle obtuse or acute?
3. In a 30-60-90 triangle, what do the 30, 60, and 90 refer to?
4. Find the measure of the missing lengths.

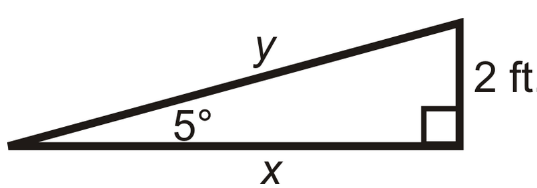
a.



b.



Know What? A restaurant needs to build a wheelchair ramp for its customers. The angle of elevation for a ramp is recommended to be 5°. If the vertical distance from the sidewalk to the front door is two feet, what is the horizontal distance that the ramp will take up (x)? How long will the ramp be (y)? Round your answers to the nearest hundredth.



What is Trigonometry?

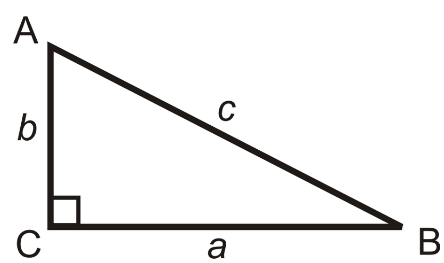
The word trigonometry comes from two words meaning *triangle* and *measure*. In this lesson we will define three trigonometric (or trig) functions. Once we have defined these functions, we will be able to solve problems like the **Know What?** above.

Trigonometry: The study of the relationships between the sides and angles of right triangles.

In trigonometry, sides are named in reference to a particular angle. The hypotenuse of a triangle is always the same, but the terms **adjacent** and **opposite** depend on which angle you are referencing. A side adjacent to an angle is the leg of the triangle that helps form the angle. A side opposite to an angle is the leg of the triangle that does not help form the angle. We never reference the right angle when referring to trig ratios.

a is $adjacent$ to $\angle B$. a is $opposite$ $\angle A$.
b is $adjacent$ to $\angle A$. b is $opposite$ $\angle B$.

c is the $hypotenuse$.



Sine, Cosine, and Tangent Ratios

The three basic trig ratios are called, sine, cosine and tangent. At this point, we will only take the sine, cosine and tangent of acute angles. However, you will learn that you can use these ratios with obtuse angles as well.

Sine Ratio: For an acute angle x in a right triangle, the $\sin x$ is equal to the ratio of the side opposite the angle over the hypotenuse of the triangle.

Using the triangle above, $\sin A = \frac{a}{c}$ and $\sin B = \frac{b}{c}$.

Cosine Ratio: For an acute angle x in a right triangle, the $\cos x$ is equal to the ratio of the side adjacent to the angle over the hypotenuse of the triangle.

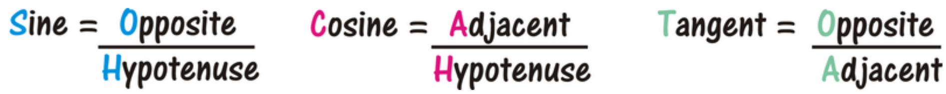
Using the triangle above, $\cos A = \frac{b}{c}$ and $\cos B = \frac{a}{c}$.

Tangent Ratio: For an acute angle x, in a right triangle, the $\tan x$ is equal to the ratio of the side opposite to the angle over the side adjacent to x.

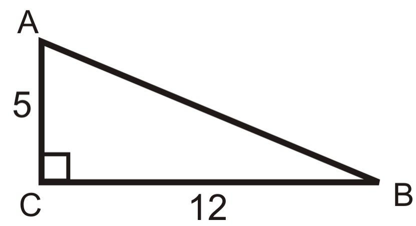
Using the triangle above, $\tan A = \frac{a}{b}$ and $\tan B = \frac{b}{a}$.

There are a few important things to note about the way we write these ratios. First, keep in mind that the abbreviations $\sin x, \cos x$, and $\tan x$ are all functions. Each ratio can be considered a function of the angle (see Chapter 10). Second, be careful when using the abbreviations that you still pronounce the full name of each function. When we write $\sin x$ it is still pronounced *sine,* with a long "*i*". When we write $\cos x$, we still say co-sine. And when we write $\tan x$, we still say tangent.

An easy way to remember ratios is to use the pneumonic SOH-CAH-TOA.



Example 1: Find the sine, cosine and tangent ratios of $\angle A$.



Solution: First, we need to use the Pythagorean Theorem to find the length of the hypotenuse.

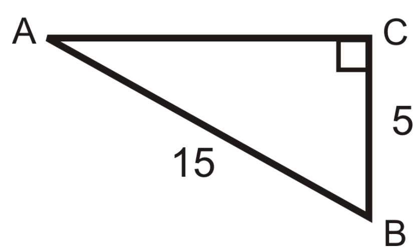
$$\begin{aligned} 5^2 + 12^2 &= h^2 \\ 13 &= h \end{aligned}$$

So, $\sin A = \frac{12}{13}, \cos A =$, and $\tan A = \frac{12}{5}$.

A few important points:

- Always reduce ratios when you can.
- Use the Pythagorean Theorem to find the missing side (if there is one).
- The tangent ratio can be bigger than 1 (the other two cannot).
- If two right triangles are similar, then their sine, cosine, and tangent ratios will be the same (because they will reduce to the same ratio).
- If there is a radical in the denominator, rationalize the denominator.

Example 2: Find the sine, cosine, and tangent of $\angle B$.

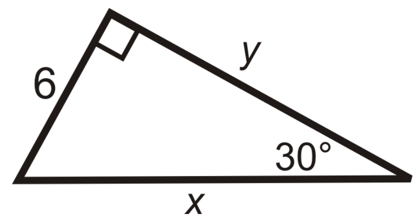


Solution: Find the length of the missing side.

$$\begin{aligned} AC^2 + 5^2 &= 15^2 \\ AC^2 &= 200 \\ AC &= 10\sqrt{2} \end{aligned}$$

Therefore, $\sin B = \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3}, \cos B = \frac{5}{15} = \frac{1}{3}$, and $\tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2}$.

Example 3: Find the sine, cosine and tangent of 30°.



Solution: This is a special right triangle, a 30-60-90 triangle. So, if the short leg is 6, then the long leg is $6\sqrt{3}$ and the hypotenuse is 12.

$\sin 30^\circ = \frac{6}{12} = \frac{1}{2}, \cos 30^\circ = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$, and $\tan 30^\circ = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

In Example 3, we knew the angle measure of the angle we were taking the sine, cosine and tangent of. This means that ***the sine, cosine and tangent for an angle are fixed***.

Sine, Cosine, and Tangent with a Calculator

We now know that the trigonometric ratios are not dependent on the sides, but the ratios. Therefore, there is one fixed value for every angle, from 0° to 90°. Your scientific (or graphing) calculator knows the values of the sine, cosine and tangent of all of these angles. Depending on your calculator, you should have [SIN], [COS], and [TAN] buttons. Use these to find the sine, cosine, and tangent of any acute angle.

Example 4: Find the indicated trigonometric value, using your calculator.

a) $\sin 78^\circ$

b) $\cos 60^\circ$

c) $\tan 15^\circ$

Solution: Depending on your calculator, you enter the degree first, and then press the correct trig button or the other way around. For TI-83s and TI-84s you press the trig button first, followed by the angle. Also, make sure the mode of your calculator is in DEGREES.

a) $\sin 78^\circ = 0.9781$

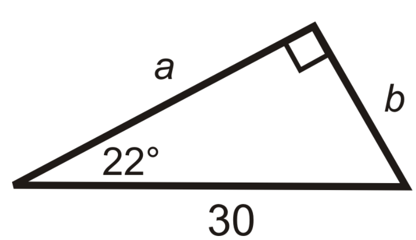
b) $\cos 60^\circ = 0.5$

c) $\tan 15^\circ = 0.2679$

Finding the Sides of a Triangle using Trig Ratios

One application of the trigonometric ratios is to use them to find the missing sides of a right triangle. All you need is one angle, other than the right angle, and one side. Let's go through a couple of examples.

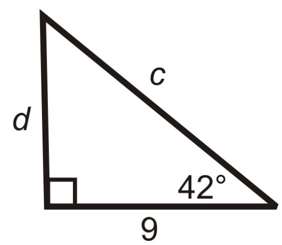
Example 5: Find the value of each variable. Round your answer to the nearest hundredth.



Solution: We are given the hypotenuse, so we would need to use the sine to find b, because it is opposite 22° and cosine to find a, because it is adjacent to 22°.

$$\begin{aligned} \sin 22^\circ &= \frac{b}{30} \\ 30 \cdot \sin 22^\circ &= b \\ b &\approx 11.24 \end{aligned} \qquad\qquad \begin{aligned} \cos 22^\circ &= \frac{a}{30} \\ 30 \cdot \cos 22^\circ &= a \\ a &\approx 27.82 \end{aligned}$$

Example 6: Find the value of each variable. Round your answer to the nearest hundredth.



Solution: Here, we are given the adjacent leg to 42°. To find c, we need to use cosine and to find d we will use tangent.

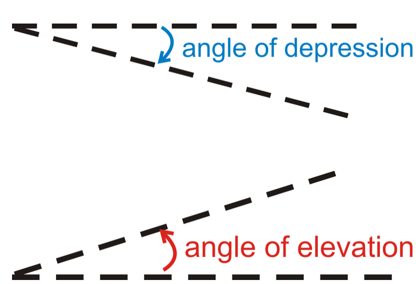
$$\begin{aligned} \cos 42^\circ &= \frac{9}{c} \\ c \cdot \cos 42^\circ &= 9 \\ c &= \frac{9}{\cos 42^\circ} \approx 12.11 \end{aligned} \qquad\qquad \begin{aligned} \tan 42^\circ &= \frac{d}{9} \\ 9 \cdot \tan 42^\circ &= d \\ d &\approx 8.10 \end{aligned}$$

Notice in both of these examples, you should only use the information that you are given. For example, you should not use the found value of b to find a (in Example 5) because b is an ***approximation***. Use exact values to give the most accurate answers. However, in both examples you could have also used the complementary angle to the one given.

Angles of Depression and Elevation

Another practical application of the trigonometric functions is to find the measure of lengths that you cannot measure. Very frequently, angles of depression and elevation are used in these types of problems.

Angle of Depression: The angle measured from the horizon or horizontal line, down.



Angle of Elevation: The angle measure from the horizon or horizontal line, up.

Example 7: An inquisitive math student is standing 25 feet from the base of the Washington Monument. The angle of elevation from her horizontal line of sight is 87.4°. If her "eye height" is 5ft, how tall is the monument?

Solution: We can find the height of the monument by using the tangent ratio and then adding the eye height of the student.

$$\tan 87.4^\circ = \frac{h}{25}$$
$$h = 25 \cdot \tan 87.4^\circ = 550.54$$

Adding 5 ft, the total height of the Washington Monument is 555.54 ft.

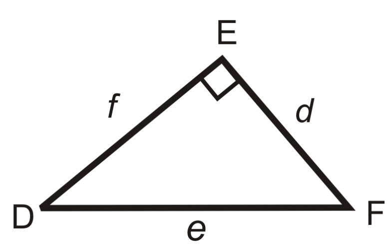
According to Wikipedia, the actual height of the monument is 555.427 ft.

Know What? Revisited To find the horizontal length and the actual length of the ramp, we need to use the tangent and sine.

$$\tan 5^\circ = \frac{2}{x} \qquad \sin 5^\circ = \frac{2}{y}$$
$$x = \frac{2}{\tan 5^\circ} = 22.86 \qquad y = \frac{2}{\sin 5^\circ} = 22.95$$

Review Questions

Use the diagram to fill in the blanks below.



1. $\tan D = \frac{?}{?}$
2. $\sin F = \frac{?}{?}$
3. $\tan F = \frac{?}{?}$
4. $\cos F = \frac{?}{?}$
5. $\sin D = \frac{?}{?}$
6. $\cos D = \frac{?}{?}$

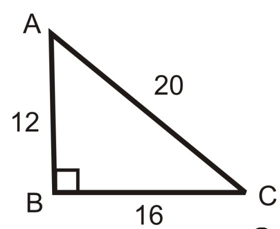
From questions 1-6, we can conclude the following. Fill in the blanks.

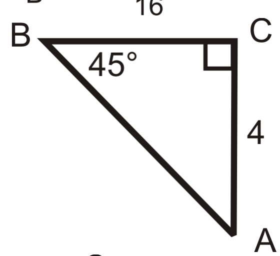
7. $\cos$ ___ $= \sin F$ and $\sin$ ___ $= \cos F$
8. The sine of an angle is ____________ to the cosine of its ______________.
9. $\tan D$ and $\tan F$ are ___________ of each other.

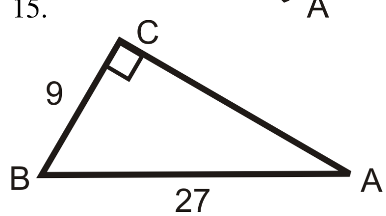
Use your calculator to find the value of each trig function below. Round to four decimal places.

10. $\sin 24^\circ$
11. $\cos 45^\circ$
12. $\tan 88^\circ$
13. $\sin 43^\circ$

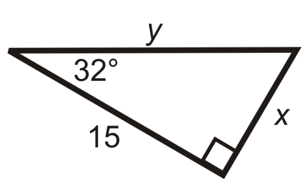
Find the sine, cosine and tangent of $\angle A$. Reduce all fractions and radicals.

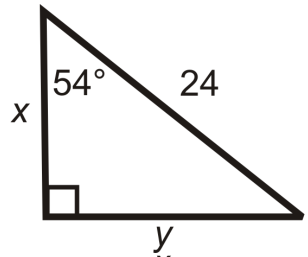
14. 

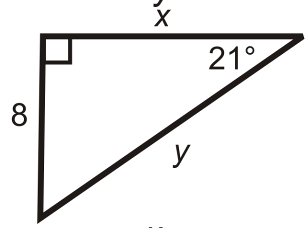
15. 

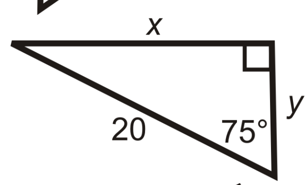
16. 

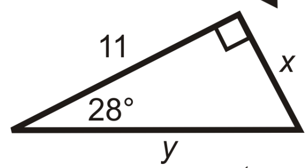
Find the length of the missing sides. Round your answers to the nearest hundredth.

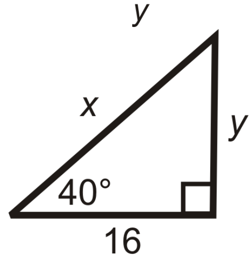
17. 

18. 

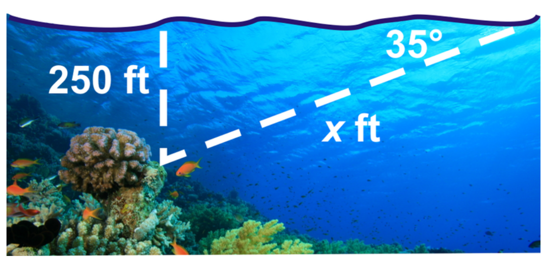
19. 

20. 

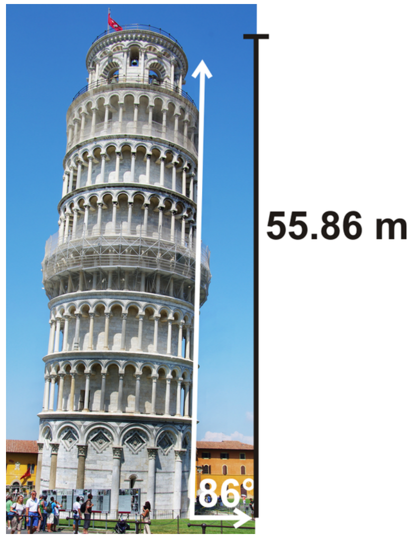
21. 

22. 

23. Kristin is swimming in the ocean and notices a coral reef below her. The angle of depression is 35° and the depth of the ocean, at that point, is 250 feet. How far away is she from the reef?



24. The Leaning Tower of Piza currently "leans" at a 4° angle and has a vertical height of 55.86 meters. How tall was the tower when it was originally built?



25. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 72°. If the building is 78 ft tall, how far away is the fountain?
26. William spots a tree directly across the river from where he is standing. He then walks 20 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 65°. How wide is the river?
27. Diego is flying his kite one afternoon and notices that he has let out the entire 120 ft of string. The angle his string makes with the ground is 52°. How high is his kite at this time?
28. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 36° angle with the ground 25 ft from the base of the tree. What was the height of the tree to the nearest foot?
29. Upon descent an airplane is 20,000 ft above the ground. The air traffic control tower is 200 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
30. ***Critical Thinking*** Why are the sine and cosine ratios always be less than 1?

Review Queue Answers

1. The hypotenuse is $14\sqrt{2}$.
2. No, $8^2 + 16^2 < 20^2$, the triangle is obtuse.
3. $30^\circ, 60^\circ$, and 90° refer to the angle measures in the special right triangle.
4. $x = 2, y = 2\sqrt{3}$
5. $x = 6\sqrt{3}, y = 18, z = 18\sqrt{3}, w = 36$

8.8 Right Triangle Trigonometry

Objective

To develop an understanding of trigonometric ratios and to use the trigonometric ratios sine, cosine and tangent along with their inverses to solve right triangles.

Review Queue

1. Given that $P(A) = 0.8, P(B) = 0.5$ and $P(A \cup B) = 0.9$, determine whether events A and B are independent.

2. Events A and B are independent and $P(A) = 0.6$ and $P(B) = 0.5$, find $P(A \cup B)'$.

3. Reduce the radical expressions:

a. $\sqrt{240}$

b. $3\sqrt{48} + 5\sqrt{75}$

c. $4\sqrt{15} \cdot \sqrt{30}$

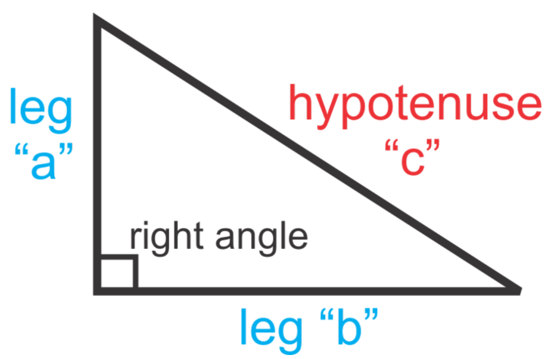
Pythagorean Theorem and its Converse

Objective

Discover, prove and apply the Pythagorean Theorem to solve for unknown sides in right triangles and prove triangles are right triangles.

Guidance

The Pythagorean Theorem refers to the relationship between the lengths of the three sides in a right triangle. It states that if a and b are the legs of the right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. For example, the lengths 3, 4, and 5 are the sides of a right triangle because $3^2 + 4^2 = 5^2 (9 + 16 = 25)$. Keep in mind that c is always the longest side.

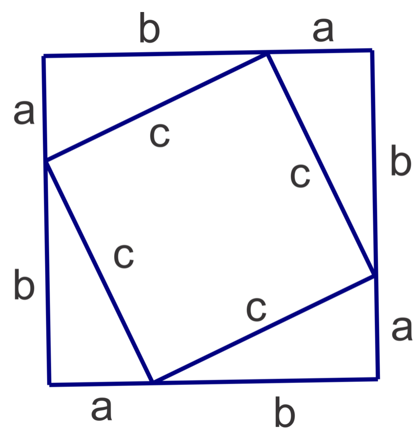


The converse of this statement is also true. If, in a triangle, c is the length of the longest side and the shorter sides have lengths a and b, and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

Proof of Pythagorean Theorem

There are many proofs of the Pythagorean Theorem and here is one of them. We will be using the concept that the area of a figure is equal to the sum of the areas of the smaller figures contained within it and algebra to derive the Pythagorean Theorem.

Using the figure below (a square with a smaller square inside), first write two equations for its area, one using the lengths of the sides of the outer square and one using the sum of the areas of the smaller square and the four triangles.



Area 1: $(a+b)^2 = a^2 + 2ab + b^2$

Area 2: $c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$

Now, equate the two areas and simplify:

$$\begin{aligned} a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + b^2 &= c^2 \end{aligned}$$

Example A

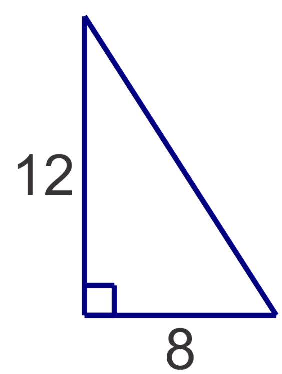
In a right triangle $a = 7$ and $c = 25$, find the length of the third side.

Solution: We can start by substituting what we know into the Pythagorean Theorem and then solve for the unknown side, b:

$$\begin{aligned} 7^2 + b^2 &= 25^2 \\ 49 + b^2 &= 625 \\ b^2 &= 576 \\ b &= 24 \end{aligned}$$

Example B

Find the length of the third side of the triangle below. Leave your answer in reduced radical form.



Solution: Since we are given the lengths of the two legs, we can plug them into the Pythagorean Theorem and find the length of the hypotenuse.

$$\begin{aligned}8^2+12^2&=c^2\\64+144&=c^2\\c^2&=208\\c&=\sqrt{208}=\sqrt{16\cdot 13}=4\sqrt{13}\end{aligned}$$

Example C

Determine whether a triangle with lengths 21, 28, 35 is a right triangle.

Solution: We need to see if these values will satisfy $a^2+b^2=c^2$. If they do, then a right triangle is formed. So,

$$\begin{aligned}21^2+28^2&=441+784=1225\\35^2&=1225\end{aligned}$$

Yes, the Pythagorean Theorem is satisfied by these lengths and a right triangle is formed by the lengths 21, 28 and 35.

Guided Practice

For the given two sides, determine the length of the third side if the triangle is a right triangle.

1. $a=10$ and $b=5$

2. $a=5$ and $c=13$

Use the Pythagorean Theorem to determine if a right triangle is formed by the given lengths.

3. 16, 30, 34

4. 9, 40, 42

5. 2, 2, 4

Answers

1. $\sqrt{10^2+5^2}=\sqrt{100+25}=\sqrt{125}=5\sqrt{5}$

2. $\sqrt{13^2-5^2}=\sqrt{169-25}=\sqrt{144}=12$

3.

$$\begin{aligned}16^2+30^2&=256+900=1156\\34^2&=1156\end{aligned}$$

Yes, this is a right triangle.

4.

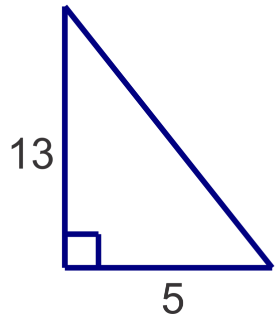
$$\begin{aligned}9^2+40^2&=81+1600=1681\\42^2&=1764\end{aligned}$$

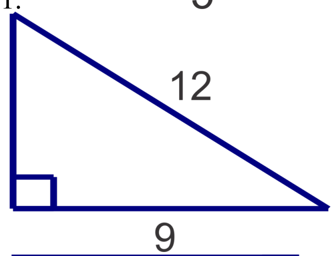
No, this is not a right triangle.

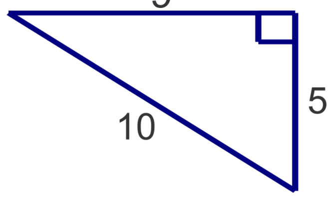
5. This one is tricky, in a triangle the lengths of any two sides must have a sum *greater* than the length of the third side. These lengths do not meet that requirement so not only do they not form a *right* triangle, they do not make a triangle at all.

Problem Set

Find the unknown side length for each right triangle below.

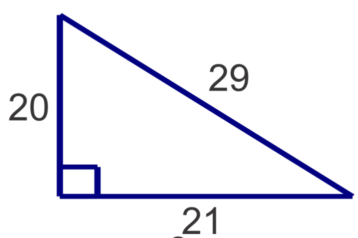
1. 

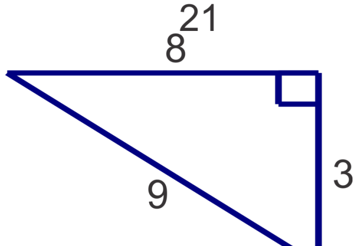
2. 

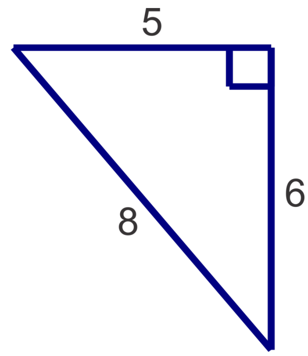
3. 

4. $a = 6, b = 8$
5. $b = 6, c = 14$
6. $a = 12, c = 18$

Determine whether the following triangles are right triangles.

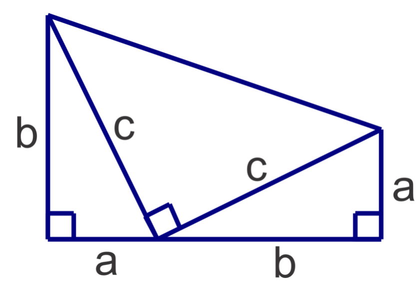
7. 

8. 

9. 

Do the lengths below form a right triangle? Remember to make sure that they form a triangle.

10. 3, 4, 5
11. 6, 6, 11
12. 11, 13, 17
13. Major General James A. Garfield (and former President of the U.S.) is credited with deriving this proof of the Pythagorean Theorem using a trapezoid. Follow the steps to recreate his proof.



(a) Find the area of the trapezoid using the trapezoid area formula: $A = \frac{1}{2}(b_1 + b_2)h$ (b) Find the sum of the areas of the three right triangles in the diagram. (c) The areas found in the previous two problems should be the same value. Set the expressions equal to each other and simplify to get the Pythagorean Theorem.

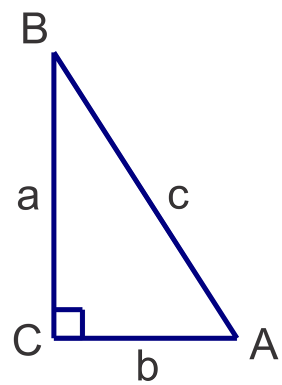
Sine, Cosine, and Tangent

Objective

Define and apply the trigonometric ratios sine, cosine and tangent to solve for the lengths of unknown sides in right triangles.

Guidance

The trigonometric ratios sine, cosine and tangent refer to the known ratios between particular sides in a right triangle based on an acute angle measure.



In this right triangle, side c is the hypotenuse.

If we consider the angle B, then we can describe each of the legs by its position relative to angle B: side a is adjacent to B; side b is opposite B

If we consider the angle A, then we can describe each of the legs by its position relative to angle A: side b is adjacent to A; side a is opposite A

Now we can define the trigonometry ratios as follows:

$$\text{Sine is } \frac{opposite}{hypotenuse} \quad \text{Cosine is } \frac{adjacent}{hypotenuse} \quad \text{Tangent is } \frac{opposite}{adjacent}$$

A shorthand way to remember these ratios is to take the letters in red above and write the phrase:

SOH CAH TOA

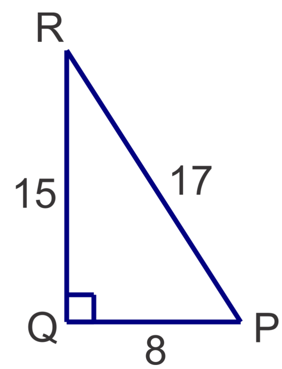
Now we can find the trigonometric ratios for each of the acute angles in the triangle above.

$$\begin{aligned} \sin A &= \frac{a}{c} & \sin B &= \frac{b}{c} \\ \cos A &= \frac{b}{c} & \cos B &= \frac{a}{c} \\ \tan A &= \frac{a}{b} & \tan B &= \frac{b}{a} \end{aligned}$$

It is important to understand that given a particular (acute) angle measure in a right triangle, these ratios are constant no matter how big or small the triangle. For example; if the measure of the angle is 25°, then $\sin 25^\circ \approx 0.4226$ and ratio of the opposite side to the hypotenuse is always 0.4226 no matter how big or small the triangle.

Example A

Find the trig ratios for the acute angles R and P in ΔPQR.



Solution: From angle R, $O = 8$; $A = 15$; and $H = 17$. Now the trig ratios are:

$$\sin R = \frac{8}{17};\ \cos R = \frac{15}{17};\ \tan R = \frac{8}{15}$$

From angle P, $O = 15$; $A = 8$; and $H = 17$. Now the trig ratios are:

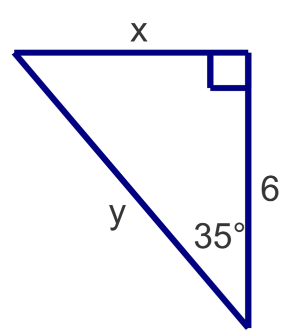
$$\sin P = \frac{15}{17};\ \cos P = \frac{8}{17};\ \tan P = \frac{15}{8}$$

Do you notice any patterns or similarities between the trigonometric ratios? The opposite and adjacent sides are switched and the hypotenuse is the same. Notice how this switch affects the ratios:

$$\sin R = \cos P \quad \cos R = \sin P \quad \tan R = \frac{1}{\tan P}$$

Example B

Use trigonometric ratios to find the x and y.



Solution: First identify or label the sides with respect to the given acute angle. So, x is opposite, y is hypotenuse (note that it is the hypotenuse because it is the side opposite the right angle, it may be adjacent to the given angle but the hypotenuse cannot be the adjacent side) and 6 is the adjacent side.

To find x, we must use the given length of 6 in our ratio too. So we are using opposite and adjacent. Since tangent is the ratio of opposite over adjacent we get:

$$\begin{aligned}
\tan 35^\circ &= \frac{x}{6} \\
x &= 6 \tan 35^\circ && \text{multiply both sides by 6} \\
x &\approx 4.20 && \text{Use the calculator to evaluate-type in 6TAN(35) ENTER}
\end{aligned}$$

NOTE: make sure that your calculator is in DEGREE mode. To check, press the MODE button and verify that DEGREE is highlighted (as opposed to RADIAN). If it is not, use the arrow buttons to go to DEGREE and press ENTER. The default mode is radian, so if your calculator is reset or the memory is cleared it will go back to radian mode until you change it.

To find y using trig ratios and the given length of 6, we have adjacent and hypotenuse so we'll use cosine:

$$\begin{aligned}
\cos 35^\circ &= \frac{6}{y} \\
\frac{\cos 35^\circ}{1} &= \frac{6}{y} && \text{set up a proportion to solve for } y \\
6 &= y \cos 35^\circ && \text{cross multiply} \\
y &= \frac{6}{\cos 35^\circ} && \text{divide by} \cos 35^\circ \\
y &= 7.32 && \text{Use the calculator to evaluate-type in 6/TAN(35) ENTER}
\end{aligned}$$

Alternatively, we could find y using the value we found for x and the Pythagorean theorem:

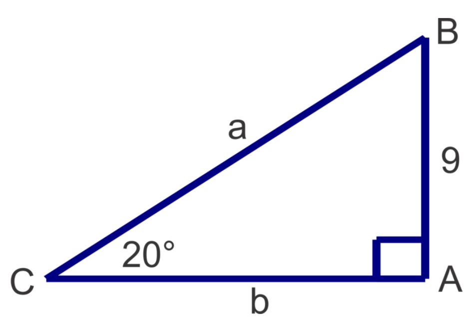
$$\begin{aligned}
4.20^2 + 6^2 &= y^2 \\
53.64 &= y^2 \\
y &\approx 7.32
\end{aligned}$$

The downside of this method is that if we miscalculated our x value, we will double down on our mistake and guarantee an incorrect y value. In general you will help avoid this kind of mistake if you use the given information whenever possible.

Example C

Given ΔABC, with $m\angle A = 90^\circ, m\angle C = 20^\circ$ and $c = 9$, find a and b.

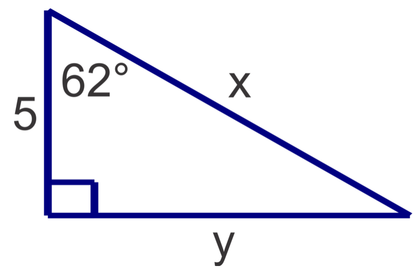
Solution: Visual learners may find it particularly useful to make a sketch of this triangle and label it with the given information:



To find a (the hypotenuse) we can use the opposite side and the sine ratio: $\sin 20^\circ = \frac{9}{a}$, solving as we did in Example B we get $a = \frac{9}{\sin 20^\circ} \approx 26.31$ To find b (the adjacent side) we can use the opposite side and the tangent ratio: $\tan 20^\circ = \frac{9}{b}$, solving for b we get $b = \frac{9}{\tan 20^\circ} \approx 24.73$.

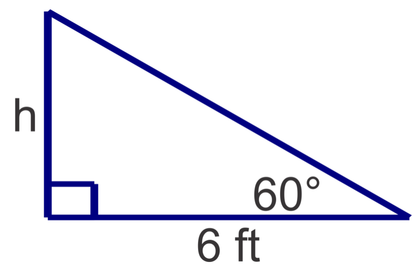
Guided Practice

1. Use trig ratios to find x and y:



2. Given ΔABC with $m\angle B = 90^\circ, m\angle A = 43^\circ$ and $a = 7$, find b and c.

3. The base of a playground slide is 6 ft from the base of the platform and the slide makes a 60° angle with the ground. To the nearest tenth of a foot, how high is the platform at the top of the slide?



Answers

1. For x:

$$\cos 62^\circ = \frac{5}{x}$$
$$x = \frac{5}{\cos 62^\circ} \approx 10.65$$

For y:

$$\tan 62^\circ = \frac{y}{5}$$
$$y = 5\tan 62^\circ \approx 9.40$$

2. For b:

$$\sin 43^\circ = \frac{7}{b}$$
$$b = \frac{7}{\sin 43^\circ} \approx 10.26$$

For c:

$$\tan 43^\circ = \frac{7}{c}$$
$$c = \frac{7}{\tan 43^\circ} \approx 7.51$$

3.

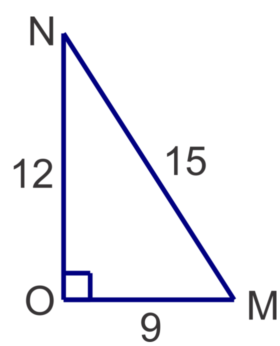
$$\tan 60^\circ = \frac{h}{6}$$
$$h = 6\tan 60^\circ \approx 10.39$$

, so the height of the platform is 10.4 ft

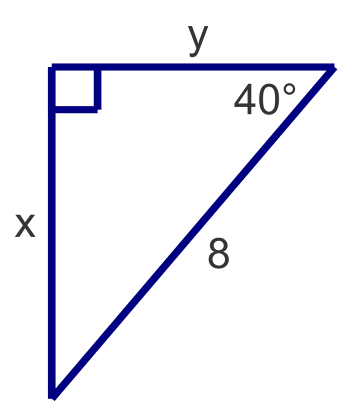
Problem Set

Use you calculator to find the following trigonometric ratios. Give answers to four decimal places.

1. $\sin 35°$
2. $\tan 72°$
3. $\cos 48°$
4. Write the three trigonometric ratios of each of the acute angles in the triangle below.



Use trigonometric ratios to find the unknown side lengths in the triangles below. Round your answers to the nearest hundredth.



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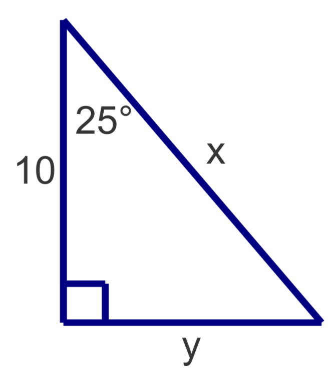

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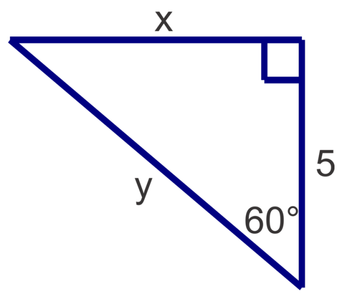
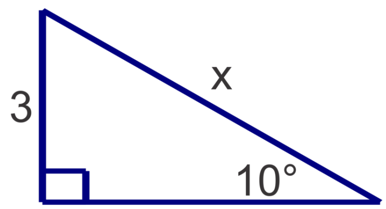


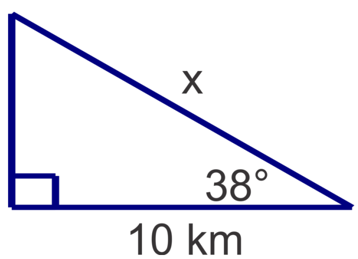
7.

For problems 8-10 use the given information about ΔABC with right angle B to find the unknown side lengths. Round your answer to the nearest hundredth.

8. $a = 12$ and $m\angle A = 43^\circ$
9. $m\angle C = 75^\circ$ and $b = 24$
10. $c = 7$ and $m\angle A = 65^\circ$
11. A ramp needs to have an angle of elevation no greater than 10 degrees. If the door is 3 ft above the sidewalk level, what is the minimum possible ramp length to the nearest tenth of a foot?



12. A ship, *Sea Dancer*, is 10 km due East of a lighthouse. A second ship, *Nelly*, is due north of the lighthouse. A spotter on the *Sea Dancer* measures the angle between the *Nelly* and the lighthouse to be 38°. How far apart are the two ships to the nearest tenth of a kilometer?



Inverse Trig Functions and Solving Right Triangles

Objective

Use the inverse trigonometric functions to find the measure of unknown acute angles in right triangles and solve right triangles.

Guidance

In the previous concept we used the trigonometric functions sine, cosine and tangent to find the ratio of particular sides in a right triangle given an angle. In this concept we will use the inverses of these functions, $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, to find the angle measure when the ratio of the side lengths is known. When we type $\sin 30^\circ$ into our calculator, the calculator goes to a table and finds the trig ratio associated with 30°, which is $\frac{1}{2}$. When we use an inverse function we tell the calculator to look up the ratio and give us the angle measure. For example: $\sin^{-1}\left(\frac{1}{2}\right) = 30^\circ$. On your calculator you would press 2^{ND}SIN to get SIN^{-1}(and then type in $\frac{1}{2}$, close the parenthesis and press ENTER. Your calculator screen should read $\text{SIN}^{-1}\left(\frac{1}{2}\right)$ when you press ENTER.

Example A

Find the measure of angle A associated with the following ratios. Round answers to the nearest degree.

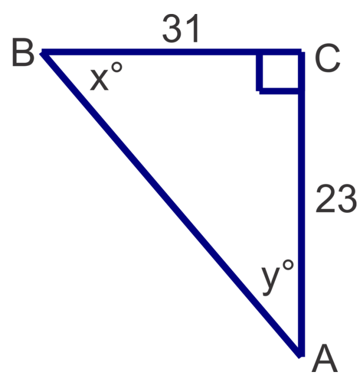
1. $\sin A = 0.8336$
2. $\tan A = 1.3527$
3. $\cos A = 0.2785$

Solution: Using the calculator we get the following:

1. $\sin^{-1}(0.8336) \approx 56^\circ$
2. $\tan^{-1}(1.3527) \approx 54^\circ$
3. $\cos^{-1}(0.2785) \approx 74^\circ$

Example B

Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.



Solution: We can solve for either x or y first. If we choose to solve for x first, the 23 is opposite and 31 is adjacent so we will use the tangent ratio.

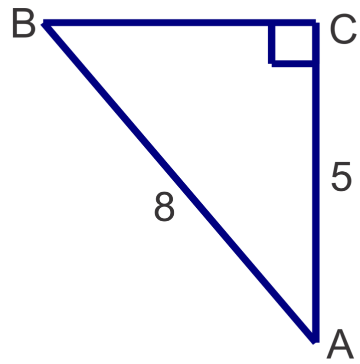
$$x = \tan^{-1}\left(\frac{23}{31}\right) \approx 37^\circ.$$

Recall that in a right triangle, the acute angles are always complementary, so $90^\circ - 37^\circ = 53^\circ$, so $y = 53^\circ$. We can also use the side lengths an a trig ratio to solve for y:

$$y = \tan^{-1}\left(\frac{31}{23}\right) \approx 53^\circ.$$

Example C

Solve the right triangle shown below. Round all answers to the nearest tenth.



Solution: We can solve for either angle A or angle B first. If we choose to solve for angle B first, then 8 is the hypotenuse and 5 is the opposite side length so we will use the sine ratio.

$$\sin B = \frac{5}{8}$$
$$m\angle B = \sin^{-1}\left(\frac{5}{8}\right) \approx 38.7^\circ$$

Now we can find A two different ways.

Method 1: We can using trigonometry and the cosine ratio:

$$\cos A = \frac{5}{8}$$
$$m\angle A = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.3^\circ$$

Method 2: We can subtract $m\angle B$ from 90°: $90^\circ - 38.7^\circ = 51.3^\circ$ since the acute angles in a right triangle are always complimentary.

Either method is valid, but be careful with Method 2 because a miscalculation of angle B would make the measure you get for angle A incorrect as well.

Guided Practice

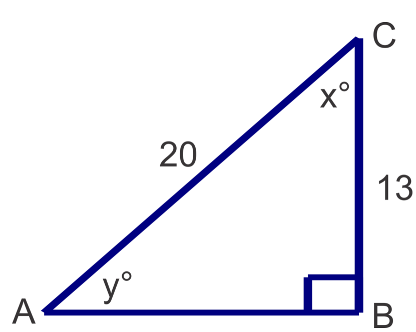
1. Find the measure of angle A to the nearest degree given the trigonometric ratios.

a. $\sin A = 0.2894$

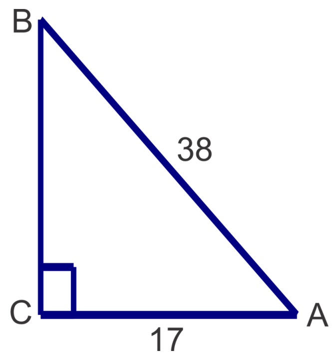
b. $\tan A = 2.1432$

c. $\cos A = 0.8911$

2. Find the measures of the unknown angles in the triangle shown. Round answers to the nearest degree.



3. Solve the triangle. Round side lengths to the nearest tenth and angles to the nearest degree.



Answers

1. a. $\sin^{-1}(0.2894) \approx 17^\circ$

b. $\tan^{-1}(2.1432) \approx 65^\circ$

c. $\cos^{-1}(0.8911) \approx 27^\circ$

2.

$$x = \cos^{-1}\left(\frac{13}{20}\right) \approx 49^\circ; \quad y = \sin^{-1}\left(\frac{13}{20}\right) \approx 41^\circ$$

3.

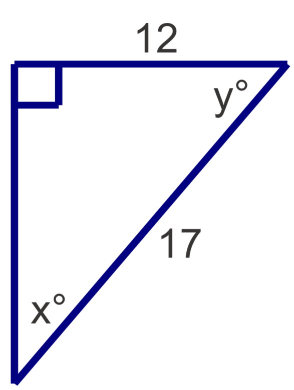
$$m\angle A = \cos^{-1}\left(\frac{17}{38}\right) \approx 63^\circ; \quad m\angle B = \sin^{-1}\left(\frac{17}{38}\right) \approx 27^\circ; \quad a = \sqrt{38^2 - 17^2} \approx 34.0$$

Problem Set

Use your calculator to find the measure of angle B. Round answers to the nearest degree.

1. $\tan B = 0.9523$
2. $\sin B = 0.8659$
3. $\cos B = 0.1568$

Find the measures of the unknown acute angles. Round measures to the nearest degree.



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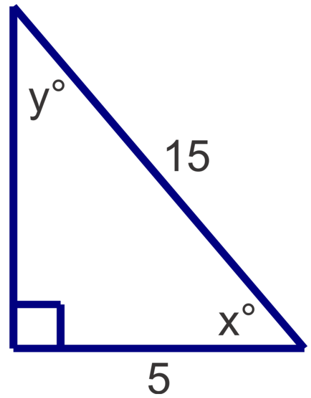
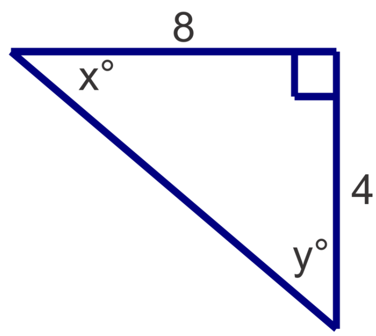
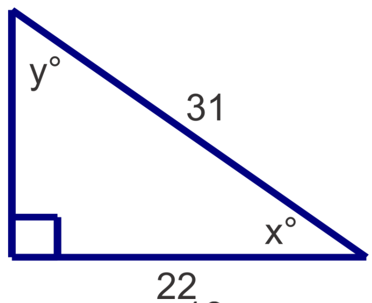
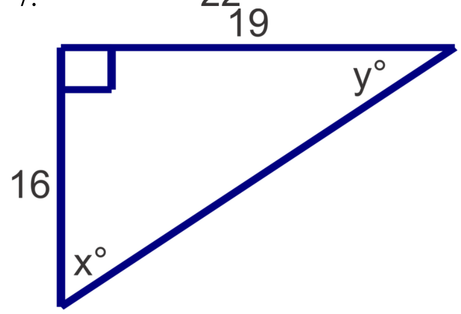
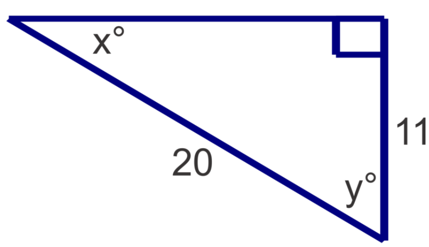
5. 

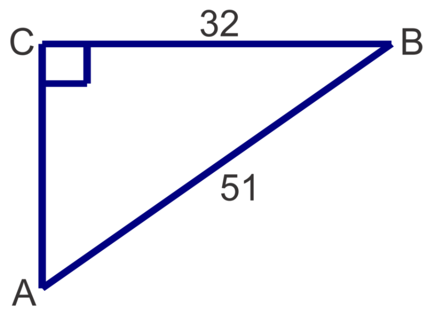
6. 

7. 

8. 

9. 

Solve the following right triangles. Round angle measures to the nearest degree and side lengths to the nearest tenth.



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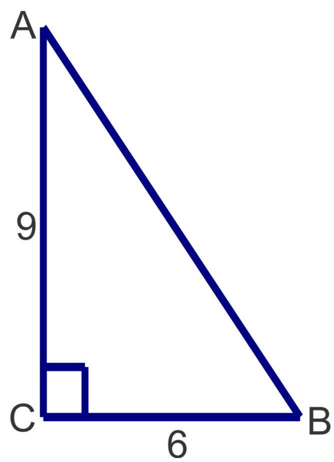

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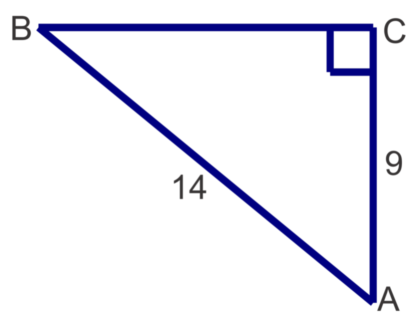


12.

Application Problems

Objective

Use the Pythagorean Theorem and trigonometric ratios to solve the real world application problems.

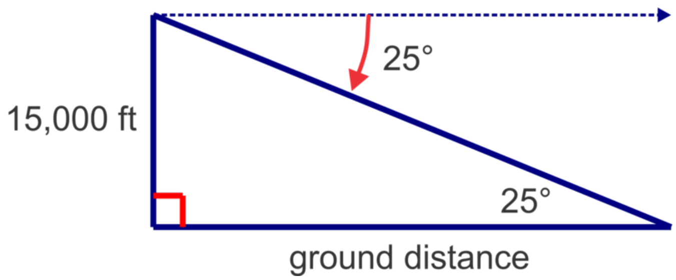
Guidance

When solving word problems, it is important to understand the terminology used to describe angles. In trigonometric problems, the terms angle of elevation and angle of depression are commonly used. Both of these angles are always measured from a horizontal line as shown in the diagrams below.

Example A

An airplane approaching an airport spots the runway at an angle of depression of 25°. If the airplane is 15,000 ft above the ground, how far (ground distance) is the plane from the runway? Give your answer to the nearest 100 ft.

Solution: Make a diagram to illustrate the situation described and then use a trigonometric ratio to solve. Keep in mind that an angle of depression is down from a horizontal line of sight-in this case a horizontal line from the pilot of the plane parallel to the ground.



Note that the angle of depression and the alternate interior angle will be congruent, so the angle in the triangle is also 25°.

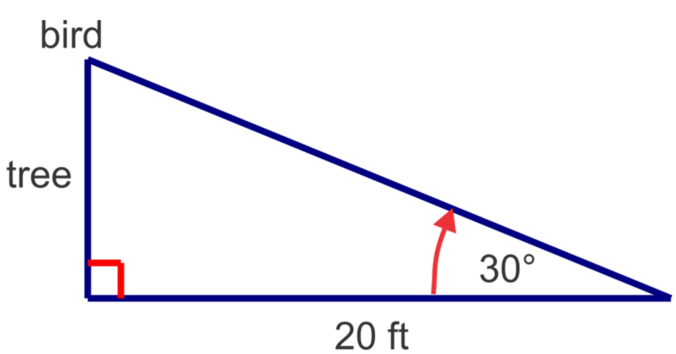
From the picture, we can see that we should use the tangent ratio to find the ground distance.

$$\tan 25^\circ = \frac{15000}{d}$$
$$d = \frac{15000}{\tan 25^\circ} \approx 32,200\ ft$$

Example B

Rachel spots a bird in a tree at an angle of elevation of 30°. If Rachel is 20 ft from the base of the tree, how high up in the tree is the bird? Give your answer to the nearest tenth of a foot.

Solution: Make a diagram to illustrate the situation. Keep in mind that there will be a right triangle and that the right angle is formed by the ground and the trunk of the tree.



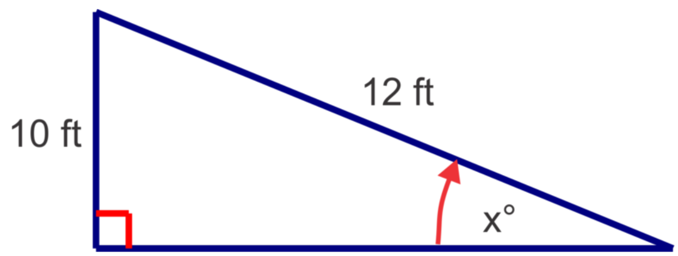
Here we can use the tangent ratio to solve for the height of the bird

$$\tan 30^\circ = \frac{h}{20}$$
$$h = 20\tan 30^\circ \approx 11.5\ ft$$

Example C

A 12 ft ladder is leaning against a house and reaches 10 ft up the side of the house. To the nearest degree, what angle does the ladder make with the ground?

Solution: In this problem, we will need to find an angle. By making a sketch of the triangle we can see which inverse trigonometric ratio to use.



$$\sin x^\circ = \frac{10}{12}$$
$$\sin^{-1}\left(\frac{10}{12}\right) \approx 56^\circ$$

Guided Practice

Use a trigonometry to solve the following application problems.

1. A ramp makes a 20° angle with the ground. If door the ramp leads to is 2 ft above the ground, how long is the ramp? Give your answer to the nearest tenth of a foot.

2. Charlie lets out 90 ft of kite string. If the angle of elevation of the string is 70°, approximately how high is the kite? Give your answer to the nearest foot.

3. A ship's sonar spots a wreckage at an angle of depression of 32°. If the depth of the ocean is about 250 ft, how far is the wreckage (measured along the surface of the water) from the ship, to the nearest foot.

Answers

1.

$$\sin 20^\circ = \frac{2}{x}$$
$$x = \frac{2}{\sin 20^\circ} \approx 5.8\ ft$$

2.

$$\sin 70^\circ = \frac{x}{90}$$
$$x = 90\sin 70^\circ \approx 85\ ft$$

3.

$$\tan 32^\circ = \frac{250}{x}$$
$$x = \frac{250}{\tan 32^\circ} \approx 400\ ft$$

Vocabulary

Angle of Elevation
An angle measured up from a horizontal line.

Angle of Depression
An angle measured down from a horizontal line.

Problem Set

Use the Pythagorean Theorem and/or trigonometry to solve the following word problems.

1. A square has sides of length 8 inches. To the nearest tenth of an inch, what is the length of its diagonal?
2. Layne spots a sailboat from her fifth floor balcony, about 25 m above the beach, at an angle of depression of 3°. To the nearest meter, how far out is the boat?
3. A zip line takes passengers on a 200 m ride from high up in the trees to a ground level platform. If the angle of elevation of the zip line is 10°, how high above ground is the tree top start platform? Give your answer to the nearest meter.
4. The angle of depression from the top of an apartment building to the base of a fountain in a nearby park is 57°. If the building is 150 ft tall, how far away, to the nearest foot, is the fountain?
5. A playground slide platform is 6 ft above ground. If the slide is 8 ft long and the end of the slide is 1 ft above ground, what angle does the slide make with the ground? Give your answer to the nearest degree.
6. Benjamin spots a tree directly across the river from where he is standing. He then walks 27 ft upstream and determines that the angle between his previous position and the tree on the other side of the river is 73°. How wide, to the nearest foot, is the river?
7. A rectangle has sides of length 6 in and 10 in. To the nearest degree, what angle does the diagonal make with the longer side?
8. Tommy is flying his kite one afternoon and notices that he has let out the entire 130 ft of string. The angle his string makes with the ground is 48°. How high, to the nearest foot, is his kite at this time?
9. A tree struck by lightning in a storm breaks and falls over to form a triangle with the ground. The tip of the tree makes a 18° angle with the ground 21 ft from the base of the tree. What was the height of the tree to the nearest foot?
10. Upon descent an airplane is 19,000 ft above the ground. The air traffic control tower is 190 ft tall. It is determined that the angle of elevation from the top of the tower to the plane is 15°. To the nearest mile, find the ground distance from the airplane to the tower.
11. Why will the sine and cosine ratios always be less than 1?

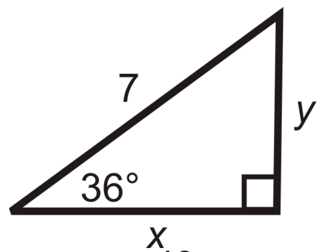
8.9 Inverse Trigonometric Ratios

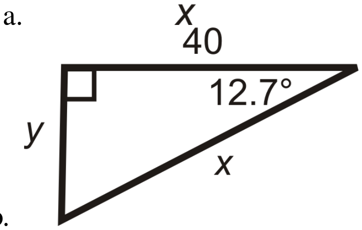
Learning Objectives

- Use the inverse trigonometric ratios to find an angle in a right triangle.
- Solve a right triangle.
- Apply inverse trigonometric ratios to real-life situation and special right triangles.

Review Queue

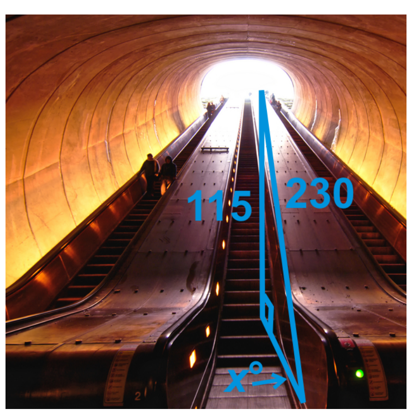
Find the lengths of the missing sides. Round your answer to the nearest hundredth.

a. 

b. 

c. Draw an isosceles right triangle with legs of length 3. What is the hypotenuse?
d. Use the triangle from #3, to find the sine, cosine, and tangent of 45°.
e. Explain why $\tan 45^\circ = 1$.

Know What? The longest escalator in North America is at the Wheaton Metro Station in Maryland. It is 230 feet long and is 115 ft high. What is the angle of elevation, x°, of this escalator?



Inverse Trigonometric Ratios

The word ***inverse*** is probably familiar to you. In mathematics, once you learn how to do an operation, you also learn how to "undo" it. For example, you may remember that addition and subtraction are considered inverse operations. Multiplication and division are also inverse operations. In algebra you used inverse operations to solve equations and inequalities.

When we apply the word inverse to the trigonometric ratios, we can find the acute angle measures within a right triangle. Normally, if you are given an angle and a side of a right triangle, you can find the other two sides, using sine, cosine or tangent. With the inverse trig ratios, you can find the angle measure, given two sides.

Inverse Tangent: If you know the opposite side and adjacent side of an angle in a right triangle, you can use inverse tangent to find the measure of the angle.

Inverse tangent is also called arctangent and is labeled $\tan^{-1}$ or *arctan*. The "-1" indicates inverse.

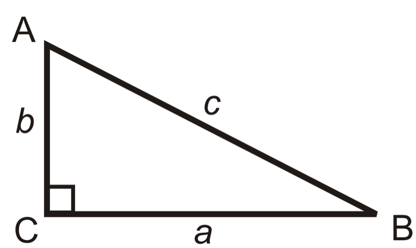
Inverse Sine: If you know the opposite side of an angle and the hypotenuse in a right triangle, you can use inverse sine to find the measure of the angle.

Inverse sine is also called arcsine and is labeled $\sin^{-1}$ or *arcsin*.

Inverse Cosine: If you know the adjacent side of an angle and the hypotenuse in a right triangle, you can use inverse cosine to find the measure of the angle.

Inverse cosine is also called arccosine and is labeled $\cos^{-1}$ or *arccos*.

Using the triangle below, the inverse trigonometric ratios look like this:



$$\tan^{-1}\left(\frac{b}{a}\right) = m\angle B \qquad \tan^{-1}\left(\frac{a}{b}\right) = m\angle A$$

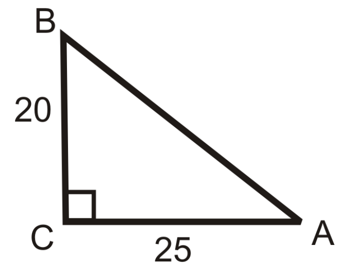
$$\sin^{-1}\left(\frac{b}{c}\right) = m\angle B \qquad \sin^{-1}\left(\frac{a}{c}\right) = m\angle A$$

$$\cos^{-1}\left(\frac{a}{c}\right) = m\angle B \qquad \cos^{-1}\left(\frac{b}{c}\right) = m\angle A$$

In order to actually find the measure of the angles, you will need you use your calculator. On most scientific and graphing calculators, the buttons look like $[\text{SIN}^{-1}], [\text{COS}^{-1}]$, and $[\text{TAN}^{-1}]$. Typically, you might have to hit a shift or 2^{nd} button to access these functions. For example, on the TI-83 and 84, $[2^{nd}][\text{SIN}]$ is $[\text{SIN}^{-1}]$. Again, make sure the mode is in degrees.

When you find the inverse of a trigonometric function, you put the word *arc* in front of it. So, the inverse of a tangent is called the arctangent (or arctan for short). Think of the arctangent as a tool you can use like any other inverse operation when solving a problem. If tangent tells you the ratio of the lengths of the sides opposite and adjacent to an angle, then tangent inverse tells you the measure of an angle with a given ratio.

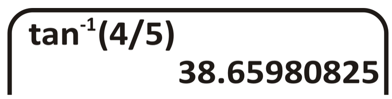
Example 1: Use the sides of the triangle and your calculator to find the value of $\angle A$. Round your answer to the nearest tenth of a degree.



Solution: In reference to $\angle A$, we are given the *opposite* leg and the *adjacent* leg. This means we should use the *tangent* ratio.

$\tan A = \frac{20}{25} = \frac{4}{5}$, therefore $\tan^{-1}\left(\frac{4}{5}\right) = m\angle A$. Use your calculator.

If you are using a TI-83 or 84, the keystrokes would be: $[2^{nd}]$[TAN] $\left(\frac{4}{5}\right)$ **[ENTER]** and the screen looks like:



So, $m\angle A = 38.7^\circ$

Example 2: $\angle A$ is an acute angle in a right triangle. Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

a) $\sin A = 0.68$

b) $\cos A = 0.85$

c) $\tan A = 0.34$

Solution:

a) $m\angle A = \sin^{-1} 0.68 = 42.8^\circ$

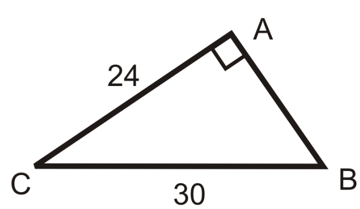
b) $m\angle A = \cos^{-1} 0.85 = 31.8^\circ$

c) $m\angle A = \tan^{-1} 0.34 = 18.8^\circ$

Solving Triangles

Now that we know how to use inverse trigonometric ratios to find the measure of the acute angles in a right triangle, we can solve right triangles. To solve a right triangle, you would need to find all sides and angles in a right triangle, using any method. When solving a right triangle, you could use sine, cosine or tangent, inverse sine, inverse cosine, or inverse tangent, or the Pythagorean Theorem. Remember when solving right triangles to only use the values that you are given.

Example 3: Solve the right triangle.



Solution: To solve this right triangle, we need to find $AB, m\angle C$ and $m\angle B$. Use AC and CB to give the most accurate answers.

$\underline{AB}$: Use the Pythagorean Theorem.

$$\begin{aligned}24^2 + AB^2 &= 30^2\\ 576 + AB^2 &= 900\\ AB^2 &= 324\\ AB &= \sqrt{324} = 18\end{aligned}$$

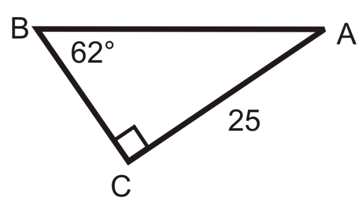
$\underline{m\angle B}$: Use the inverse sine ratio.

$$\begin{aligned}\sin B &= \frac{24}{30} = \frac{4}{5}\\ \sin^{-1}\left(\frac{4}{5}\right) &= 53.1^\circ = m\angle B\end{aligned}$$

$\underline{m\angle C}$: Use the inverse cosine ratio.

$$\begin{aligned}\cos C &= \frac{24}{30} = \frac{4}{5}\\ \cos^{-1}\left(\frac{4}{5}\right) &= 36.9^\circ = m\angle C\end{aligned}$$

Example 4: Solve the right triangle.



Solution: To solve this right triangle, we need to find AB, BC and $m\angle A$.

$\underline{AB}$: Use sine ratio.

$$\begin{aligned}\sin 62^\circ &= \frac{25}{AB}\\ AB &= \frac{25}{\sin 62^\circ}\\ AB &\approx 28.31\end{aligned}$$

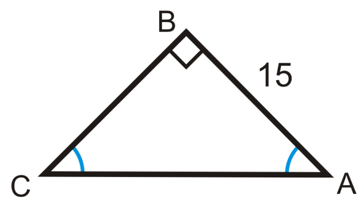
$\underline{BC}$: Use tangent ratio.

$$\begin{aligned}\tan 62^\circ &= \frac{25}{BC}\\ BC &= \frac{25}{\tan 62^\circ}\\ BC &\approx 13.30\end{aligned}$$

$m\angle A$: Use Triangle Sum Theorem

$$62^\circ + 90^\circ + m\angle A = 180^\circ$$
$$m\angle A = 28^\circ$$

Example 5: Solve the right triangle.



Solution: Even though, there are no angle measures given, we know that the two acute angles are congruent, making them both 45°. Therefore, this is a 45-45-90 triangle. You can use the trigonometric ratios or the special right triangle ratios.

Trigonometric Ratios

$$\tan 45^\circ = \frac{15}{BC} \qquad \sin 45^\circ = \frac{15}{AC}$$
$$BC = \frac{15}{\tan 45^\circ} = 15 \qquad AC = \frac{15}{\sin 45^\circ} \approx 21.21$$

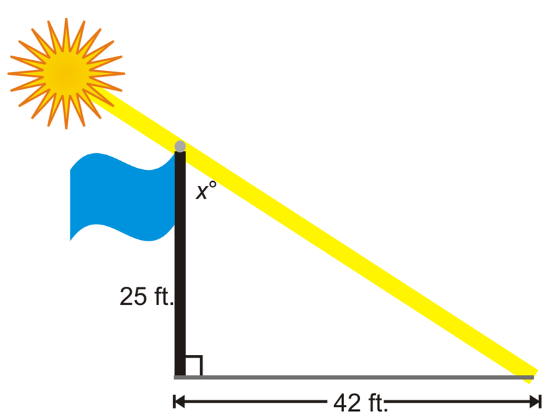
45-45-90 Triangle Ratios

$$BC = AB = 15, AC = 15\sqrt{2} \approx 21.21$$

Real-Life Situations

Much like the trigonometric ratios, the inverse trig ratios can be used in several real-life situations. Here are a couple examples.

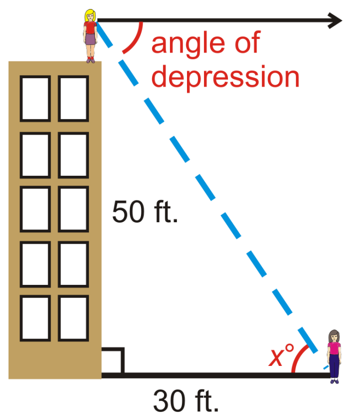
Example 6: A 25 foot tall flagpole casts a 42 feet shadow. What is the angle that the sun hits the flagpole?



Solution: First, draw a picture. The angle that the sun hits the flagpole is the acute angle at the top of the triangle, x°. From the picture, we can see that we need to use the inverse tangent ratio.

$$\tan x = \frac{42}{25}$$
$$\tan^{-1}\frac{42}{25} \approx 59.2^\circ = x$$

Example 7: Elise is standing on the top of a 50 foot building and spots her friend, Molly across the street. If Molly is 35 feet away from the base of the building, what is the angle of depression from Elise to Molly? Elise's eye height is 4.5 feet.



Solution: Because of parallel lines, the angle of depression is equal to the angle at Molly, or x°. We can use the inverse tangent ratio.

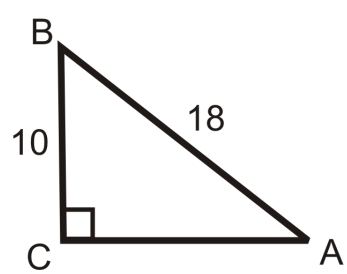
$$\tan^{-1}\left(\frac{54.5}{30}\right) = 61.2^\circ = x$$

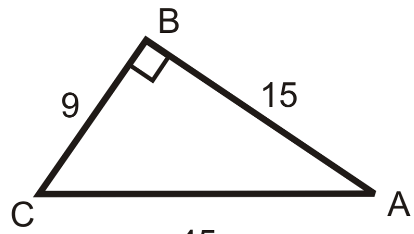
Know What? Revisited To find the escalator's angle of elevation, we need to use the inverse sine ratio.

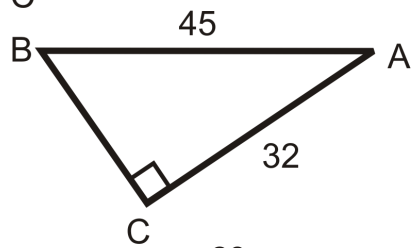
$$\sin^{-1}\left(\frac{115}{230}\right) = 30^\circ \qquad \text{The angle of elevation is } 30^\circ.$$

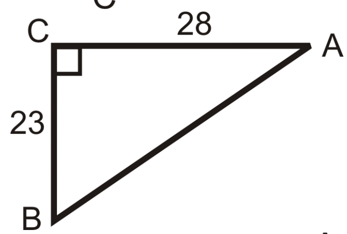
Review Questions

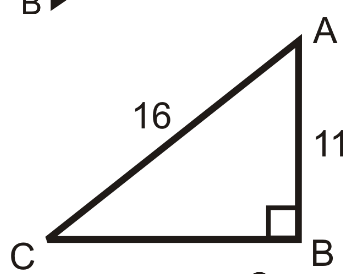
Use your calculator to find $m\angle A$ to the nearest tenth of a degree.

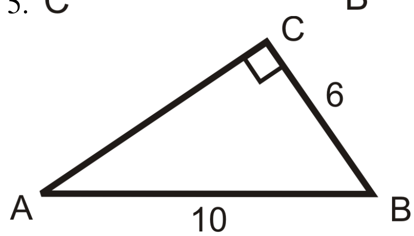
1. 

2. 

3. 

4. 

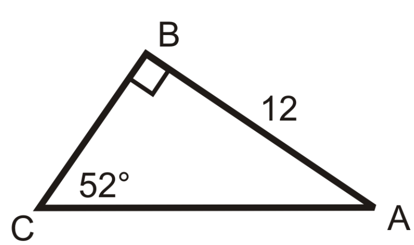
5. 

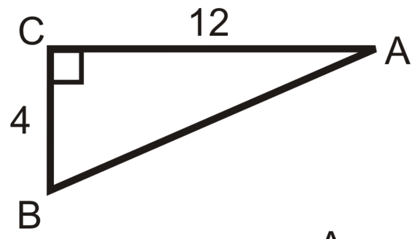
6. 

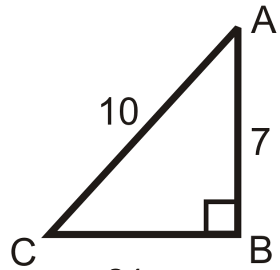
Let $\angle A$ be an acute angle in a right triangle. Find $m\angle A$ to the nearest tenth of a degree.

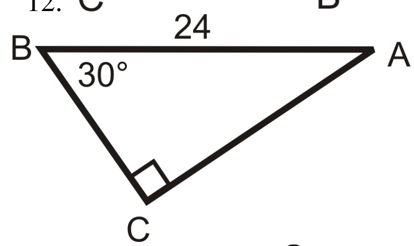
7. $\sin A = 0.5684$
8. $\cos A = 0.1234$
9. $\tan A = 2.78$

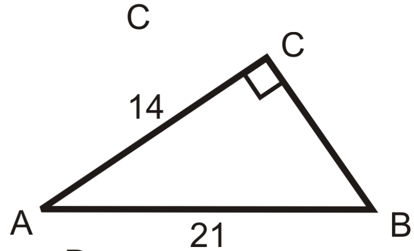
Solving the following right triangles. Find all missing sides and angles.

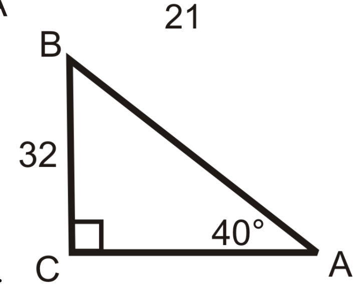
10. 

11. 

12. 

13. 

14. 

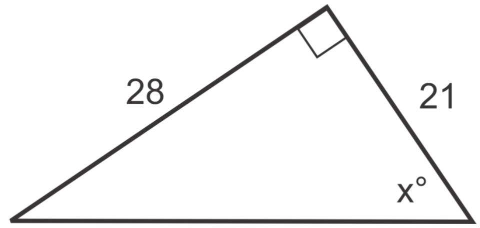
15. 

16. ***Writing*** Explain when to use a trigonometric ratio to find a side length of a right triangle and when to use the Pythagorean Theorem.

Real-Life Situations Use what you know about right triangles to solve for the missing angle. If needed, draw a picture. Round all answers to the nearest tenth of a degree.

17. A 75 foot building casts an 82 foot shadow. What is the angle that the sun hits the building?
18. Over 2 miles (horizontal), a road rises 300 feet (vertical). What is the angle of elevation?
19. A boat is sailing and spots a shipwreck 650 feet below the water. A diver jumps from the boat and swims 935 feet to reach the wreck. What is the angle of depression from the boat to the shipwreck?
20. Elizabeth wants to know the angle at which the sun hits a tree in her backyard at 3 pm. She finds that the length of the tree's shadow is 24 ft at 3 pm. At the same time of day, her shadow is 6 ft 5 inches. If Elizabeth is 4 ft 8 inches tall, find the height of the tree and hence the angle at which the sunlight hits the tree.
21. Alayna is trying to determine the angle at which to aim her sprinkler nozzle to water the top of a 5 ft bush in her yard. Assuming the water takes a straight path and the sprinkler is on the ground 4 ft from the tree, at what angle of inclination should she set it?
22. ***Science Connection*** Would the answer to number 20 be the same every day of the year? What factors would influence this answer? How about the answer to number 21? What factors might influence the path of the water?

23. Tommy was solving the triangle below and made a mistake. What did he do wrong?



$$\tan^{-1}\left(\frac{21}{28}\right) \approx 36.9^\circ$$

24. Tommy then continued the problem and set up the equation: $\cos 36.9^\circ = \frac{21}{h}$. By solving this equation he found that the hypotenuse was 26.3 units. Did he use the correct trigonometric ratio here? Is his answer correct? Why or why not?
25. How could Tommy have found the hypotenuse in the triangle another way and avoided making his mistake?

Examining Patterns Below is a table that shows the sine, cosine, and tangent values for eight different angle measures. Answer the following questions.

TABLE 8.5:

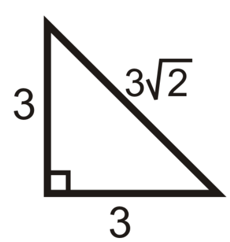
	10°	20°	30°	40°	50°	60°	70°	80°
Sine	0.1736	0.3420	0.5	0.6428	0.7660	0.8660	0.9397	0.9848
Cosine	0.9848	0.9397	0.8660	0.7660	0.6428	0.5	0.3420	0.1736
Tangent	0.1763	0.3640	0.5774	0.8391	1.1918	1.7321	2.7475	5.6713

26. What value is equal to $\sin 40^\circ$?
27. What value is equal to $\cos 70^\circ$?
28. Describe what happens to the sine values as the angle measures increase.
29. Describe what happens to the cosine values as the angle measures increase.
30. What two numbers are the sine and cosine values between?
31. Find $\tan 85^\circ, \tan 89^\circ$, and $\tan 89.5^\circ$ using your calculator. Now, describe what happens to the tangent values as the angle measures increase.
32. Explain why all of the sine and cosine values are less than one. (hint: think about the sides in the triangle and the relationships between their lengths)

Review Queue Answers

a. $\sin 36^\circ = \frac{y}{7}$ $\quad$ $\cos 36^\circ = \frac{x}{7}$
$y = 4.11$ $\quad$ $x = 5.66$

b. $\cos 12.7^\circ = \frac{40}{x}$ $\quad$ $\tan 12.7^\circ = \frac{y}{40}$
$x = 41.00$ $\quad$ $y = 9.01$

c.


d. $\sin 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\cos 45^\circ = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$
$\tan 45^\circ = \frac{3}{3} = 1$
e. The tangent of 45° equals one because it is the ratio of the opposite side over the adjacent side. In an isosceles right triangle, or 45-45-90 triangle, the opposite and adjacent sides are the same, making the ratio always 1.

8.10 Extension: Laws of Sines and Cosines

Learning Objectives

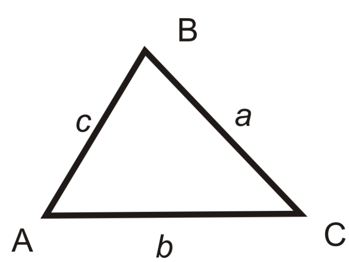
- Identify and use the Law of Sines and Cosines.

In this chapter, we have only applied the trigonometric ratios to right triangles. However, you can extend what we know about these ratios and derive the Law of Sines and the Law of Cosines. Both of these laws can be used with any type of triangle to find any angle or side within it. That means we can find the sine, cosine and tangent of angle that are greater than 90°, such as the obtuse angle in an obtuse triangle.

Law of Sines

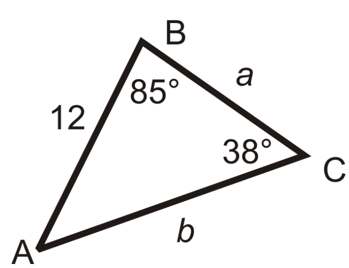
Law of Sines: If $\triangle ABC$ has sides of length, a, b, and c, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Looking at a triangle, the lengths a, b, and c are opposite the angles of the same letter. Let's use the Law of Sines on a couple of examples.



We will save the proof for a later course.

Example 1: Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.



Solution: First, to find $m\angle A$, we can use the Triangle Sum Theorem.

$$m\angle A + 85^\circ + 38^\circ = 180^\circ$$
$$m\angle A = 57^\circ$$

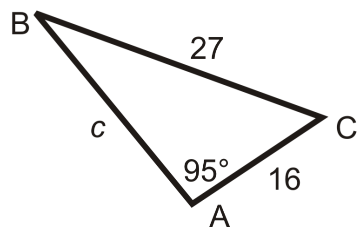
Now, use the Law of Sines to set up ratios for a and b.

$$\frac{\sin 57^\circ}{a} = \frac{\sin 85^\circ}{b} = \frac{\sin 38^\circ}{12}$$

$$\begin{aligned} \frac{\sin 57^\circ}{a} &= \frac{\sin 38^\circ}{12} \\ a \cdot \sin 38^\circ &= 12 \cdot \sin 57^\circ \\ a &= \frac{12 \cdot \sin 57^\circ}{\sin 38^\circ} \approx 16.4 \end{aligned}$$

$$\begin{aligned} \frac{\sin 85^\circ}{b} &= \frac{\sin 38^\circ}{12} \\ b \cdot \sin 38^\circ &= 12 \cdot \sin 85^\circ \\ b &= \frac{12 \cdot \sin 85^\circ}{\sin 38^\circ} \approx 19.4 \end{aligned}$$

Example 2: Solve the triangle using the Law of Sines. Round decimal answers to the nearest tenth.



Solution: Set up the ratio for $\angle B$ using Law of Sines.

$$\begin{aligned} \frac{\sin 95^\circ}{27} &= \frac{\sin B}{16} \\ 27 \cdot \sin B &= 16 \cdot \sin 95^\circ \\ \sin B &= \frac{16 \cdot \sin 95^\circ}{27} \to \sin^{-1}\left(\frac{16 \cdot \sin 95^\circ}{27}\right) = 36.2^\circ \end{aligned}$$

To find $m\angle C$ use the Triangle Sum Theorem. $m\angle C + 95^\circ + 36.2^\circ = 180^\circ \to m\angle C = 48.8^\circ$

To find c, use the Law of Sines again. $\frac{\sin 95^\circ}{27} = \frac{\sin 48.8^\circ}{c}$

$$\begin{aligned} c \cdot \sin 95^\circ &= 27 \cdot \sin 48.8^\circ \\ c &= \frac{27 \cdot \sin 48.8^\circ}{\sin 95^\circ} \approx 20.4 \end{aligned}$$

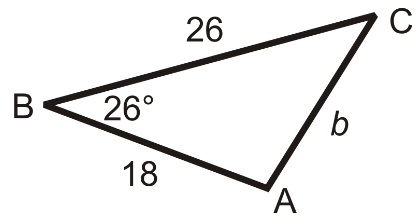
Law of Cosines

Law of Cosines: If $\triangle ABC$ has sides of length a, b, and c, then $a^2 = b^2 + c^2 - 2bc \cos A$

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\ \cos B \\ c^2 &= a^2 + b^2 - 2ab\ \cos C \end{aligned}$$

Even though there are three formulas, they are all very similar. First, notice that whatever angle is in the cosine, the opposite side is on the other side of the equal sign.

Example 3: Solve the triangle using Law of Cosines. Round your answers to the nearest hundredth.



Solution: Use the second equation to solve for $\angle B$.

$$\begin{aligned} b^2 &= 26^2 + 18^2 - 2(26)(18)\cos 26^\circ \\ b^2 &= 1000 - 936\cos 26^\circ \\ b^2 &= 158.7288 \\ b &\approx 12.60 \end{aligned}$$

To find $m\angle A$ or $m\angle C$, you can use either the Law of Sines or Law of Cosines. Let's use the Law of Sines.

$$\frac{\sin 26^\circ}{12.60} = \frac{\sin A}{18}$$

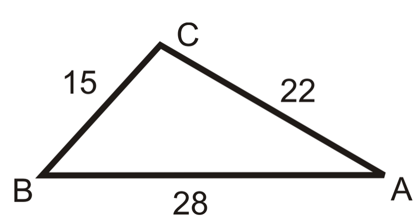
$$\begin{aligned} 12.60 \cdot \sin A &= 18 \cdot \sin 26^\circ \\ \sin A &= \frac{18 \cdot \sin 26^\circ}{12.60} \end{aligned}$$

$\sin^{-1}\left(\frac{18 \cdot \sin 26^\circ}{12.60}\right) \approx 38.77^\circ$ To find $m\angle C$, use the Triangle Sum Theorem.

$$\begin{aligned} 26^\circ + 38.77^\circ + m\angle C &= 180^\circ \\ m\angle C &= 115.23^\circ \end{aligned}$$

Unlike the previous sections in this chapter, with the Laws of Sines and Cosines, we have been using values that we have found to find other values. With these two laws, you have to use values that are not given. Just keep in mind to always wait until the very last step to put anything into your calculator. This will ensure that you have the most accurate answer.

Example 4: Solve the triangle. Round your answers to the nearest hundredth.



Solution: When you are given only the sides, you have to use the Law of Cosines to find one angle and then you can use the Law of Sines to find another.

$$
\begin{aligned}
15^2 &= 22^2 + 28^2 - 2(22)(28)\cos A \\
225 &= 1268 - 1232\cos A \\
-1043 &= -1232\cos A \\
\frac{-1043}{-1232} &= \cos A \to \cos^{-1}\left(\frac{1043}{1232}\right) \approx 32.16^\circ
\end{aligned}
$$

Now that we have an angle and its opposite side, we can use the Law of Sines.

$$\frac{\sin 32.16^\circ}{15} = \frac{\sin B}{22}$$

$$
\begin{aligned}
15 \cdot \sin B &= 22 \cdot \sin 32.16^\circ \\
\sin B &= \frac{22 \cdot \sin 32.16^\circ}{15}
\end{aligned}
$$

$\sin^{-1}\left(\frac{22 \cdot \sin 32.16^\circ}{15}\right) \approx 51.32^\circ$ To find $m\angle C$, use the Triangle Sum Theorem.

$$
\begin{aligned}
32.16^\circ + 51.32^\circ + m\angle C &= 180^\circ \\
m\angle C &= 96.52^\circ
\end{aligned}
$$

To Summarize

Use Law of Sines when given:

- An angle and its opposite side.
- Any two angles and one side.
- Two sides and the non-included angle.

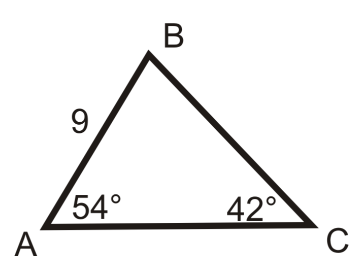
Use Law of Cosines when given:

- Two sides and the included angle.
- All three sides.

Review Questions

Use the Law of Sines or Cosines to solve $\triangle ABC$. If you are not given a picture, draw one. Round all decimal answers to the nearest tenth.

1.



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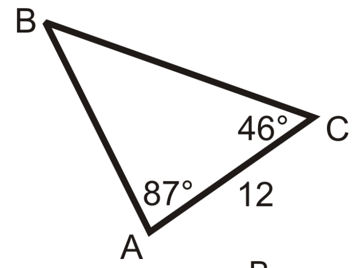

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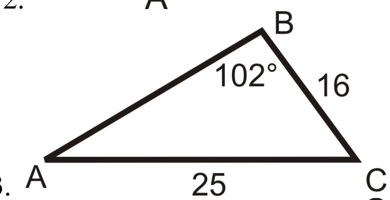

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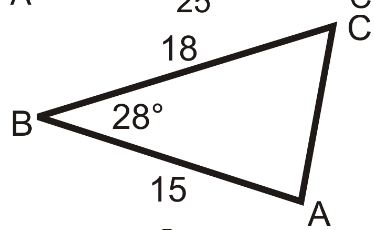

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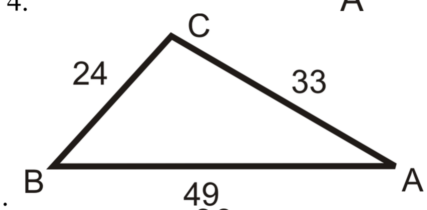

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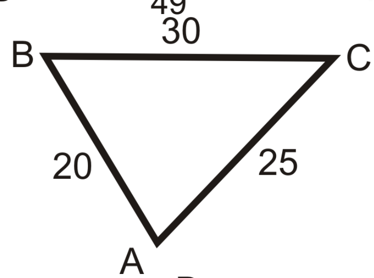

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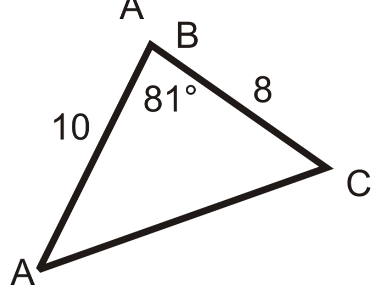

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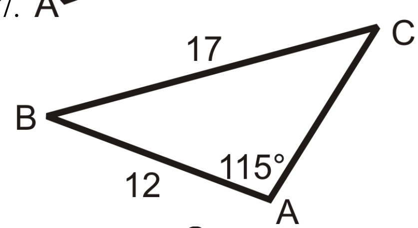

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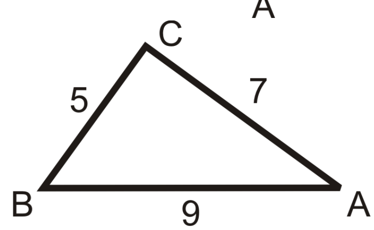


10. $m\angle A = 74^\circ, m\angle B = 11^\circ, BC = 16$

11. $m\angle A = 64^\circ, AB = 29, AC = 34$
12. $m\angle C = 133^\circ, m\angle B = 25^\circ, AB = 48$

Use the Law of Sines to solve $\triangle ABC$ below.

13. $m\angle A = 20^\circ, AB = 12, BC = 5$

Recall that when we learned how to prove that triangles were congruent we determined that SSA (two sides and an angle not included) did not determine a unique triangle. When we are using the Law of Sines to solve a triangle and we are given two sides and the angle not included, we may have two possible triangles. Problem 14 illustrates this.

14. Let's say we have $\triangle ABC$ as we did in problem 13. In problem 13 you were given two sides and the not included angle. This time, you have two angles and the side between them (ASA). Solve the triangle given that $m\angle A = 20^\circ, m\angle C = 125^\circ, AC = 8.4$
15. Does the triangle that you found in problem 14 meet the requirements of the given information in problem 13? How are the two different $m\angle C$ related? Draw the two possible triangles overlapping to visualize this relationship.

It is beyond the scope of this text to determine when there will be two possible triangles, but the concept of the possibility is something worth noting at this time.

8.11 The Law of Sines

Objective

Use the Law of Sines proportion to solve non right triangles and find the area of triangles.

Review Queue

Evaluate the following trig functions. Give exact answers.

1. $\sin 225^\circ$
2. $\csc 300^\circ$
3. $\sec \frac{5\pi}{6}$
4. $\tan \pi$

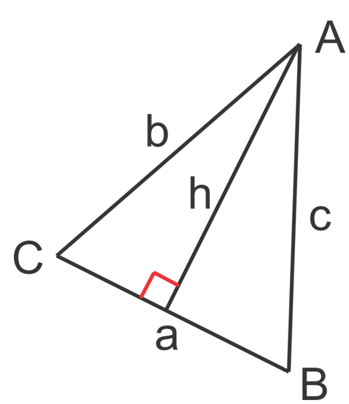
Law of Sines with AAS and ASA

Objective

Derive the Law of Sines proportion and use it to solve non right triangles in which two angles and one side are given.

Guidance

Consider the non right triangle below. We can construct an altitude from any one of the vertices to divide the triangle into two right triangles as show below.



from the diagram we can write two trigonometric functions involving h:

$$\begin{aligned} \sin C &= \frac{h}{b} \qquad \textit{and} \qquad & \sin B &= \frac{h}{c} \\ b \sin C &= h & c \sin B &= h \end{aligned}$$

Since both are equal to h, we can set them equal to each other to get:

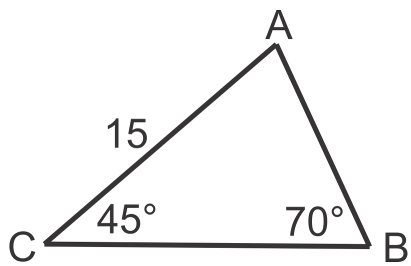
$b \sin C = c \sin B$ and finally divide both sides by bc to create the proportion:

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

If we construct the altitude from a different vertex, say B, we would get the proportion: $\frac{\sin A}{a} = \frac{\sin C}{c}$. Now, the transitive property allows us to conclude that $\frac{\sin A}{a} = \frac{\sin B}{b}$. We can put them all together as the Law of Sines: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. In the examples that follow we will use the Law of Sines to solve triangles.

Example A

Solve the triangle.



Solution: Since we are given two of the three angles in the triangle, we can find the third angle using the fact that the three angles must add up to 180°. So, $m\angle A = 180^\circ - 45^\circ - 70^\circ = 650^\circ$. Now we can substitute the known values into the Law of Sines proportion as shown:

$$\frac{\sin 65^\circ}{a} = \frac{\sin 70^\circ}{15} = \frac{\sin 45^\circ}{c}$$

Taking two ratios at a time we can solve the proportions to find a and c using cross multiplication.

To find a:

$$\frac{\sin 65^\circ}{a} = \frac{\sin 70^\circ}{15}$$
$$a = \frac{15 \sin 65^\circ}{\sin 70^\circ} \approx 14.5$$

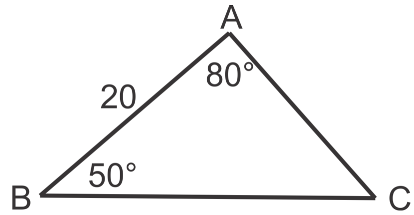
To find c:

$$\frac{\sin 70^\circ}{15} = \frac{\sin 45^\circ}{c}$$
$$c = \frac{15 \sin 45^\circ}{\sin 70^\circ} \approx 11.3$$

This particular triangle is an example in which we are given two angles and the non-included side or AAS (also SAA).

Example B

Solve the triangle.



Solution: In this example we are given two angles and a side as well but the side is between the angles. We refer to this arrangement as ASA. In practice, in doesn't really matter whether we are given AAS or ASA. We will follow the same procedure as Example A. First, find the third angle: $m\angle A = 180^\circ - 50^\circ - 80^\circ = 50^\circ$.

Second, write out the appropriate proportions to solve for the unknown sides, a and b.

To find a:

$$\frac{\sin 80^\circ}{a} = \frac{\sin 50^\circ}{20}$$
$$a = \frac{20 \sin 80^\circ}{\sin 50^\circ} \approx 25.7$$

To find b:

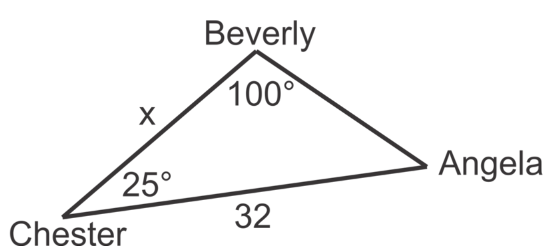
$$\frac{\sin 50^\circ}{b} = \frac{\sin 50^\circ}{20}$$
$$b = \frac{20 \sin 50^\circ}{\sin 50^\circ} = 20$$

Notice that $c = b$ and $m\angle C = m\angle B$. This illustrates a property of isosceles triangles that states that the base angles (the angles opposite the congruent sides) are also congruent.

Example C

Three fishing ships in a fleet are out on the ocean. The Chester is 32 km from the Angela. An officer on the Chester measures the angle between the Angela and the Beverly to be 25°. An officer on the Beverly measures the angle between the Angela and the Chester to be 100°. How far apart, to the nearest kilometer are the Chester and the Beverly?

Solution: First, draw a picture. Keep in mind that when we say that an officer on one of the ships is measuring an angle, the angle she is measuring is at the vertex where her ship is located.



Now that we have a picture, we can determine the angle at the Angela and then use the Law of Sines to find the distance between the Beverly and the Chester.

The angle at the Angela is $180^\circ - 100^\circ - 25^\circ = 55^\circ$.

Now find x,

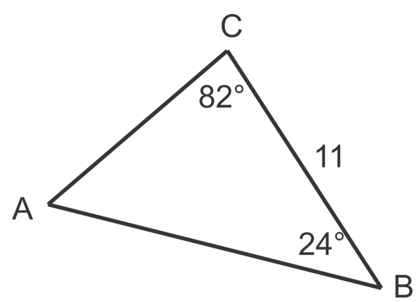
$$\frac{\sin 55^\circ}{x} = \frac{\sin 100^\circ}{32}$$
$$x = \frac{32\sin 55^\circ}{\sin 100^\circ} \approx 27$$

The Beverly and the Chester are about 27 km apart.

Guided Practice

Solve the triangles.

1.



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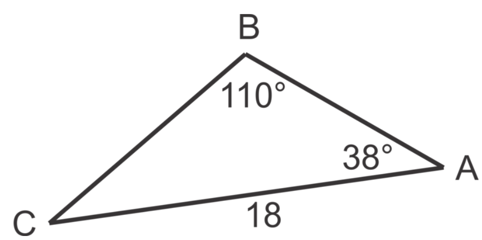


3. A surveying team is measuring the distance between point A on one side of a river and point B on the far side of the river. One surveyor is positioned at point A and the second surveyor is positioned at point C, 65 m up the riverbank from point A. The surveyor at point A measures the angle between points B and C to be 103°. The surveyor at point C measures the angle between points A and B to be 42°. Find the distance between points A and B.

Answers

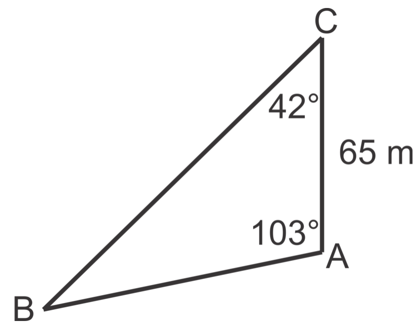
1. $m\angle A = 180^\circ - 82^\circ - 24^\circ = 74^\circ$

$$\frac{\sin 24^\circ}{b} = \frac{\sin 74^\circ}{11}, \text{ so } b = \frac{11\sin 24^\circ}{\sin 74^\circ} \approx 4.7$$
$$\frac{\sin 82^\circ}{c} = \frac{\sin 74^\circ}{11}, \text{ so } c = \frac{11\sin 82^\circ}{\sin 74^\circ} \approx 11.3$$

2. $m\angle C = 180^\circ - 110^\circ - 38^\circ = 32^\circ$

$$\frac{\sin 38^\circ}{a} = \frac{\sin 110^\circ}{18}, \text{ so } a = \frac{18\sin 38^\circ}{\sin 110^\circ} \approx 11.8$$
$$\frac{\sin 32^\circ}{c} = \frac{\sin 110^\circ}{18}, \text{ so } c = \frac{18\sin 32^\circ}{\sin 110^\circ} \approx 10.2$$

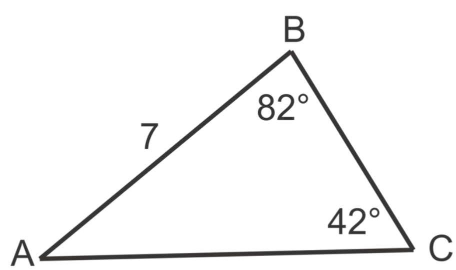
3.

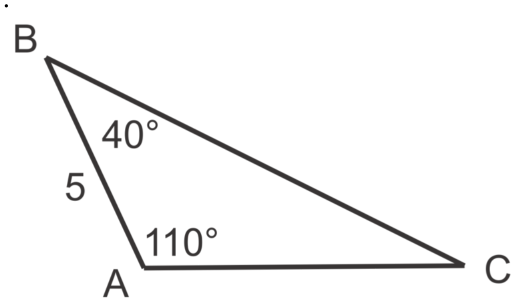


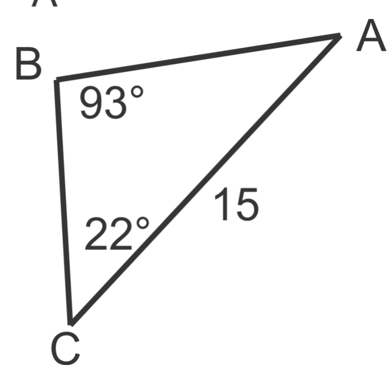
$$
\begin{aligned}
m\angle B &= 180^\circ - 103^\circ - 42^\circ = 35^\circ \\
\frac{\sin 35^\circ}{65} &= \frac{\sin 42^\circ}{c} \\
c &= \frac{65 \sin 42^\circ}{\sin 35^\circ} \approx 75.8\ m
\end{aligned}
$$

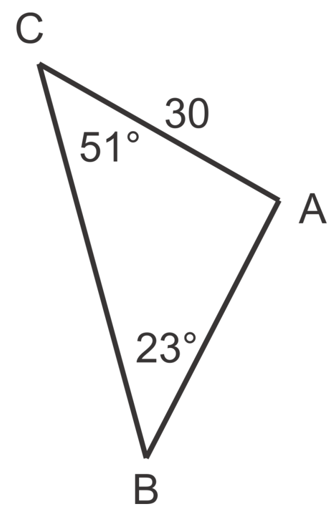
Problem Set

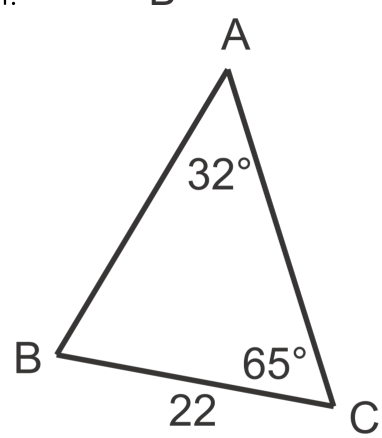
Solve the triangles. Round your answers to the nearest tenth.

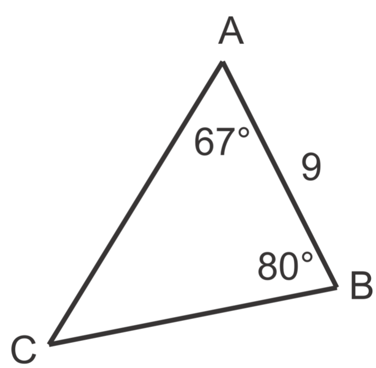
1. 

2. 

3. 

4. 

5. 

6. 

Using the given information, solve ΔABC.

7.

$$m\angle A = 85^\circ$$
$$m\angle C = 40^\circ$$
$$a = 12$$

8.

$$m\angle B = 60^\circ$$
$$m\angle C = 25^\circ$$
$$a = 28$$

9.

$$m\angle B = 42^\circ$$
$$m\angle A = 36^\circ$$
$$b = 8$$

10.

$$m\angle B = 30^\circ$$
$$m\angle A = 125^\circ$$
$$c = 45$$

Use the Law of Sines to solve the following world problems.

11. A surveyor is trying to find the distance across a ravine. He measures the angle between a spot on the far side of the ravine, X, and a spot 200 ft away on his side of the ravine, Y, to be 100°. He then walks to Y the angle between X and his previous location to be 20°. How wide is the ravine?
12. A triangular plot of land has angles 46° and 58°. The side opposite the 46° angle is 35 m long. How much fencing, to the nearest half meter, is required to enclose the entire plot of land?

The Ambiguous Case - SSA

Objective

When given two sides and the non-included angle, identify triangles in which there could be two solutions and find both if applicable.

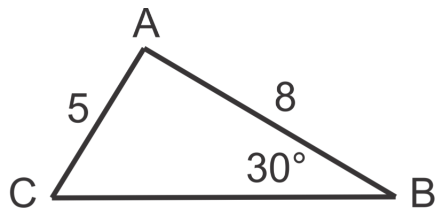
Guidance

Recall that the sine ratios for an angle and its supplement will always be equal. In other words, $\sin\theta = \sin(180-\theta)$. In Geometry you learned that two triangles could not be proven congruent using SSA and you investigated cases in which there could be two triangles. In Example A, we will explore how the Law of Sines can be used to find two possible triangles when given two side lengths of a triangle and a non-included angle.

Example A

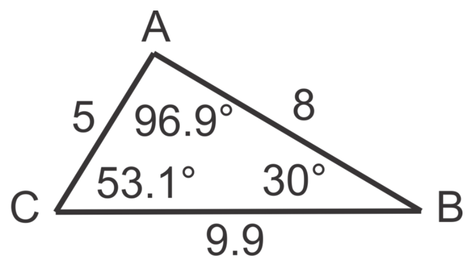
Given ΔABC with $m\angle A = 30^\circ$, $a = 5$, and $b = 8$, solve for the other angle and side measures.

Solution: First, let's make a diagram to show the relationship between the given sides and angles. Then we can set up a proportion to solve for angle C:



$$\frac{\sin 30^\circ}{5} = \frac{\sin C}{8}$$
$$\sin C = \frac{8\sin 30^\circ}{5}$$
$$C = \sin^{-1}\left(\frac{8\sin 30^\circ}{5}\right) \approx 53.1^\circ$$

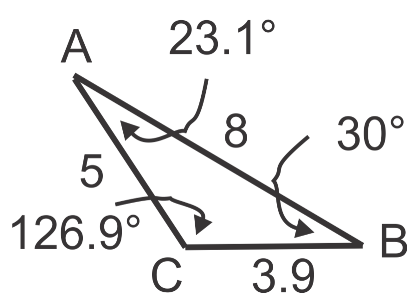
From here we can find $m\angle A = 96.9^\circ$, since the three angles must add up to 180°. We can also find the third side using another Law of Sines ratio:



$$\frac{\sin 30^\circ}{5} = \frac{\sin 96.9^\circ}{a}$$
$$a = \frac{5\sin 96.9^\circ}{\sin 30^\circ} \approx 9.9$$

Putting these measures in the triangle, we get:

But, we know that $\sin\theta = \sin(180-\theta)$ so when we solved for C we only got one of the two possible angles. The other angle will be $180^\circ - 53.1^\circ = 126.9^\circ$. Next we need to determine the measure of angle A for and the length of the third side in this second possible triangle. The sum of the three angles must still be 180°, so $m\angle A = 23.1^\circ$. Now set up a proportion to solve for the third side just as before:



$$\frac{\sin 30^\circ}{5} = \frac{\sin 23.1^\circ}{a}$$
$$a = \frac{5\sin 23.1^\circ}{\sin 30^\circ} \approx 3.9$$

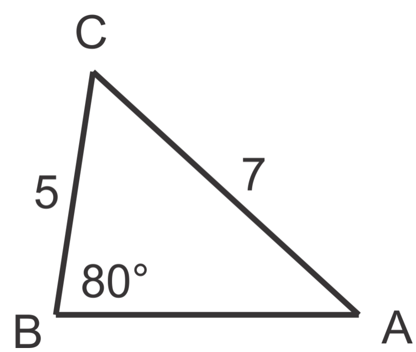
The second triangle would look like this:

In this instance there were two possible triangles.

Example B

Given ΔABC with $m\angle B = 80^\circ$, $a = 5$ and $b = 7$, solve for the other angle and side measures.

Solution: Again we will start with a diagram and use the law of sines proportion to find a second angle measure in the triangle.



$$\begin{aligned}
\frac{\sin 80^\circ}{7} &= \frac{\sin A}{5} \\
\sin A &= \frac{5\sin 80^\circ}{7} \\
A &= \sin^{-1}\left(\frac{5\sin 80^\circ}{7}\right) \approx 44.7^\circ
\end{aligned}$$

Now find the third angle, $180^\circ - 80^\circ - 44.7^\circ = 55.3^\circ$ and solve for the third side:

$$\begin{aligned}
\frac{\sin 80^\circ}{7} &= \frac{\sin 55.3^\circ}{c} \\
c &= \frac{7\sin 55.3^\circ}{\sin 80^\circ} \approx 5.8
\end{aligned}$$

Because we used the inverse sine function to determine the measure of angle A, the angle could be the supplement of 44.7° or 135.3° so we need to check for a second triangle. If we let $m\angle A = 135.3^\circ$ and then attempt to find the third angle, we will find that the sum of the two angles we have is greater than 180° and thus no triangle can be formed.

$$\begin{aligned}
m\angle A + m\angle B + m\angle C &= 180^\circ \\
135.3^\circ + 80^\circ + m\angle C &= 180^\circ \\
215.3^\circ + m\angle C &> 180^\circ
\end{aligned}$$

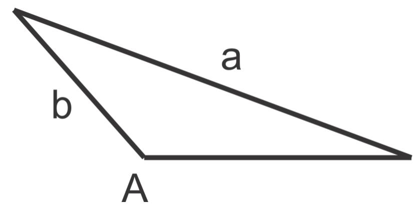
This example shows that two triangles are not always possible. Note that if the given angle is obtuse, there will only be one possible triangle for this reason.

More Guidance

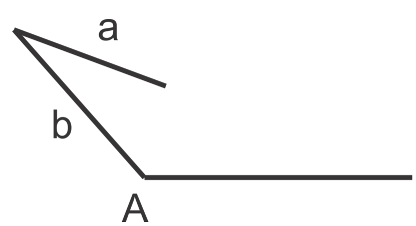
In both examples we simply tested to see if there would be a second triangle. There are, however, guidelines to follow to determine when a second triangle exists and when it does not. The "check and see" method always works and therefore it is not necessary to memorize the following table. It is interesting, however, to see to pictures and make the connection between the inequalities and what if any triangle can be formed.

First, consider when A is obtuse:

If $a > b$, then **one** triangle can be formed.

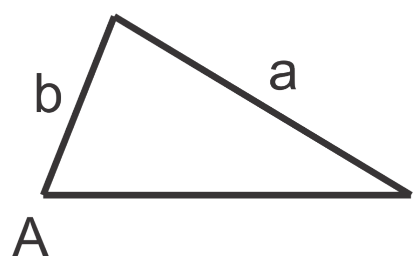


If $a \leq b$, then **no** triangle can be formed.



Now, consider the possible scenarios when A is acute.

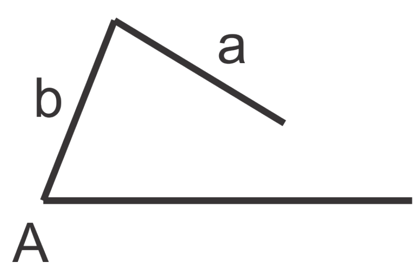
If $a > b$, the **one** triangle can be formed.



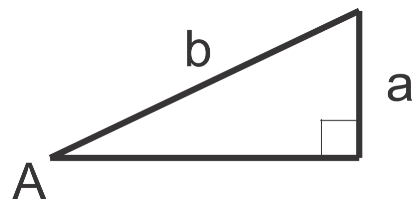
For the following cases, where $a < b$, keep in mind that we would be using the proportion:

$\frac{\sin A}{a} = \frac{\sin B}{b}$ and that $\sin B = \frac{b \sin A}{a}$

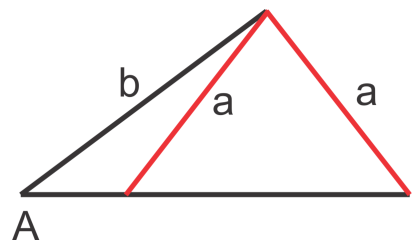
If $b \sin A > a$, **no** triangle can be formed because $B > 1$.



If $b \sin A = a$, **one right** triangle can be formed because $\sin B = 1$.



If $b \sin A < a$ (and $a < b$), **two** triangles can be formed because $\sin B < 1$.



Example C

Given ΔABC with $m\angle A = 42^\circ$, $b = 10$ and $a = 8$, use the rules to determine how many, if any, triangles can be formed and then solve the possible triangle(s).

Solution: In this case, A is acute and $a < b$, so we need to look at the value of $b \sin a$. Since $b \sin A = 10 \sin 42^\circ \approx 6.69 < a$, there will be two triangles. To solve for these triangles, use the Law of Sines extended proportion instead of making a diagram. Plugging in what we know, we have:

$$\frac{\sin 42^\circ}{8} = \frac{\sin B}{b} = \frac{\sin C}{10}$$

Take the first and last ratios to solve a proportion to find the measure of angle A.

$$\frac{\sin C}{10} = \frac{\sin 42^\circ}{8}$$
$$C = \sin^{-1}\left(\frac{10 \sin 42^\circ}{8}\right) \approx 56.8^\circ$$

So, the $m\angle C \approx 56.8^\circ$ or 123.2° and $m\angle B \approx 81.2^\circ$ or 14.8° respectively.

Solve for the measure of side b in each triangle:

$$\frac{\sin 42^\circ}{8} = \frac{\sin 81.2^\circ}{b} \qquad \textit{and} \qquad \frac{\sin 42^\circ}{8} = \frac{\sin 14.8^\circ}{b}$$
$$b = \frac{8 \sin 81.2^\circ}{\sin 42^\circ} \approx 11.8 \qquad\qquad b = \frac{8 \sin 14.8^\circ}{\sin 42^\circ} \approx 3.1$$

Putting it all together, we have:

Triangle 1: $m\angle A \approx 42^\circ, m\angle B \approx 81.2^\circ, m\angle C = 56.8^\circ, a = 8, b \approx 11.8, c = 10$

Triangle 2: $m\angle A \approx 42^\circ, m\angle B \approx 14.8^\circ, m\angle C = 123.2^\circ, a = 8, b \approx 3.1, c = 10$

Guided Practice

1. Use the given side lengths and angle measure to determine whether zero, one or two triangles exists.

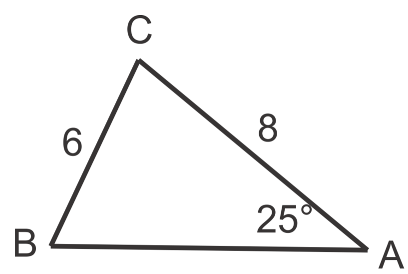
a. $m\angle A = 100^\circ, a = 3, b = 4$.

b. $m\angle A = 50^\circ, a = 8, b = 10$.

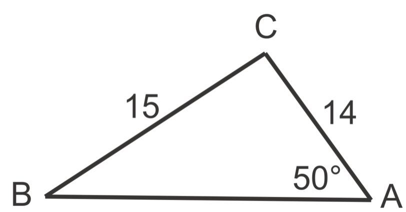
c. $m\angle A = 72^\circ, a = 7, b = 6$.

2. Solve the following triangles.

a.



b.



3. Given $m\angle A = 30^\circ$, $a = 80$ and $b = 150$, find $m\angle C$.

Answers

1. a. Since A is obtuse and $a \le b$, no triangle can be formed.

b. Since A is acute, $a < b$ and $b \sin A < a$, two triangles can be formed.

c. Since A is acute and $a > b$, there is one possible triangle.

2. a. There will be two triangles in this case because A is acute, $a < b$ and $b \sin A < a$.

Using the extended proportion: $\frac{\sin 25^\circ}{6} = \frac{\sin B}{8} = \frac{\sin C}{c}$, we get:

$$\begin{aligned} m\angle B &\approx 34.3^\circ & \quad or \quad & m\angle B \approx 145.7^\circ \\ m\angle C &\approx 120.7^\circ & & m\angle C \approx 9.3^\circ \\ c &\approx 12.2 & & c \approx 2.3 \end{aligned}$$

b. Since A is acute and $a > b$, there is one possible triangle.

Using the extended proportion: $\frac{\sin 50^\circ}{15} = \frac{\sin B}{14} = \frac{\sin C}{c}$, we get:

$$\begin{aligned} m\angle B &\approx 45.6^\circ \\ m\angle C &\approx 84.4^\circ \\ c &\approx 19.5 \end{aligned}$$

3. In this instance A is acute, $a < b$ and $b \sin A < a$ so two triangles can be formed. So, once we find the two possible measures of angle B, we will find the two possible measures of angle C. First find $m\angle B$:

$$\begin{aligned}\frac{\sin 30^\circ}{80} &= \frac{\sin B}{150}\\ \sin B &= \frac{150\sin 30^\circ}{80}\\ B &\approx 69.6^\circ, 110.4^\circ\end{aligned}$$

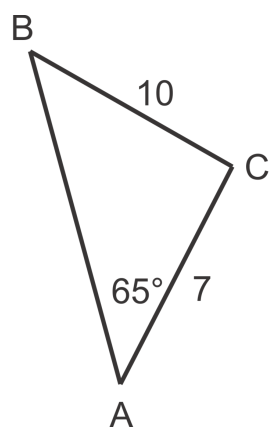
Now that we have B, use the triangle sum to find $m\angle C \approx 80.4^\circ, 39.9^\circ$.

Problem Set

For problems 1-5, use the rules to determine if there will be one, two or no possible triangle with the given measurements.

1. $m\angle A = 65^\circ, a = 10, b = 11$
2. $m\angle A = 25^\circ, a = 8, b = 15$
3. $m\angle A = 100^\circ, a = 6, b = 4$
4. $m\angle A = 75^\circ, a = 25, b = 30$
5. $m\angle A = 48^\circ, a = 41, b = 50$

Solve the following triangles, if possible. If there is a second possible triangle, solve it as well.



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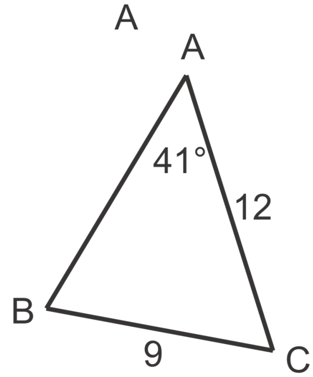

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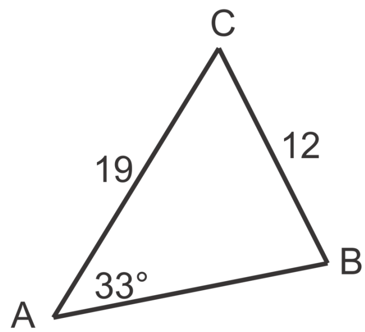

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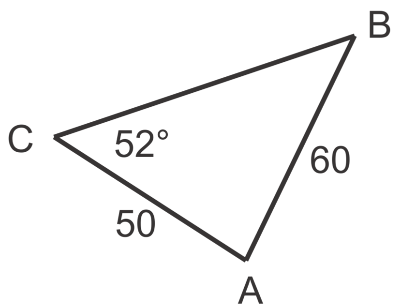

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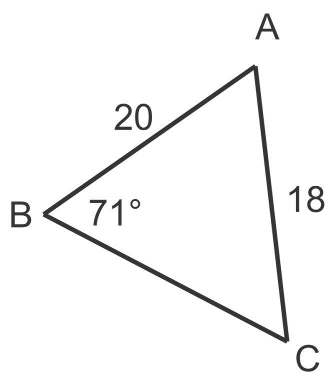

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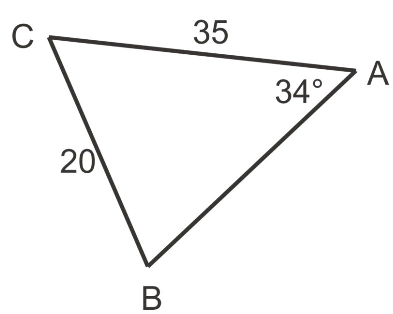


11.

Area of a Triangle

Objective

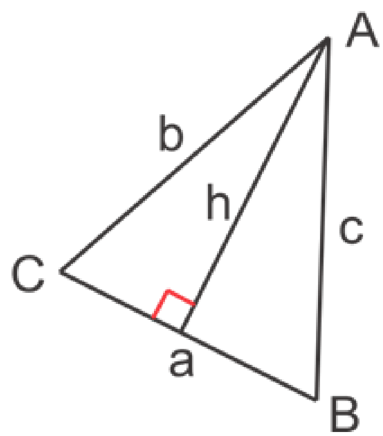
Use the sine ratio to find the area of non-right triangles in which two sides and the included angle measure are known.

Guidance

Recall the non right triangle for which we derived the law of sine.

We are most familiar with the area formula: $A = \frac{1}{2}bh$ where the base, b, is the side length which is perpendicular to the altitude. If we consider angle C in the diagram, we can write the following trigonometric expression for the altitude of the triangle, h:

$$\sin C = \frac{h}{b}$$
$$b \sin C = h$$



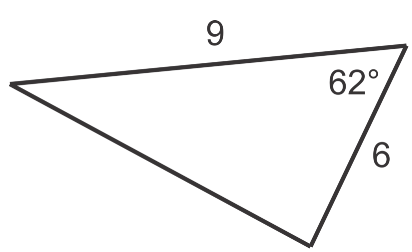
No we can replace h in the formula with $b \sin C$ and the side perpendicular to h is the base, a. Our new area formula is thus:

$$A = \frac{1}{2}ab \ \sin C.$$

It is important to note that C is the angle between sides a and b and that any two sides and the included angle can be used in the formula.

Example A

Find the area of the triangle.

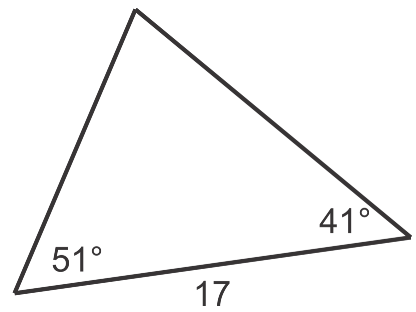


Solution: We are given two sides and the included angle so let $a = 6$, $b = 9$ and $C = 62^\circ$. Now we can use the formula to find the area of the triangle:

$$A = \frac{1}{2}(6)(9)\sin(62^\circ) \approx 23.8 \text{ } \textit{square units}$$

Example B

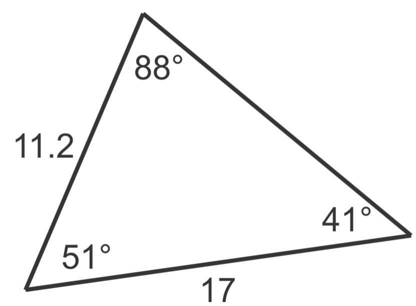
Find the area of the triangle.



Solution: In this triangle we do not have two sides and the included angle. We must first find another side length using the Law of Sines. We can find the third angle using the triangle sum: $180^\circ - 51^\circ - 41^\circ = 88^\circ$. Use the Law of Sines to find the side length opposite 41°:

$$\frac{\sin 88^\circ}{17} = \frac{\sin 41^\circ}{x}$$
$$x = \frac{17 \sin 41^\circ}{\sin 88^\circ} \approx 11.2$$

Put these measures in the triangle:



We now have two sides and the included angle and can use the area formula:

$$A = \frac{1}{2}(11.2)(17)\sin(51^\circ) \approx 74 \textit{ square units}$$

Example C

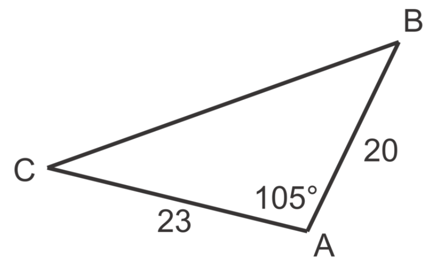
Given $c = 25\ cm$, $a = 31\ cm$ and $B = 78^\circ$, find the area of ΔABC.

Solution: Here we are given two sides and the included angle. We can adjust the formula to represent the sides and angle we are given: $A = \frac{1}{2}ac\ \sin B$. It really doesn't matter which "letters" are in the formula as long as they represent **two sides and the included angle** (the angle *between* the two sides.) Now put in our values to find the area: $A = \frac{1}{2}(31)(25)\sin(78^\circ) \approx 379\ cm^2$.

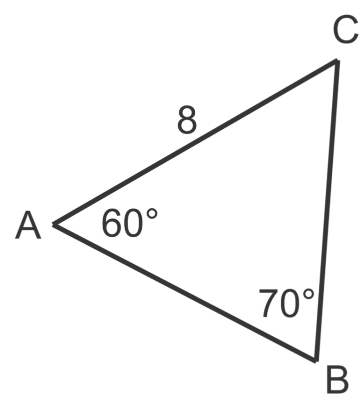
Guided Practice

Find the area of each of the triangles below. Round answers to the nearest square unit.

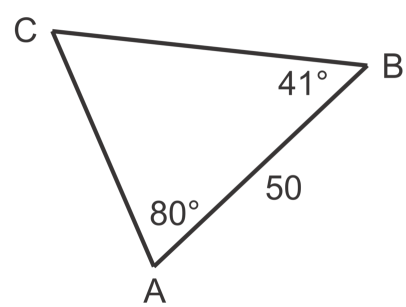
1.



2.



3.



Answers

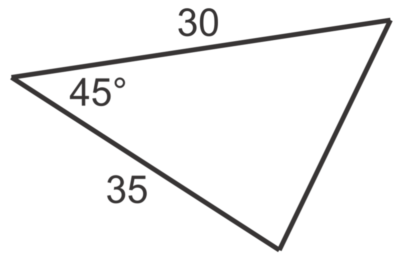
1. Two sides and the included angle are given so $A = \frac{1}{2}(20)(23)\sin 105^\circ \approx 222$ *sq units*.

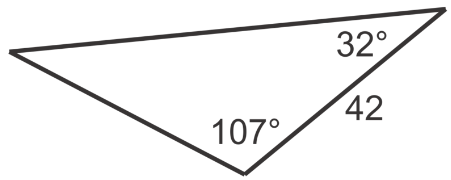
2. Find side a first: $\frac{\sin 70^\circ}{8} = \frac{\sin 60^\circ}{a}$, so $a = \frac{8 \sin 60^\circ}{\sin 70^\circ} \approx 7.4$. Next find $m\angle C = 180^\circ - 60^\circ - 70^\circ = 50^\circ$. Using the area formula, $A = \frac{1}{2}(7.4)(8)\sin 50^\circ \approx 22.7$ *sq units*.

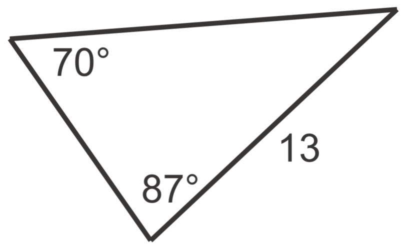
3. Find $m\angle C = 180^\circ - 80^\circ - 41^\circ = 59^\circ$. Find a second side: $\frac{\sin 59^\circ}{50} = \frac{\sin 80^\circ}{a}$, so $a = \frac{50 \sin 80^\circ}{\sin 59^\circ} \approx 57.4$. Using the area formula, $A = \frac{1}{2}(57.4)(50)\sin 41^\circ \approx 941$ *sq units*.

Problem Set

Find the area of each of the triangles below. Round your answers to the nearest square unit.

1. 

2. 

3. 

4. $m\angle A = 71^\circ, b = 15, c = 19$
5. $m\angle C = 120^\circ, b = 22, a = 16$
6. $m\angle B = 60^\circ, a = 18, c = 12$
7. $m\angle A = 28^\circ, m\angle C = 73^\circ, b = 45$
8. $m\angle B = 56^\circ, m\angle C = 81^\circ, c = 33$
9. $m\angle A = 100^\circ, m\angle B = 30^\circ, a = 100$
10. The area of ΔABC is 66 square units. If two sides of the triangle are 11 and 21 units, what is the measure of the included angle? Is there more than one possible value? Explain.
11. A triangular garden is bounded on one side by a 20 ft long barn and a second side is bounded by a 25 ft long fence. If the barn and the fence meet at a 50° angle, what is the area of the garden if the third side is the length of the segment between the ends of the fence and the barn?
12. A contractor is constructing a counter top in the shape of an equilateral triangle with side lengths 3 ft. If the countertop material costs $25 per square foot, how much will the countertop cost?

8.12 The Law of Cosines

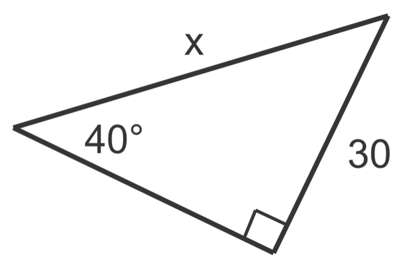
Objective

Use the Law of Cosines equation to solve non right triangles and find the area of triangles using Heron's Formula.

Review Queue

Find the value of x in the following triangles.

1.



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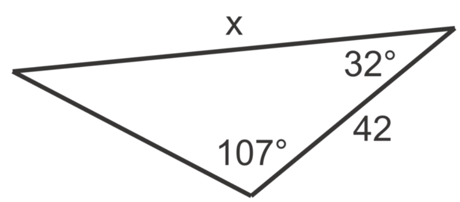

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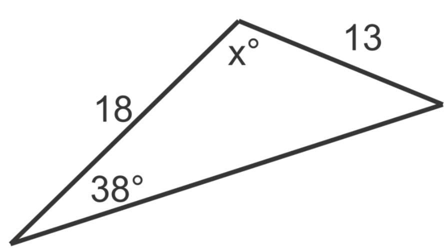


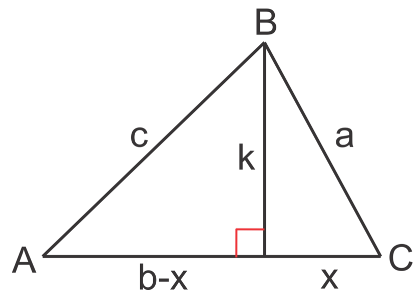
Using the Law of Cosines with SAS (to find the third side)

Objective

Use the Law of Cosines to determine the length of the third side of a triangle when two sides and the included angle are known.

Guidance

The Law of Cosines can be used to solve for the third side of a triangle when two sides and the included angle are known in a triangle. consider the non right triangle below in which we know a, b and C. We can draw an altitude from B to create two smaller right triangles as shown where x represents the length of the segment from C to the foot of the altitude and $b-x$ represents the length of remainder of the side opposite angle B.



Now we can use the Pythagorean Theorem to relate the lengths of the segments in each of the right triangles shown.

Triangle 1: $x^2+k^2=a^2$ or $k^2=a^2-x^2$

Triangle 2: $(b-x)^2+k^2=c^2$ or $k^2=c^2-(b-x)^2$

Since both equations are equal to k^2, we can set them equal to each other and simplify:

$$\begin{aligned}a^2-x^2&=c^2-(b-x)^2\\a^2-x^2&=c^2-(b^2-2bx+x^2)\\a^2-x^2&=c^2-b^2+2bx-x^2\\a^2&=c^2-b^2+2bx\\a^2+b^2-2bx&=c^2\end{aligned}$$

Recall that we know the values of a and b and the measure of angle C. We don't know the measure of x. We can use the cosine ratio as show below to find an expression for x in terms of what we already know.

$$\cos C=\frac{x}{a}\quad\text{so}\quad x=a\cos C$$

Finally, we can replace x in the equation to get the Law of Cosines: $a^2+b^2-2ab\cos C=c^2$

Keep in mind that a and b are the sides of angle C in the formula.

Example A

Find c when $m\angle C=80^\circ, a=6$ and $b=12$.

Solution: Replacing the variables in the formula with the given information and solve for c:

$$\begin{aligned}c^2&=6^2+12^2-2(6)(12)\cos 80^\circ\\c^2&\approx 154.995\\c&\approx 12.4\end{aligned}$$

Example B

Find a, when $m\angle A=43^\circ$, $b=16$ and $c=22$.

Solution: This time we are given the sides surrounding angle A and the measure of angle A. We can rewrite the formula as: $a^2=c^2+b^2-2cb\cos A$. Just remember that the length by itself on one side should be the side opposite the angle in the cosine ratio. Now we can plug in our values and solve for a.

$$a^2 = 16^2 + 22^2 - 2(16)(22)\cos 43^\circ$$
$$a^2 \approx 225.127$$
$$a \approx 15$$

Example C

Rae is making a triangular flower garden. One side is bounded by her porch and a second side is bounded by her fence. She plans to put in a stone border on the third side. If the length of the porch is 10 ft and the length of the fence is 15 ft and they meet at a 100° angle, how many feet of stone border does she need to create?

Solution: Let the two known side lengths be a and b and the angle between is C. Now we can use the formula to find c, the length of the third side.

$$c^2 = 10^2 + 15^2 - 2(10)(15)\cos 100^\circ$$
$$c^2 \approx 377.094$$
$$c \approx 19.4$$

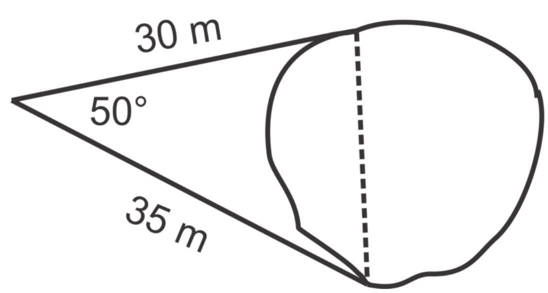
So Rae will need to create a 19.4 ft stone border.

Guided Practice

1. Find c when $m\angle C = 75^\circ, a = 32$ and $b = 40$.

2. Find b when $m\angle B = 120^\circ, a = 11$ and $c = 17$.

3. Dan likes to swim laps across a small lake near his home. He swims from a pier on the north side to a pier on the south side multiple times for a workout. One day he decided to determine the length of his swim. He determines the distances from each of the piers to a point on land and the angles between the piers from that point to be 50°. How many laps does Dan need to swim to cover 1000 meters?



Answers

1.

$$c^2 = 32^2 + 40^2 - 2(32)(40)\cos 75^\circ$$
$$c^2 \approx 1961.42$$
$$c \approx 44.3$$

2.

$$b^2 = 11^2 + 17^2 - 2(11)(17)\cos 120^\circ$$
$$b^2 \approx 597$$
$$b \approx 24.4$$

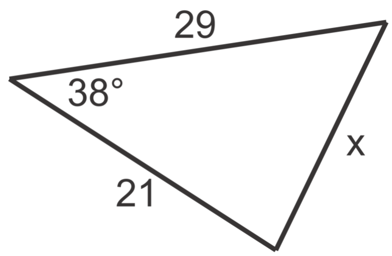
3.

$$\begin{aligned} c^2 &= 30^2 + 35^2 - 2(30)(35)\cos 50^\circ \\ c^2 &\approx 775.146 \\ c &\approx 27.84 \end{aligned}$$

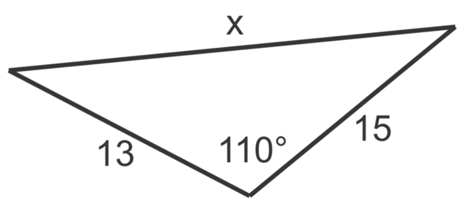
Since each lap is 27.84 meters, Dan must swim $\frac{1000}{27.84} \approx 36$ laps.

Problem Set

Use the Law of Cosines to find the value of x, to the nearest tenth, in problems 1 through 6.



1.



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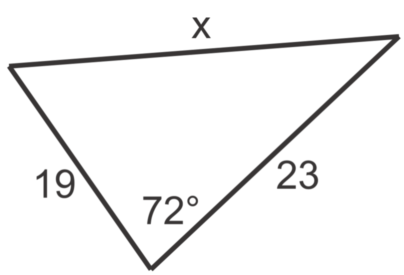

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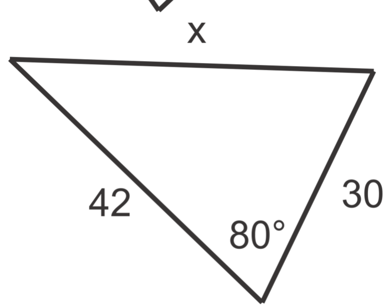

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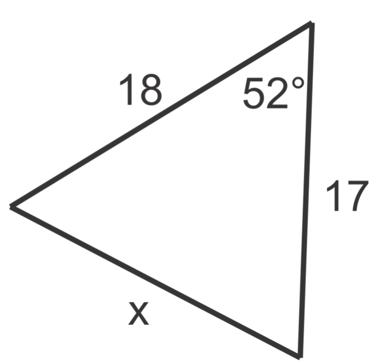

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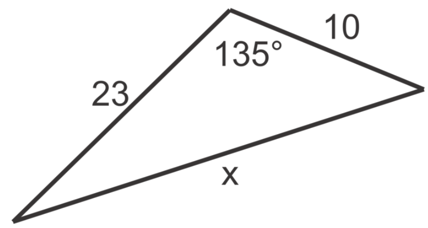


6.

For problems 7 through 10, find the unknown side of the triangle. Round your answers to the nearest tenth.

7. Find c, given $m\angle C = 105^\circ$, $a = 55$ and $b = 61$.
8. Find b, given $m\angle B = 26^\circ$, $a = 33$ and $c = 24$.
9. Find a, given $m\angle A = 77^\circ$, $b = 12$ and $c = 19$.
10. Find b, given $m\angle B = 95^\circ$, $a = 28$ and $c = 13$.
11. Explain why when $m\angle C = 90^\circ$, the Law of Cosines becomes the Pythagorean Theorem.
12. Luis is designing a triangular patio in his backyard. One side, 20 ft long, will be up against the side of his house. A second side is bordered by his wooden fence. If the fence and the house meet at a 120° angle and the fence is 15 ft long, how long is the third side of the patio?

Using the Law of Cosines with SSS (to find an angle)

Objective

Use the Law of Cosines to find the measure of an angle in a triangle in which all three side lengths are known.

Guidance

The Law of Cosines, $a^2 + b^2 - 2ab\cos C$, can be rearranged to facilitate the calculation of the measure of angle C when a, b and c are all known lengths.

$$\begin{aligned} a^2 + b^2 - 2ab\cos C &= c^2 \\ a^2 + b^2 - c^2 &= 2ab\cos C \\ \frac{a^2 + b^2 - c^2}{2ab} &= \cos C \end{aligned}$$

which can be further manipulated to $C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$.

Example A

Find the measure of the largest angle in the triangle with side lengths 12, 18 and 21.

Solution: First, we must determine which angle will be the largest. Recall from Geometry that the longest side is opposite the largest angle. The longest side is 21 so we will let $c = 21$ since C is the angle we are trying to find. Let $a = 12$ and $b = 18$ and use the formula to solve for C as shown. It doesn't matter which sides we assign to a and b. They are interchangeable in the formula.

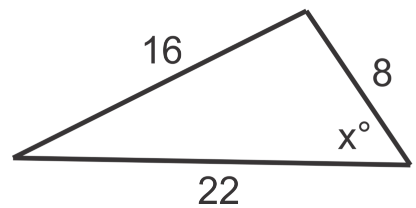
$$m\angle C = \cos^{-1}\left(\frac{12^2 + 18^2 - 21^2}{2(12)(18)}\right) \approx 86^\circ$$

Note: Be careful to put parenthesis around the entire numerator and entire denominator on the calculator to ensure the proper order of operations. Your calculator screen should look like this:

$$\cos^{-1}((12^2+18^2-21^2)/(2(12)(18)))$$

Example B

Find the value of x, to the nearest degree.



Solution: The angle with measure x° will be angle C so $c = 16, a = 22$ and $b = 8$. Remember, a and b are interchangeable in the formula. Now we can replace the variables with the known measures and solve.

$$\cos^{-1}\left(\frac{22^2+8^2-16^2}{2(22)(8)}\right) \approx 34^\circ$$

Example C

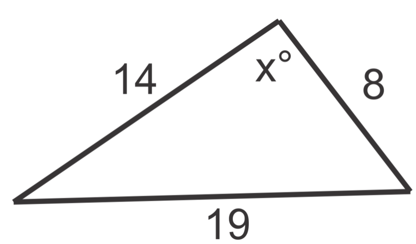
Find the $m\angle A$, if $a = 10, b = 15$ and $c = 21$.

Solution: First, let's rearrange the formula to reflect the sides given and requested angle:

$\cos A = \left(\frac{b^2+c^2-a^2}{2(b)(c)}\right)$, now plug in our values $m\angle A = \cos^{-1}\left(\frac{15^2+21^2-10^2}{2(15)(21)}\right) \approx 26^\circ$

Guided Practice

1. Find the measure of x in the diagram:



2. Find the measure of the smallest angle in the triangle with side lengths 47, 54 and 72.

3. Find $m\angle B$, if $a = 68, b = 56$ and $c = 25$.

Answers

1. $\cos^{-1}\left(\frac{14^2+8^2-19^2}{2(14)(8)}\right) \approx 117^\circ$

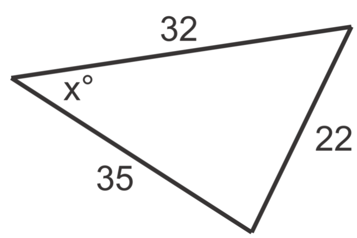
2. The smallest angle will be opposite the side with length 47, so this will be our c in the equation.

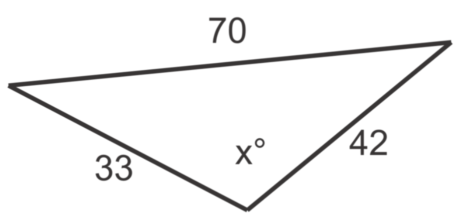
$$\cos^{-1}\left(\frac{54^2+72^2-47^2}{2(54)(72)}\right) \approx 41^\circ$$

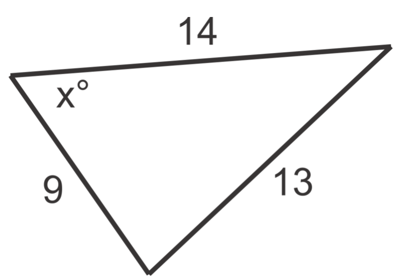
3. Rearrange the formula to solve for $m\angle B, \cos B = \left(\frac{a^2+c^2-b^2}{2(a)(c)}\right); \cos^{-1}\left(\frac{68^2+25^2-56^2}{2(68)(25)}\right) \approx 52^\circ$

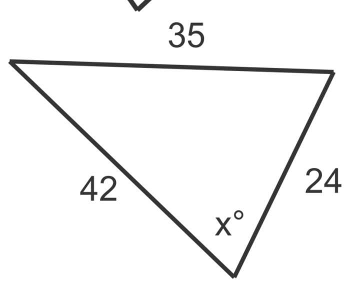
Problem Set

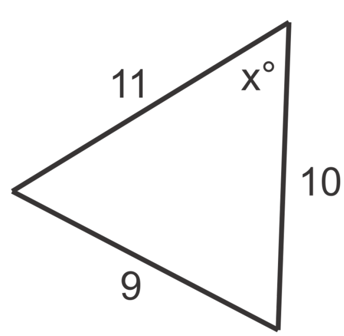
Use the Law of Cosines to find the value of x, to the nearest degree, in problems 1 through 6.

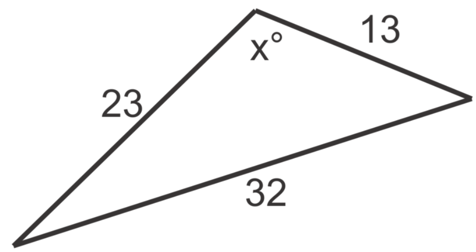
1. 

2. 

3. 

4. 

5. 



6.
7. Find the measure of the smallest angle in the triangle with side lengths 150, 165 and 200 meters.
8. Find the measure of the largest angle in the triangle with side length 59, 83 and 100 yards.
9. Find the $m\angle C$ if $a = 6, b = 9$ and $c = 13$.
10. Find the $m\angle B$ if $a = 15, b = 8$ and $c = 9$.
11. Find the $m\angle A$ if $a = 24, b = 20$ and $c = 14$.
12. A triangular plot of land is bordered by a road, a fence and a creek. If the stretch along the road is 100 meters, the length of the fence is 115 meters and the side along the creek is 90 meters, at what angle do the fence and road meet?

Heron's Formula for the Area of a Triangle and Problem Solving with Trigonometry

Objective

Use Heron's formula for area of a triangle when the side lengths are known and solve real world application problems using Law of Sines, Law of Cosines or the area formulas.

Guidance

Heron's Formula, named after Hero of Alexandria 2000 years ago, can be used to find the area of a triangle given the three side lengths. The formula requires the semi-perimeter, s, or $\frac{1}{2}(a+b+c)$, where a, b and c are the lengths of the sides of the triangle.

Heron's Formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Example A

Use Heron's formula to find the area of a triangle with side lengths 13 cm, 16 cm and 23 cm.

Solution: First, find the semi-perimeter or s: $s = \frac{1}{2}(13+16+23) = 26$. Next, substitute our values into the formula as shown and evaluate:

$$A = \sqrt{26(26-13)(26-16)(26-23)} = \sqrt{26(13)(10)(3)} = \sqrt{10140} \approx 101\ cm^2$$

Example B

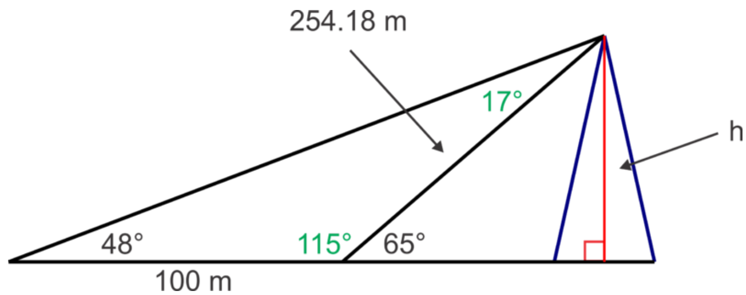
Alena is planning a garden in her yard. She is using three pieces of wood as a border. If the pieces of wood have lengths 4 ft, 6ft and 3 ft, what is the area of her garden?

Solution: The garden will be triangular with side lengths 4 ft, 6 ft and 3 ft. Find the semi-perimeter and then use Heron's formula to find the area.

$$s = \frac{1}{2}(4+6+3) = \frac{13}{2}$$
$$A = \sqrt{\frac{13}{2}\left(\frac{13}{2}-4\right)\left(\frac{13}{2}-6\right)\left(\frac{13}{2}-3\right)} = \sqrt{\frac{13}{2}\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)\left(\frac{7}{2}\right)} = \sqrt{\frac{455}{16}} \approx 28\ ft^2$$

Example C

Caroline wants to measure the height of a radio tower. From some distance away from the tower, the angle of elevation from her spot to the top of the tower is 65°. Caroline walks 100 m further away from the tower and measures the angle of elevation to the top of tower to be 48°. How tall is the tower?



Solution: First, make a diagram to illustrate the situation.

We can use angle properties (linear pair and triangle sum) to find the angles shown in green in the diagram.

$180^\circ - 65^\circ = 115^\circ$ and $180^\circ - 48^\circ - 115^\circ = 17^\circ$

Next, we can use the Law of Sines in the obtuse triangle to find the hypotenuse in the right triangle:

$$\frac{\sin 17^\circ}{100} = \frac{\sin 48^\circ}{x}$$
$$x = \frac{100 \sin 48^\circ}{\sin 17^\circ} \approx 254.18$$

Finally we can use the sine ratio in the right triangle to find the height of the tower:

$\sin 65^\circ = \frac{h}{254.18}, h = 254.18 \sin 65^\circ \approx 230.37\ m$

Guided Practice

Use the most appropriate rule or formula (Law of Sines, Law of Cosines, area formula with sine or Heron's formula) to answer the following questions.

1. Find the area of a triangle with side lengths 50 m, 45 m and 25 m.

2. Matthew is planning to fertilize his grass. Each bag of fertilizer claims to cover 500 sq ft of grass. His property of land is approximately in the shape of a triangle. He measures two sides of his yard to be 75 ft and 100 ft and the angle between them is 72°. How many bags of fertilizer must he buy?

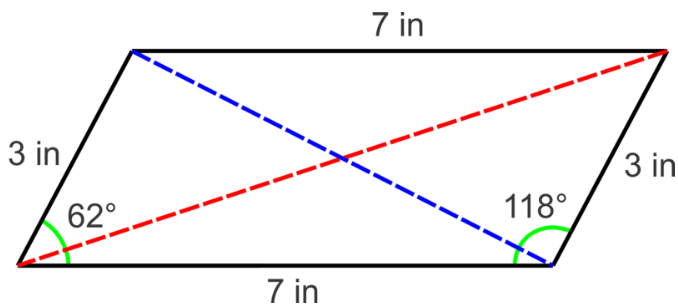
3. A pair of adjacent sides in a parallelogram are 3 in and 7 in and the angle between them is 62°, find the length of the diagonals.

Answers

1. Heron's Formula: $s = \frac{1}{2}(50+45+25) = 60, A = \sqrt{60(60-50)(60-45)(60-25)} \approx 561\ m^2$.

2. Area formula with sine: $\frac{1}{2}(75)(100)\sin 72^\circ \approx 3566\ ft^2$, Number of bags $\frac{3566}{500} \approx 7.132 \approx 8$ bags. We round up because 7 bags is not quite enough.

3.



Law of Cosines to find the blue diagonal:

$$c^2 = 3^2 + 7^2 - 2(3)(7)\cos 62^\circ$$
$$c = \sqrt{38.28} \approx 6.19$$

So, 6.19 in

To find the green diagonal we can use the Law of Cosines with the adjacent angle: $180^\circ - 62^\circ - 118^\circ$:

$$c^2 = 7^2 + 3^2 - 2(7)(3)\cos 118^\circ$$
$$c = \sqrt{77.72} \approx 8.82$$

So, 8.82 in

Problem Set

Use the Law of Sines, Law of Cosine, area of triangle with sine or Heron's Formula to solve the real world application problems.

1. Two observers, Rachel and Luis, are standing on the shore, 0.5 miles apart. They each measure the angle between the shoreline and a sailboat out on the water at the same time. If Rachel's angle is 63° and Luis' angle is 56°, find the distance between Luis and the sailboat to the nearest hundredth of a mile.
2. Two pedestrians walk from opposite ends of a city block to a point on the other side of the street. The angle formed by their paths is 125°. One pedestrian walks 300 ft and the other walks 320 ft. How long is the city block to the nearest foot?
3. Two sides and the included angle of a parallelogram have measures 3.2 cm, 4.8 cm and 54.3° respectively. Find the lengths of the diagonals to the nearest tenth of a centimeter.
4. A bridge is supported by triangular braces. If the sides of each brace have lengths 63 ft, 46 ft and 40 ft, find the measure of the largest angle to the nearest degree.
5. Find the triangular area, to the nearest square meter, enclosed by three pieces of fencing 123 m, 150 m and 155 m long.
6. Find the area, to the nearest square inch, of a parallelogram with sides of length 12 in and 15 in and included angle of 78°.
7. A person at point A looks due east and spots a UFO with an angle of elevation of 40°. At the same time, another person, 1 mi due west of A looks due east and sights the same UFO with an angle of elevation of 25°. Find the distance between A and the UFO. How far is the UFO above the ground? Give answers to the nearest hundredth of a mile.
8. Find the area of a triangular playground, to the nearest square meter, with sides of length 10 m, 15 m and 16 m.

9. A yard is bounded on two sides with fences of length 80 ft and 60 ft. If these fences meet at a 75° angle, how many feet of fencing are required to completely enclosed a triangular region?
10. When a boy stands on the bank of a river and looks across to the other bank, the angle of depression is 12°. If he climbs to the top of a 10 ft tree and looks across to other bank, the angle of depression is 15°. What is the distance from the first position of the boy to the other bank of the river? How wide is the river? Give your answers to the nearest foot.

8.13 Chapter 8 Review

Keywords & Theorems

The Pythagorean Theorem

- Pythagorean Theorem
- Pythagorean Triple
- Distance Formula

The Pythagorean Theorem Converse

- Pythagorean Theorem Converse
- Theorem 8-3
- Theorem 8-4

Similar Right Triangles

- Theorem 8-5
- Geometric Mean

Special Right Triangles

- Isosceles Right (45-45-90) Triangle
- 30-60-90 Triangle
- 45-45-90 Theorem
- 30-60-90 Theorem

Tangent, Sine and Cosine Ratios

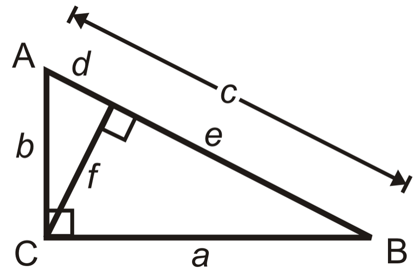
- Trigonometry
- Adjacent (Leg)
- Opposite (Leg)
- Sine Ratio
- Cosine Ratio
- Tangent Ratio
- Angle of Depression
- Angle of Elevation

Solving Right Triangles

- Inverse Tangent
- Inverse Sine
- Inverse Cosine

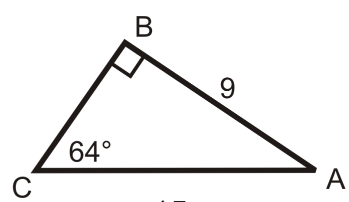
Review

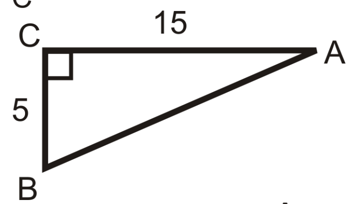
Fill in the blanks using right triangle $\triangle ABC$.

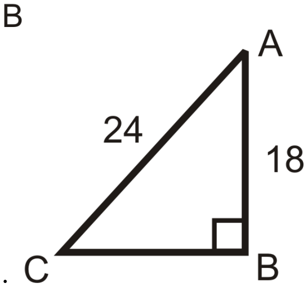


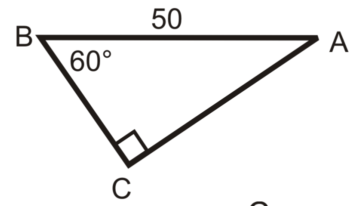
1. $a^2 + \_\_\_\_^2 = c^2$
2. $\sin \_\_\_\_ = \frac{b}{c}$
3. $\tan \_\_\_\_ = \frac{f}{d}$
4. $\cos \_\_\_\_ = \frac{b}{c}$
5. $\tan^{-1}\left(\frac{f}{e}\right) = \_\_\_\_$
6. $\sin^{-1}\left(\frac{f}{b}\right) = \_\_\_\_$
7. $\_\_\_\_^2 + d^2 = b^2$
8. $\frac{?}{b} = \frac{b}{c}$
9. $\frac{e}{?} = \frac{?}{c}$
10. $\frac{d}{f} = \frac{f}{?}$

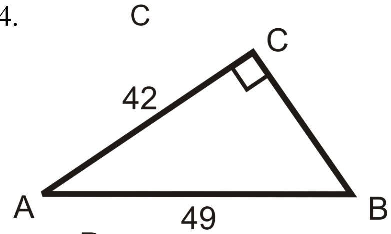
Solve the following right triangles using the Pythagorean Theorem, the trigonometric ratios, and the inverse trigonometric ratios. When possible, simplify the radical. If not, round all decimal answers to the nearest tenth.

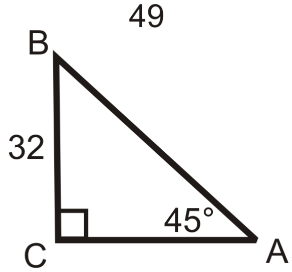
11. 

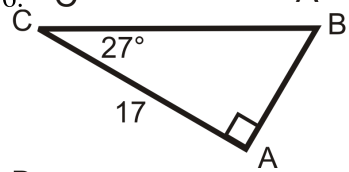
12. 

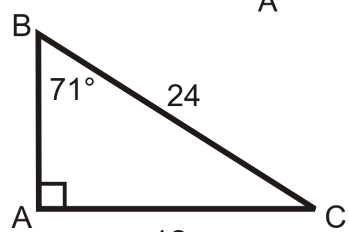
13. 

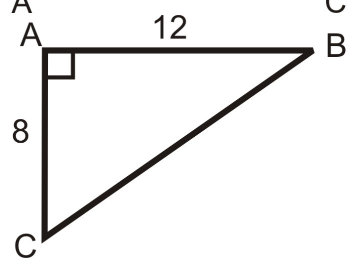
14. 

15. 

16. 

17. 

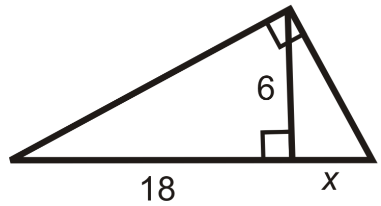
18. 

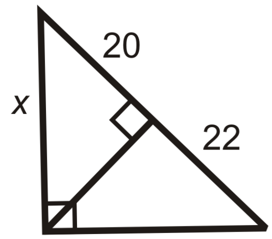
19. 

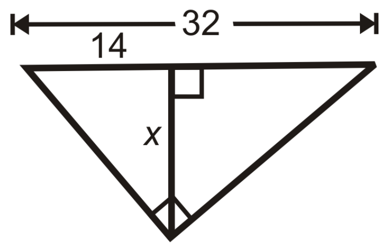
Determine if the following lengths make an acute, right, or obtuse triangle. If they make a right triangle, determine if the lengths are a Pythagorean triple.

20. 11, 12, 13
21. 16, 30, 34
22. 20, 25, 42
23. $10\sqrt{6}, 30, 10\sqrt{15}$
24. 22, 25, 31
25. 47, 27, 35

Find the value of x.

26. 

27. 

28. 

29. The angle of elevation from the base of a mountain to its peak is 76°. If its height is 2500 feet, what is the length to reach the top? Round the answer to the nearest tenth.
30. Taylor is taking an aerial tour of San Francisco in a helicopter. He spots ATT Park (baseball stadium) at a horizontal distance of 850 feet and down (vertical) 475 feet. What is the angle of depression from the helicopter to the park? Round the answer to the nearest tenth.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9693 .

CHAPTER 9 Transformations

Chapter Outline

The final chapter of Geometry explores transformations. A transformation is a move, flip, or rotation of an image. First, we will look at different types of symmetry and then discuss the different types of transformations. Finally, we will compose transformations and look at tessellations.

9.1 Exploring Symmetry

Learning Objectives

- Learn about lines of symmetry.
- Discuss line and rotational symmetry.
- Learn about the center of symmetry.

Review Queue

a. Define symmetry in your own words.
b. Draw a regular hexagon. How many degrees does each angle have?
c. Draw all the diagonals in your hexagon. What is the measure of each central angle?
d. Plot the points $A(1,3), B(3,1), C(5,3)$, and $D(3,5)$. What kind of shape is this? Prove it using the distance formula and/or slope.

Know What? Symmetry exists all over nature. One example is a starfish, like the one below. Draw in the line(s) of symmetry, center of symmetry and the angle of rotation for this starfish.

Lines of Symmetry

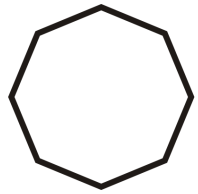
Line of Symmetry: A line that passes through a figure such that it splits the figure into two congruent halves.

Many figures have a line of symmetry, but some do not have any lines of symmetry. Figures can also have more than one line of symmetry.

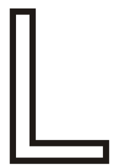
Example 1: Find all lines of symmetry for the shapes below.

a)

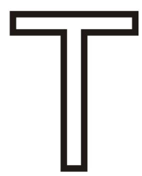
b)



c)

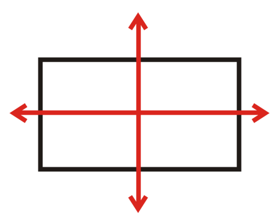


d)

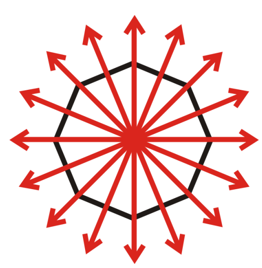


Solution: For each figure, draw lines that cut the figure in half perfectly. Figure a) has two lines of symmetry, b) has eight, c) has no lines of symmetry, and d) has one.

a)



b)



c)

d)

Figures a), b), and d) all have ***line symmetry***.

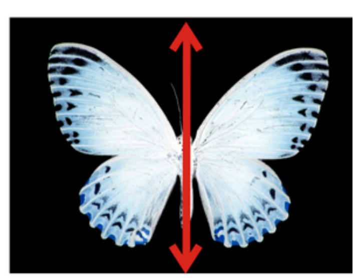
Line Symmetry: When a figure has one or more lines of symmetry.

Example 2: Do the figures below have line symmetry?

a)

b)

Solution: Yes, both of these figures have line symmetry. One line of symmetry is shown for the flower; however it has several more lines of symmetry. The butterfly only has one line of symmetry.



Rotational Symmetry

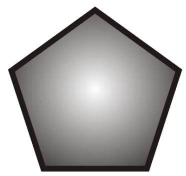
Rotational Symmetry: When a figure can be rotated (less that 360°) and it looks the same way it did before the rotation.

Center of Rotation: The point at which the figure is rotated around such that the rotational symmetry holds. Typically, the center of rotation is the center of the figure.

Along with rotational symmetry and a center of rotation, figures will have an ***angle of rotation***. The angle of rotation, tells us how many degrees we can rotate a figure so that it still looks the same.

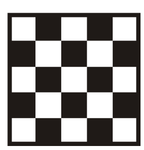
Example 3: Determine if each figure below has rotational symmetry. If it does, determine the angle of rotation.

a)



b)

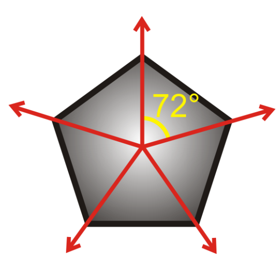
c)



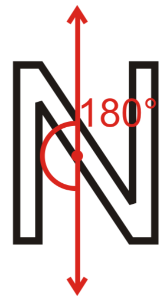
Solution:

a) The regular pentagon can be rotated 5 times so that each vertex is at the top. This means the angle of rotation is $\frac{360^\circ}{5} = 72^\circ$.

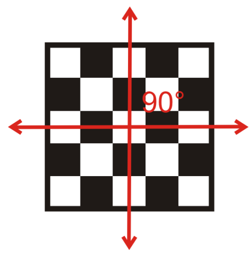
The pentagon can be rotated $72^\circ, 144^\circ, 216^\circ$, and 288° so that it still looks the same.



b) The "*N*" can be rotated twice, 180°, so that it still looks the same.

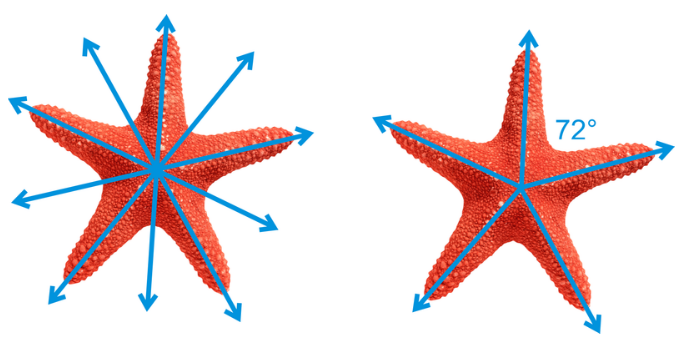


c) The checkerboard can be rotated 4 times so that the angle of rotation is $\frac{360^\circ}{4} = 90^\circ$. It can be rotated 180° and 270° as well. The final rotation is always 360° to get the figure back to its original position.



In general, ***if a shape can be rotated n times, the angle of rotation is*** $\frac{360^\circ}{n}$. Then, multiply the angle of rotation by 1, 2, 3..., and n to find the additional angles of rotation.

Know What? Revisited The starfish has 5 lines of symmetry and has rotational symmetry of 72°. Therefore, the starfish can be rotated $72^\circ, 144^\circ, 216^\circ$, and 288° and it will still look the same. The center of rotation is the center of the starfish.



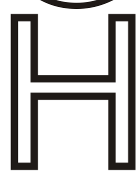
Review Questions

Determine if the following questions are ALWAYS true, SOMETIMES true, or NEVER true.

1. Right triangles have line symmetry.
2. Isosceles triangles have line symmetry.
3. Every rectangle has line symmetry.
4. Every rectangle has exactly two lines of symmetry.
5. Every parallelogram has line symmetry.
6. Every square has exactly two lines of symmetry.
7. Every regular polygon has three lines of symmetry.
8. Every sector of a circle has a line of symmetry.

9. Every parallelogram has rotational symmetry.
10. A rectangle has $90^\circ, 180^\circ$, and 270° angles of rotation.
11. Draw a quadrilateral that has two pairs of congruent sides and exactly one line of symmetry.
12. Draw a figure with infinitely many lines of symmetry.
13. Draw a figure that has one line of symmetry and no rotational symmetry.
14. Fill in the blank: A regular polygon with n sides has ______ lines of symmetry.

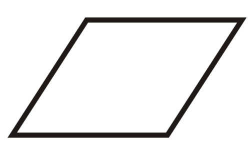
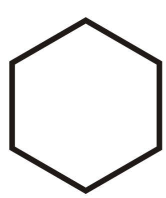
Find all lines of symmetry for the letters below.

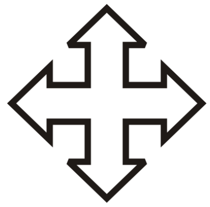
15. 
16. 
17. 
18. 
19. 
20. Do any of the letters above have rotational symmetry? If so, which one(s) and what are the angle(s) of rotation?

Determine if the words below have line symmetry or rotational symmetry.

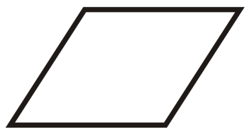
21. OHIO
22. **MOW**
23. **WOW**
24. **KICK**
25. **pod**

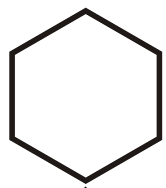
Trace each figure and then draw in all lines of symmetry.

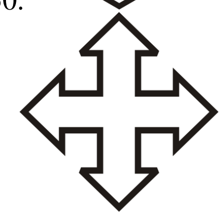
26. 
27. 

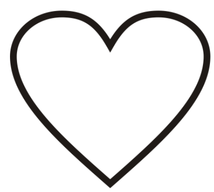
28. 

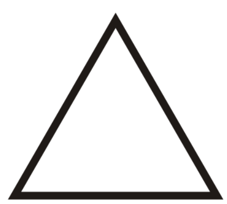
Find the angle(s) of rotation for each figure below.

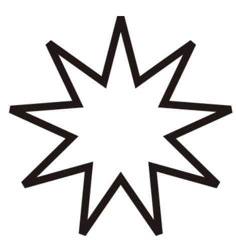
29. 

30. 

31. 

32. 

33. 

34. 

Determine if the figures below have line symmetry or rotational symmetry. Identify all lines of symmetry and all angles of rotation.

35. 

36. 

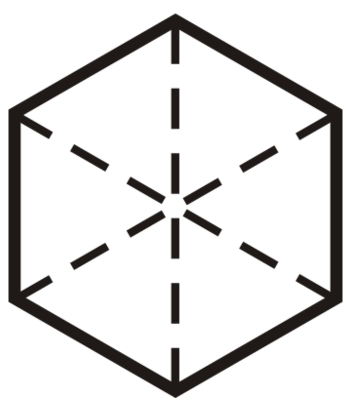
37. 

Review Queue Answers

1. Where one side of an object matches the other side; answers will vary.

2 and 3. each angle has $\frac{(n-1)180^\circ}{n} = \frac{5(180^\circ)}{6} = 120^\circ$

each central angle has $\frac{360^\circ}{6} = 60^\circ$



4. The figure is a square.

9.2 Translations and Vectors

Learning Objectives

- Graph a point, line, or figure and translate it x and y units.
- Write a translation rule.
- Use vector notation.

Review Queue

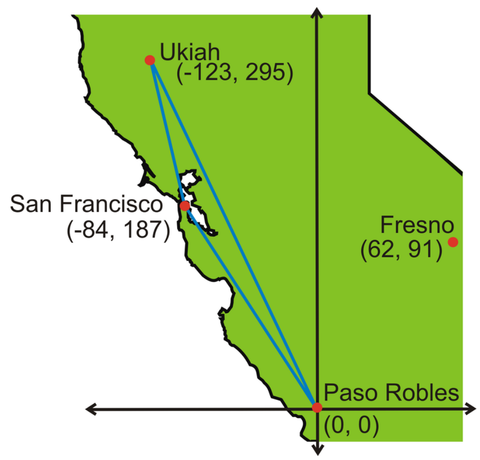
a. Find the equation of the line that contains (9, -1) and (5, 7).
b. What type of quadrilateral is formed by $A(1,-1), B(3,0), C(5,-5)$ and $D(-3,0)$?
c. Find the equation of the line parallel to #1 that passes through (4, -3).
d. Find the equation of the line perpendicular to #1 that passes through (4, -3).

Know What? Lucy currently lives in San Francisco, S, and her parents live in Paso Robles, P. She will be moving to Ukiah, U, in a few weeks. All measurements are in miles. Find:

a) The component form of $\overrightarrow{PS}, \overrightarrow{SU}$ and $\overrightarrow{PU}$.

b) Lucy's parents are considering moving to Fresno, F. Find the component form of $\overrightarrow{PF}$ and $\overrightarrow{UF}$.

c) Is Ukiah or Paso Robles closer to Fresno?



Transformations

Recall from Lesson 7.6, we learned about dilations, which is a type of transformation. Now, we are going to continue learning about other types of transformations. All of the transformations in this chapter are rigid transformations.

Transformation: An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation: A transformation that preserves size and shape.

The rigid transformations are: translations, reflections, and rotations. The new figure created by a transformation is called the ***image***. The original figure is called the ***preimage***. Another word for a rigid transformation is an ***isometry***. Rigid transformations are also called congruence transformations.

Also in Lesson 7.6, we learned how to label an image. If the preimage is A, then the image would be labeled A', said "a prime." If there is an image of A', that would be labeled A'', said "a double prime."

Translations

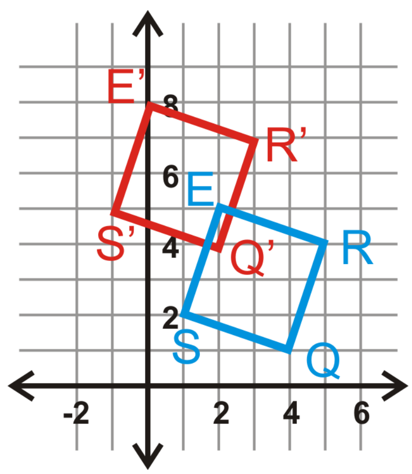
The first of the rigid transformations is a translation.

Translation: A transformation that moves every point in a figure the same distance in the same direction.

In the coordinate plane, we say that a translation moves a figure x units and y units.

Example 1: Graph square $S(1,2), Q(4,1), R(5,4)$ and $E(2,5)$. Find the image after the translation $(x,y) \rightarrow (x-2, y+3)$. Then, graph and label the image.

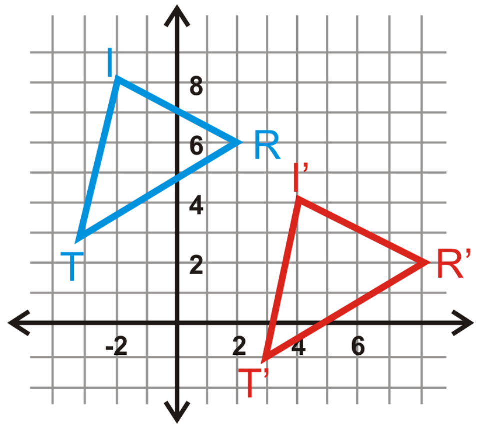
Solution: The translation notation tells us that we are going to move the square to the left 2 and up 3.



$$
\begin{aligned}
(x,y) &\rightarrow (x-2, y+3)\\
S(1,2) &\rightarrow S'(-1,5)\\
Q(4,1) &\rightarrow Q'(2,4)\\
R(5,4) &\rightarrow R'(3,7)\\
E(2,5) &\rightarrow E'(0,8)
\end{aligned}
$$

Example 2: Find the translation rule for $\triangle TRI$ to $\triangle T'R'I'$.

Solution: Look at the movement from T to T'. T is (-3, 3) and T' is (3, -1). The change in x is 6 units to the right and the change in y is 4 units down. Therefore, the translation rule is $(x,y) \rightarrow (x+6, y-4)$.



From both of these examples, we see that a translation preserves congruence. Therefore, ***a translation is an isometry***. We can show that each pair of figures is congruent by using the distance formula.

Example 3: Show $\triangle TRI \cong \triangle T'R'I'$ from Example 2.

Solution: Use the distance formula to find all the lengths of the sides of the two triangles.

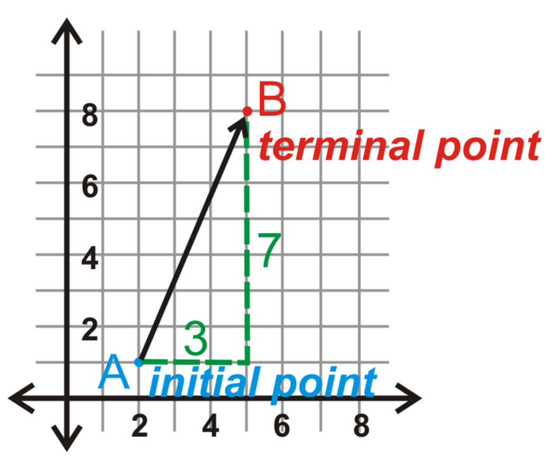
$\triangle TRI$	$\triangle T'R'I'$
$TR = \sqrt{(-3-2)^2+(3-6)^2} = \sqrt{34}$	$T'R' = \sqrt{(3-8)^2+(-1-2)^2} = \sqrt{34}$
$RI = \sqrt{(2-(-2))^2+(6-8)^2} = \sqrt{20}$	$R'I' = \sqrt{(8-4)^2+(2-4)^2} = \sqrt{20}$
$TI = \sqrt{(-3-(-2))^2+(3-8)^2} = \sqrt{26}$	$T'I' = \sqrt{(3-4)^2+(-1-4)^2} = \sqrt{26}$

Vectors

Another way to write a translation rule is to use vectors.

Vector: A quantity that has direction and size.

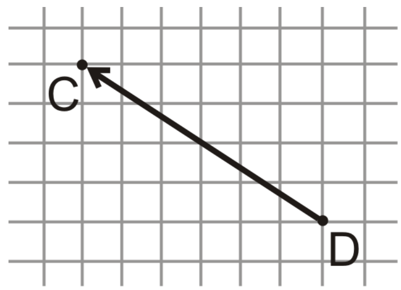
In the graph below, the line from A to B, or the distance traveled, is the vector. This vector would be labeled $\overrightarrow{AB}$ because A is the ***initial point*** and B is the ***terminal point***. The terminal point always has the arrow pointing towards it and has the half-arrow over it in the label.



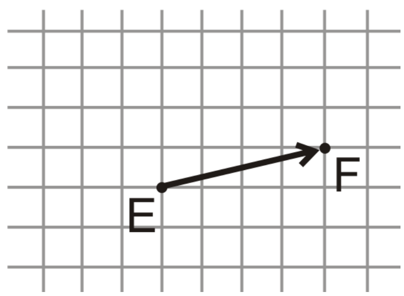
The ***component form*** of $\overrightarrow{AB}$ combines the horizontal distance traveled and the vertical distance traveled. We write the component form of $\overrightarrow{AB}$ as $\langle 3,7 \rangle$ because $\overrightarrow{AB}$ travels 3 units to the right and 7 units up. Notice the brackets are pointed, $\langle 3,7 \rangle$, not curved.

Example 4: Name the vector and write its component form.

a)



b)

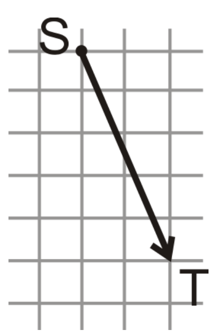


Solution:

a) The vector is $\overrightarrow{DC}$. From the initial point D to terminal point C, you would move 6 units to the left and 4 units up. The component form of $\overrightarrow{DC}$ is $\langle -6,4 \rangle$.

b) The vector is $\overrightarrow{EF}$. The component form of $\overrightarrow{EF}$ is $\langle 4,1 \rangle$.

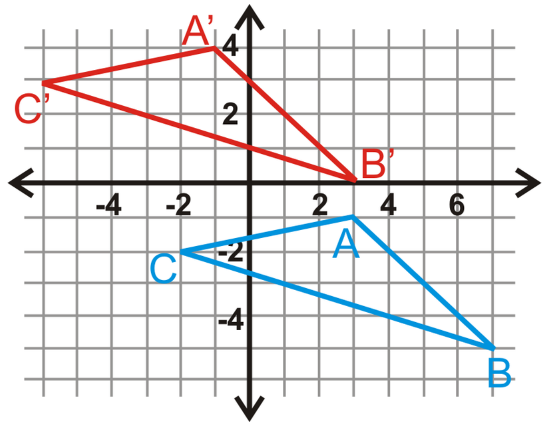
Example 5: Draw the vector $\overrightarrow{ST}$ with component form $\langle 2,-5 \rangle$.



Solution: The graph above is the vector $\overrightarrow{ST}$. From the initial point S it moves down 5 units and to the right 2 units.

The positive and negative components of a vector always correlate with the positive and negative parts of the coordinate plane. We can also use vectors to translate an image.

Example 6: Triangle $\triangle ABC$ has coordinates $A(3,-1), B(7,-5)$ and $C(-2,-2)$. Translate $\triangle ABC$ using the vector $\langle -4,5 \rangle$. Determine the coordinates of $\triangle A'B'C'$.



Solution: It would be helpful to graph $\triangle ABC$. To translate $\triangle ABC$, add each component of the vector to each point to find $\triangle A'B'C'$.

$$A(3,-1)+\langle -4,5\rangle = A'(-1,4)$$
$$B(7,-5)+\langle -4,5\rangle = B'(3,0)$$
$$C(-2,-2)+\langle -4,5\rangle = C'(-6,3)$$

Example 7: Write the translation rule for the vector translation from Example 6.

Solution: To write $\langle -4,5\rangle$ as a translation rule, it would be $(x,y) \rightarrow (x-4,y+5)$.

Know What? Revisited

a) $\overrightarrow{PS} = \langle -84,187\rangle, \overrightarrow{SU} = \langle -39,108\rangle, \overrightarrow{PU} = \langle -123,295\rangle$

b) $\overrightarrow{PF} = \langle 62,91\rangle, \overrightarrow{UF} = \langle 185,-204\rangle$

c) You can plug the vector components into the Pythagorean Theorem to find the distances. Paso Robles is closer to Fresno than Ukiah.

$$UF = \sqrt{185^2+(-204)^2} \cong 275.4 \ miles, PF = \sqrt{62^2+91^2} \cong 110.1 \ miles$$

Review Questions

1. What is the difference between a vector and a ray?

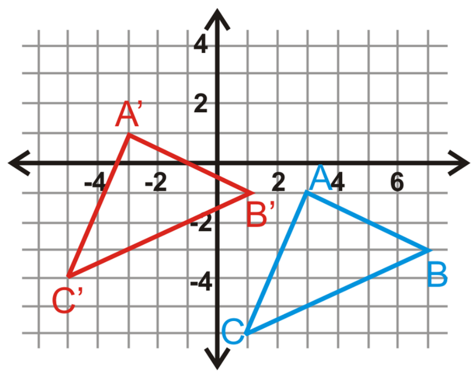
Use the translation $(x,y) \rightarrow (x+5,y-9)$ for questions 2-8.

2. What is the image of $A(-6,3)$?
3. What is the image of $B(4,8)$?
4. What is the preimage of $C'(5,-3)$?
5. What is the image of A'?
6. What is the preimage of $D'(12,7)$?
7. What is the image of A''?
8. Plot A, A', A'', and A''' from the questions above. What do you notice? Write a conjecture.

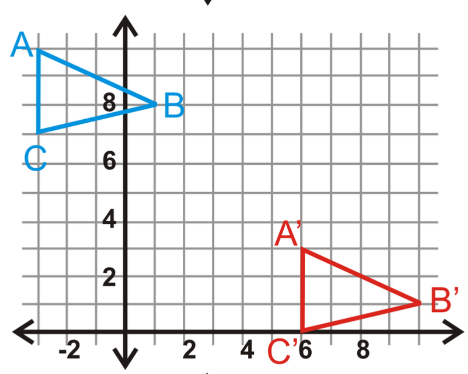
The vertices of $\triangle ABC$ are $A(-6,-7), B(-3,-10)$ and $C(-5,2)$. Find the vertices of $\triangle A'B'C'$, given the translation rules below.

9. $(x,y) \rightarrow (x-2,y-7)$
10. $(x,y) \rightarrow (x+11,y+4)$
11. $(x,y) \rightarrow (x,y-3)$
12. $(x,y) \rightarrow (x-5,y+8)$

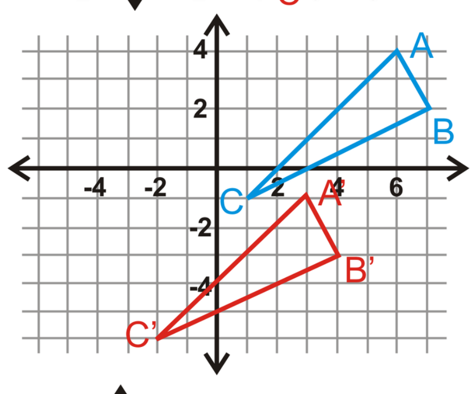
In questions 13-16, $\triangle A'B'C'$ is the image of $\triangle ABC$. Write the translation rule.



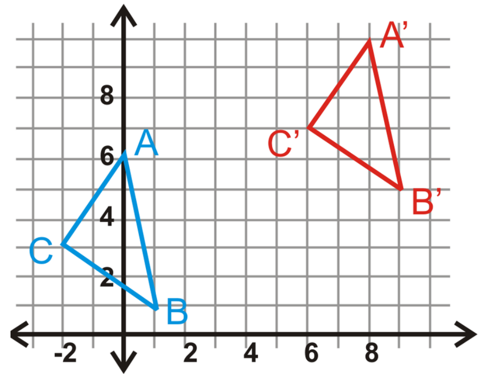
13.



14.



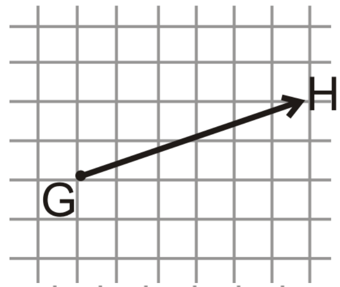
15.

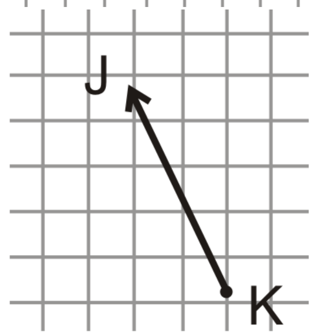


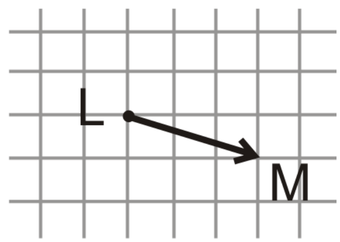
16.

17. Verify that a translation is an isometry using the triangle from #15.
18. If $\triangle A'B'C'$ was the *preimage* and $\triangle ABC$ was the image, write the translation rule for #16.

For questions 19-21, name each vector and find its component form.

19. 

20. 

21. 

For questions 22-24, plot and correctly label each vector.

22. $\overrightarrow{AB} = \langle 4, -3 \rangle$
23. $\overrightarrow{CD} = \langle -6, 8 \rangle$
24. $\overrightarrow{FE} = \langle -2, 0 \rangle$
25. The coordinates of $\triangle DEF$ are $D(4,-2), E(7,-4)$ and $F(5,3)$. Translate $\triangle DEF$ using the vector $\langle 5, 11 \rangle$ and find the coordinates of $\triangle D'E'F'$.
26. The coordinates of quadrilateral $QUAD$ are $Q(-6,1), U(-3,7), A(4,-2)$ and $D(1,-8)$. Translate $QUAD$ using the vector $\langle -3, -7 \rangle$ and find the coordinates of $Q'U'A'D'$.

For problems 27-29, write the translation rule as a translation vector.

27. $(x,y) \rightarrow (x-3, y+8)$
28. $(x,y) \rightarrow (x+9, y-12)$
29. $(x,y) \rightarrow (x, y-7)$

For problems 30-32, write the translation vector as a translation rule.

30. $\langle -7, 2 \rangle$
31. $\langle 11, 25 \rangle$
32. $\langle 15, -9 \rangle$

Review Queue Answers

a. $y = -2x + 17$
b. Kite
c. $y = -2x + 5$
d. $y = \frac{1}{2}x - 5$

9.3 Reflections

Learning Objectives

- Reflect a figure over a given line.
- Determine the rules of reflections in the coordinate plane.

Review Queue

a. Define reflection in your own words.
b. Plot $A(-3,2)$. Translate A such that $(x,y) \rightarrow (x+6,y)$.
c. What line is halfway between A and A' ?
d. Translate A such that $(x,y) \rightarrow (x,y-4)$. Call this point A''.
e. What line is halfway between A and A'' ?

Know What? A lake can act like a mirror in nature. Describe the line of reflection in the photo below. If this image were on the coordinate plane, what could the equation of the line of reflection be? (There could be more than one correct answer, depending on where you place the origin.)

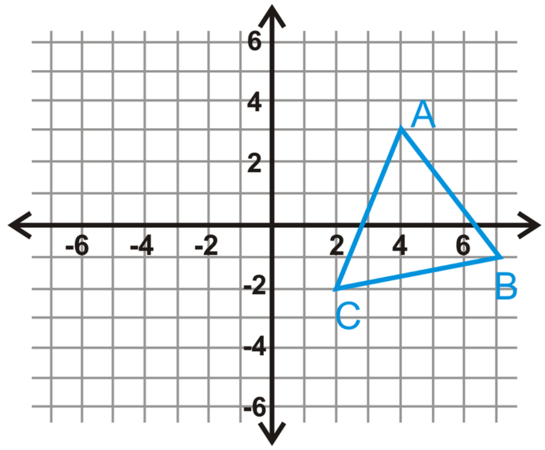
Reflections over an Axis

The next transformation is a reflection. Another way to describe a reflection is a "flip."

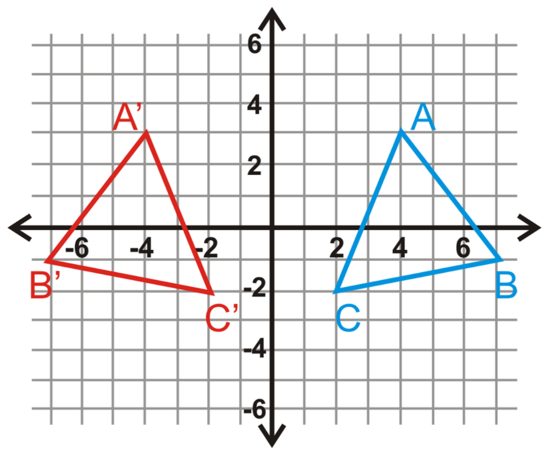
Reflection: A transformation that turns a figure into its mirror image by flipping it over a line.

Line of Reflection: The line that a figure is reflected over.

Example 1: Reflect $\triangle ABC$ over the $y-$axis. Find the coordinates of the image.



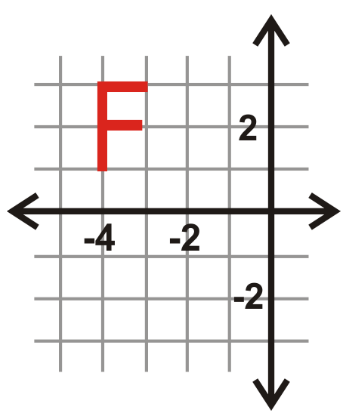
Solution: To reflect $\triangle ABC$ over the $y-$axis the $y-$coordinates will remain the same. The $x-$coordinates will be the same distance away from the $y-$axis, but on the other side of the $y-$axis.



$$A(4,3) \rightarrow A'(-4,3)$$
$$B(7,-1) \rightarrow B'(-7,-1)$$
$$C(2,-2) \rightarrow C'(-2,-2)$$

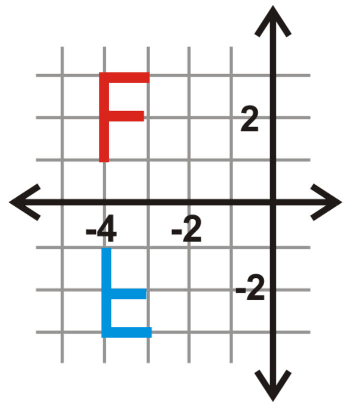
From this example, we can generalize a rule for reflecting a figure over the $y-$axis.

Reflection over the $y-$**axis:** If (x,y) is reflected over the $y-$axis, then the image is $(-x,y)$.



Example 2: Reflect the letter "F" over the $x-$axis.

Solution: To reflect the letter F over the $x-$axis, now the $x-$coordinates will remain the same and the $y-$coordinates will be the same distance away from the $x-$axis on the other side.



The generalized rule for reflecting a figure over the $x-$axis:

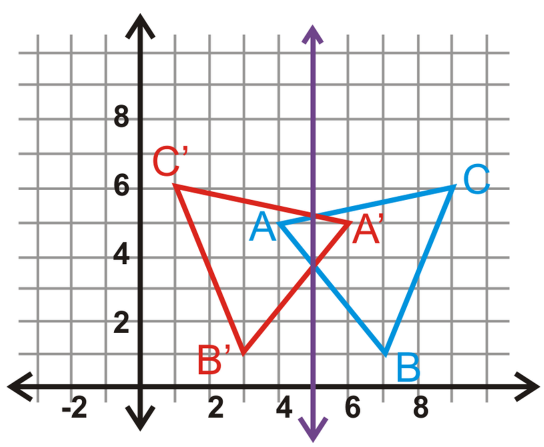
Reflection over the $x-$axis: If (x,y) is reflected over the $x-$axis, then the image is $(x,-y)$.

Reflections over Horizontal and Vertical Lines

Other than the x and y axes, we can reflect a figure over any vertical or horizontal line.

Example 3: Reflect the triangle $\triangle ABC$ with vertices $A(4,5), B(7,1)$ and $C(9,6)$ over the line $x = 5$.

Solution: Notice that this vertical line is through our preimage. Therefore, the image's vertices are the same distance away from $x = 5$ as the preimage. As with reflecting over the $y-$axis (or $x = 0$), the $y-$coordinates will stay the same.

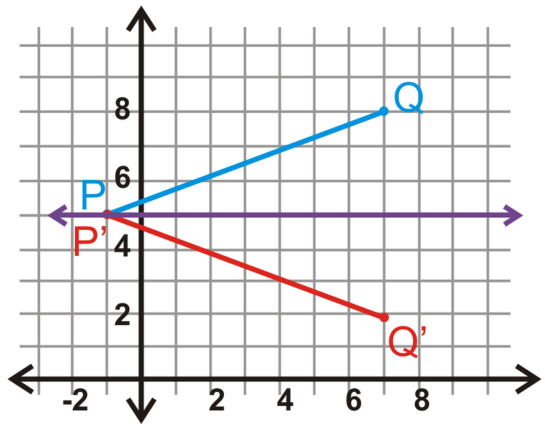


$$
\begin{aligned}
A(4,5) &\rightarrow A'(6,5)\\
B(7,1) &\rightarrow B'(3,1)\\
C(9,6) &\rightarrow C'(1,6)
\end{aligned}
$$

Example 4: Reflect the line segment $\overline{PQ}$ with endpoints $P(-1,5)$ and $Q(7,8)$ over the line $y = 5$.

Solution: Here, the line of reflection is on P, which means P' has the same coordinates. Q' has the same $x-$coordinate as Q and is the same distance away from $y = 5$, but on the other side.

$$P(-1,5) \rightarrow P'(-1,5)$$
$$Q(7,8) \rightarrow Q'(7,2)$$

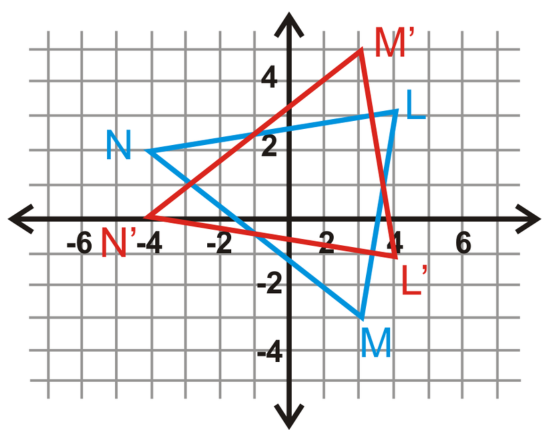


Reflection over $x = a$ **:** If (x,y) is reflected over the vertical line $x = a$, then the image is $(2a - x, y)$.

Reflection over $y = b$ **:** If (x,y) is reflected over the horizontal line $y = b$, then the image is $(x, 2b - y)$.

From these examples we also learned that if a point is on the line of reflection then the image is the same as the original point.

Example 5: A triangle $\triangle LMN$ and its reflection, $\triangle L'M'N'$ are to the left. What is the line of reflection?



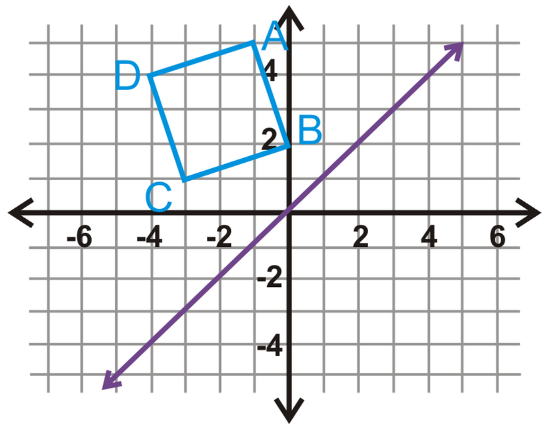
Solution: Looking at the graph, we see that the preimage and image intersect when $y = 1$. Therefore, this is the line of reflection.

If the image does not intersect the preimage, find the midpoint between a preimage and its image. This point is on the line of reflection. You will need to determine if the line is vertical or horizontal.

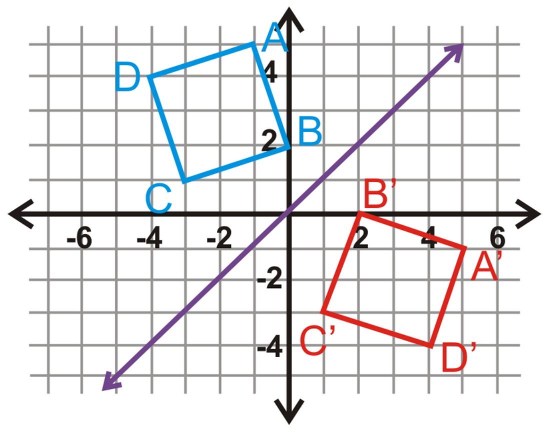
Reflections over $y = x$ **and** $y = -x$

Technically, any line can be a line of reflection. We are going to study two more cases of reflections, reflecting over $y = x$ and over $y = -x$.

Example 6: Reflect square $ABCD$ over the line $y = x$.



Solution: The purple line is $y = x$. To reflect an image over a line that is not vertical or horizontal, you can fold the graph on the line of reflection.

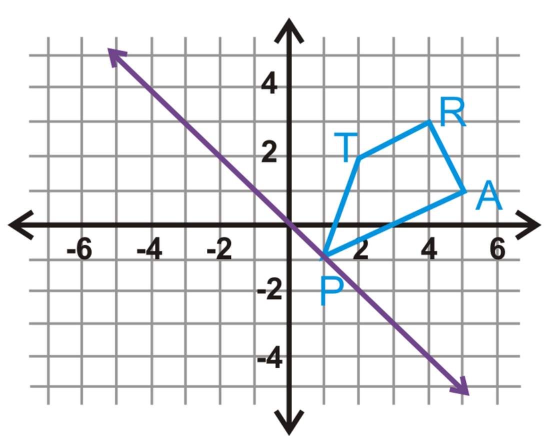


$$\begin{gathered} A(-1,5) \rightarrow A'(5,-1) \\ B(0,2) \rightarrow B'(2,0) \\ C(-3,1) \rightarrow C'(1,-3) \\ D(-4,4) \rightarrow D'(4,-4) \end{gathered}$$

From this example, we see that the x and y values are switched when a figure is reflected over the line $y = x$.

Reflection over $y = x$: If (x,y) is reflected over the line $y = x$, then the image is (y,x).

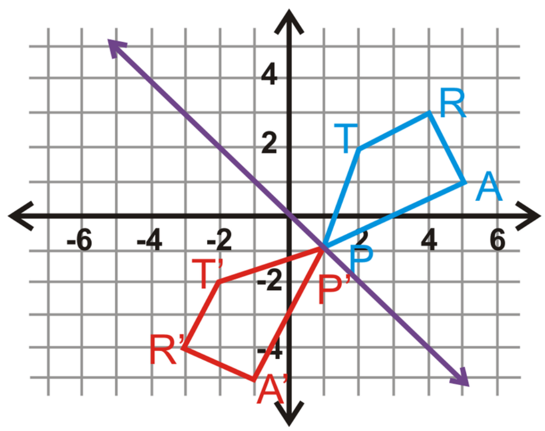
Example 7: Reflect the trapezoid *TRAP* over the line $y = -x$.



Solution: The purple line is $y = -x$. You can reflect the trapezoid over this line just like we did in Example 6.

$$\begin{aligned}
T(2,2) &\rightarrow T'(-2,-2)\\
R(4,3) &\rightarrow R'(-3,-4)\\
A(5,1) &\rightarrow A'(-1,-5)\\
P(1,-1) &\rightarrow P'(1,-1)
\end{aligned}$$

From this example, we see that the x and y values are switched and the signs are changed when a figure is reflected over the line $y = x$.



Reflection over $y = -x$: If (x,y) is reflected over the line $y = -x$, then the image is $(-y,-x)$.

At first glance, it does not look like P and P' follow the rule above. However, when you switch 1 and -1 you would have (-1, 1). Then, take the opposite sign of both, (1, -1). Therefore, when a point is on the line of reflection, it will be its own reflection.

From all of these examples, we notice that ***a reflection is an isometry***.

Know What? Revisited The white line in the picture is the line of reflection. This line coincides with the water's edge. If we were to place this picture on the coordinate plane, the line of reflection would be any horizontal line. One example could be the $x-$axis.

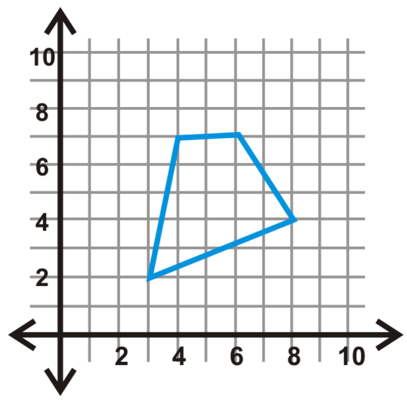
Review Questions

1. Which letter is a reflection over a vertical line of the letter "b"?

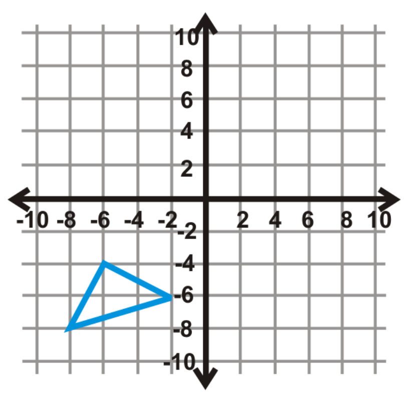
2. Which letter is a reflection over a horizontal line of the letter "b"?

Reflect each shape over the given line.

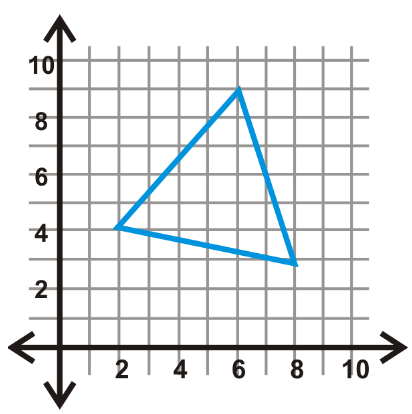
3. $y-$axis



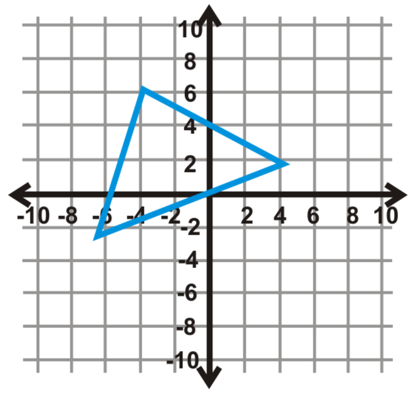
4. $x-$axis



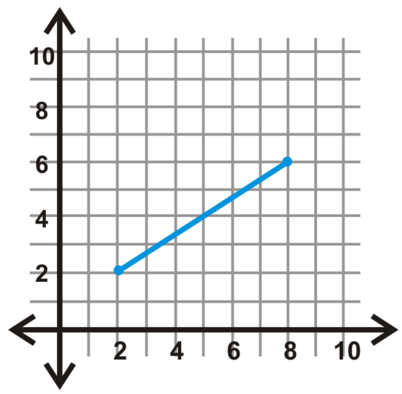
5. $y = 3$



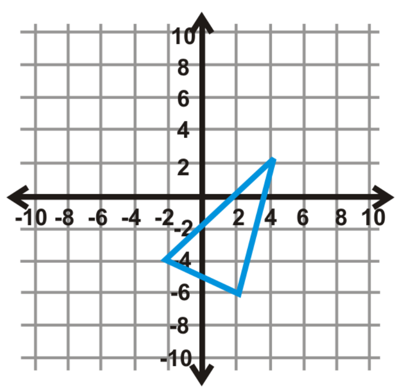
6. $x = -1$



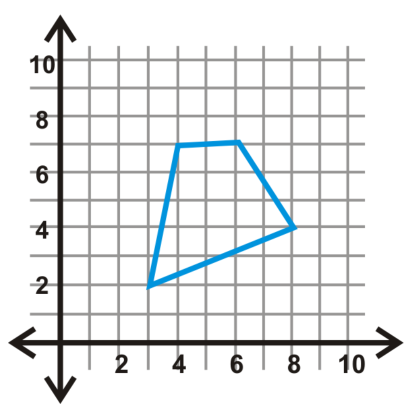
7. $x-$axis



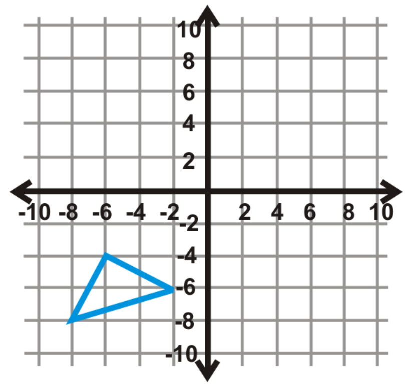
8. $y-$axis



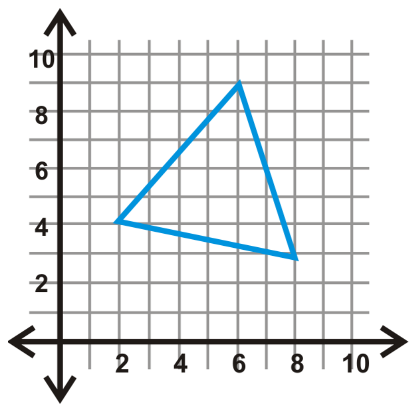
9. $y = x$



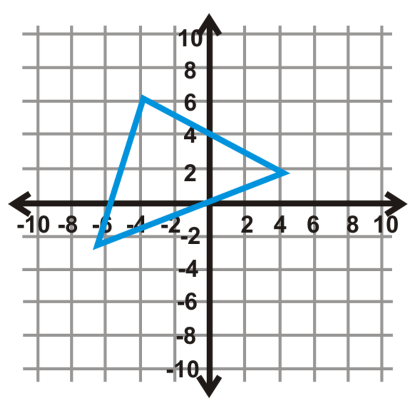
10. $y = -x$



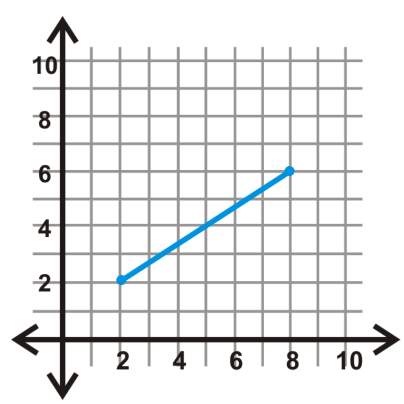
11. $x = 2$



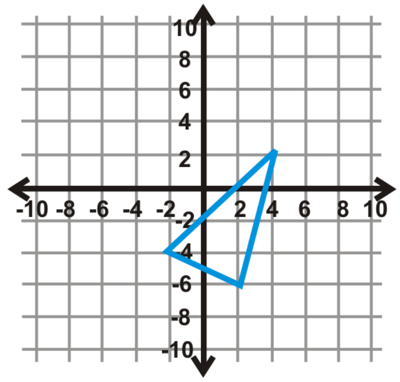
12. $y = -4$



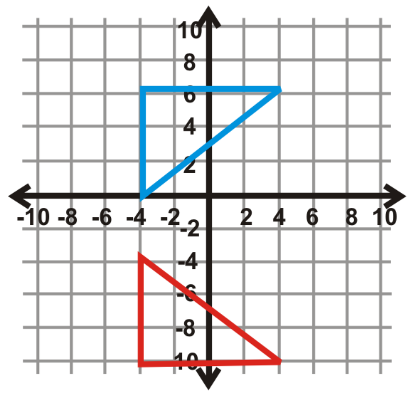
13. $y = -x$

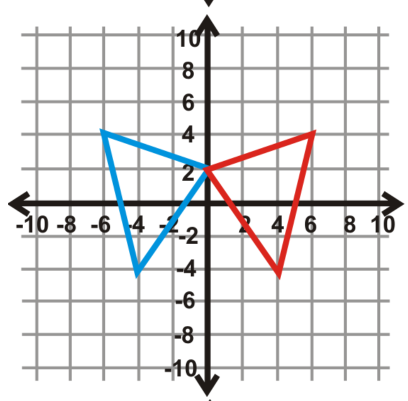


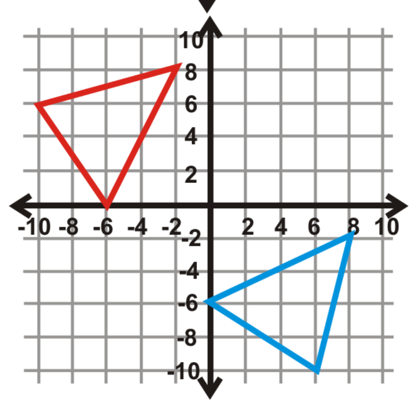
14. $y = x$



Find the line of reflection of the blue triangle (preimage) and the red triangle (image).

15. 

16. 

17. 

Two Reflections The vertices of $\triangle ABC$ are $A(-5,1), B(-3,6)$, and $C(2,3)$. Use this information to answer questions 18-21.

18. Plot $\triangle ABC$ on the coordinate plane.
19. Reflect $\triangle ABC$ over $y = 1$. Find the coordinates of $\triangle A'B'C'$.
20. Reflect $\triangle A'B'C'$ over $y = -3$. Find the coordinates of $\triangle A''B''C''$.
21. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle DEF$ are $D(6,-2), E(8,-4)$, and $F(3,-7)$. Use this information to answer questions 22-25.

22. Plot $\triangle DEF$ on the coordinate plane.
23. Reflect $\triangle DEF$ over $x = 2$. Find the coordinates of $\triangle D'E'F'$.
24. Reflect $\triangle D'E'F'$ over $x = -4$. Find the coordinates of $\triangle D''E''F''$.
25. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle GHI$ are $G(1,1), H(5,1)$, and $I(5,4)$. Use this information to answer questions 26-29.

26. Plot $\triangle GHI$ on the coordinate plane.
27. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.
28. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.
29. What ***one*** transformation would be the same as this double reflection?
30. Following the steps to reflect a triangle using a compass and straightedge.
    a. Make a triangle on a piece of paper. Label the vertices A, B and C.
    b. Make a line next to your triangle (this will be your line of reflection).
    c. Construct perpendiculars from each vertex of your triangle through the line of reflection.
    d. Use your compass to mark off points on the other side of the line that are the same distance from the line as the original A, B and C. Label the points A', B' and C'.
    e. Connect the new points to make the image $\triangle A'B'C'$.
31. Describe the relationship between the line of reflection and the segments connecting the preimage and image points.
32. Repeat the steps from problem 28 with a line of reflection that passes ***through*** the triangle.

Review Queue Answers

a. *Examples are:* To flip an image over a line; A mirror image.
b. $A'(3,2)$
c. the $y-$axis
d. $A''(-3,-2)$
e. the $x-$axis

9.4 Rotations

Learning Objectives

- Find the image of a figure in a rotation in a coordinate plane.
- Recognize that a rotation is an isometry.

Review Queue

a. Reflect $\triangle XYZ$ with vertices $X(9,2), Y(2,4)$ and $Z(7,8)$ over the $y-$axis. What are the vertices of $\triangle X'Y'Z'$?
b. Reflect $\triangle X'Y'Z'$ over the $x-$axis. What are the vertices of $\triangle X''Y''Z''$?
c. How do the coordinates of $\triangle X''Y''Z''$ relate to $\triangle XYZ$?

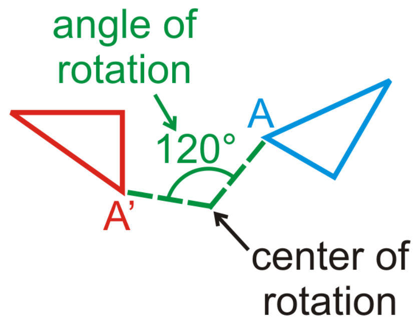
Know What? The international symbol for recycling appears below. It is three arrows rotated around a point. Let's assume that the arrow on the top is the preimage and the other two are its images. Find the center of rotation and the angle of rotation for each image.

Defining Rotations

Rotation: A transformation by which a figure is turned around a fixed point to create an image.

Center of Rotation: The fixed point that a figure is rotated around.

Lines can be drawn from the preimage to the center of rotation, and from the center of rotation to the image. The angle formed by these lines is the angle of ***rotation.***

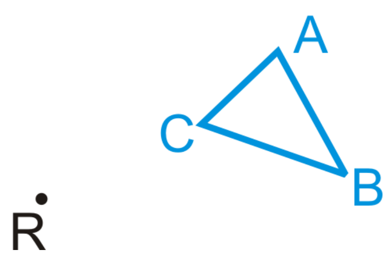


In this section, our center of rotation will always be the ***origin.*** Rotations can also be clockwise or counterclockwise. We will only do ***counterclockwise*** rotations, to go along with the way the quadrants are numbered.

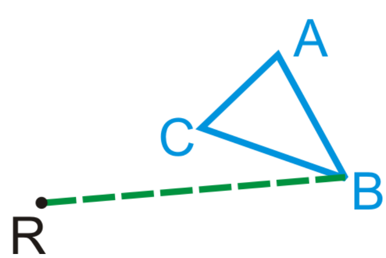
Investigation 12-1: Drawing a Rotation of 100°

Tools Needed: pencil, paper, protractor, ruler

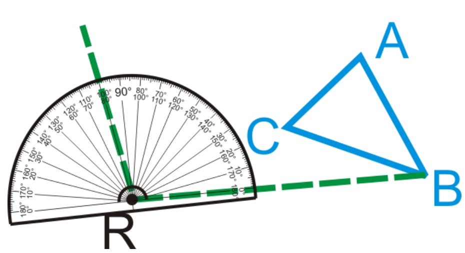
a. Draw $\triangle ABC$ and a point R outside the circle.



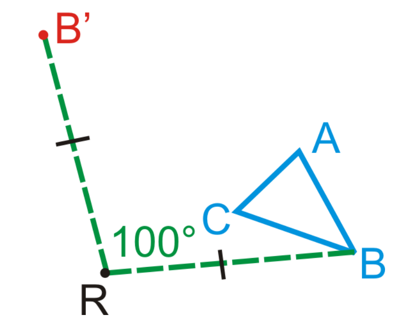
b. Draw the line segment $\overline{RB}$.

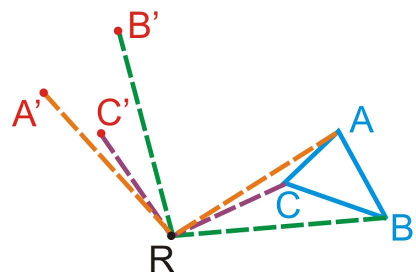


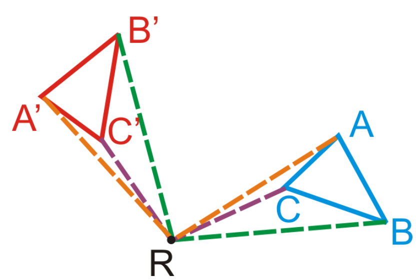
c. Take your protractor, place the center on R and the initial side on $\overline{RB}$. Mark a 100° angle.



d. Find B' such that $RB = RB'$.
e. Repeat steps 2-4 with points A and C.
f. Connect A', B', and C' to form $\triangle A'B'C'$.





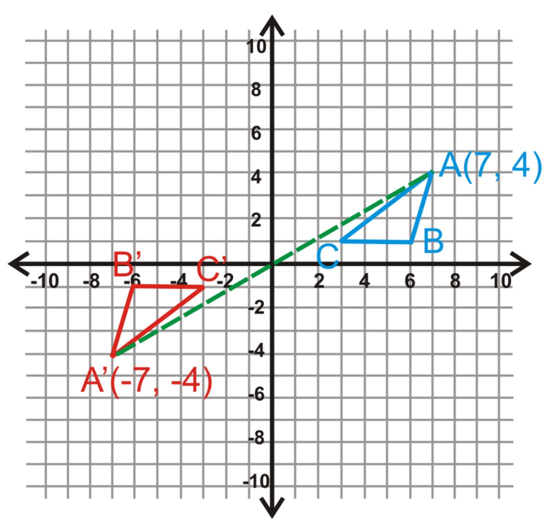


This is the process you would follow to rotate any figure 100° counterclockwise. If it was a different angle measure, then in Step 3, you would mark a different angle. You will need to repeat steps 2-4 for every vertex of the shape.

180° **Rotation**

To rotate a figure 180° in the coordinate plane, we use the origin as the center of the rotation. Recall, that a 180° angle is the same as a straight line. So, a rotation of a point over the origin of 180° will be on the same line and the same distance away from the origin.

Example 1: Rotate $\triangle ABC$, with vertices $A(7,4), B(6,1)$, and $C(3,1)$ 180°. Find the coordinates of $\triangle A'B'C'$.



Solution: You can either use Investigation 12-1 or the hint given above to find $\triangle A'B'C'$. It is very helpful to graph the triangle. Using the hint, if A is $(7,4)$, that means it is 7 units to the right of the origin and 4 units up. A' would then be 7 units to the ***left*** of the origin and 4 units ***down.*** The vertices are:

$$A(7,4) \rightarrow A'(-7,-4)$$
$$B(6,1) \rightarrow B'(-6,-1)$$
$$C(3,1) \rightarrow C'(-3,-1)$$

The image has vertices that are the negative of the preimage. This will happen every time a figure is rotated 180°.

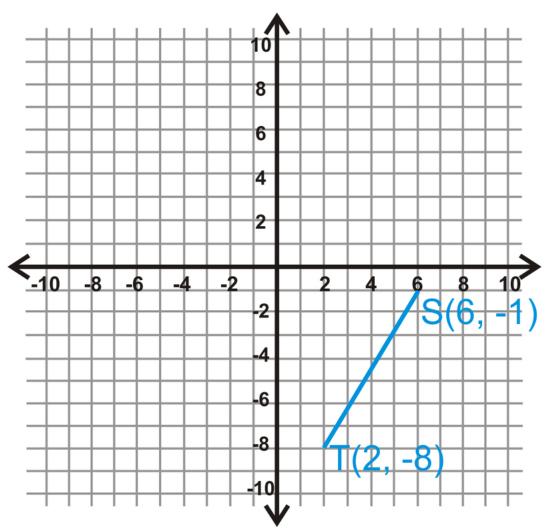
Rotation of 180°**:** If (x,y) is rotated 180° around the origin, then the image will be $(-x,-y)$.

From this example, we can also see that ***a rotation is an isometry.*** This means that $\triangle ABC \cong \triangle A'B'C'$. You can use the distance formula to verify that our assertion holds true.

90° **Rotation**

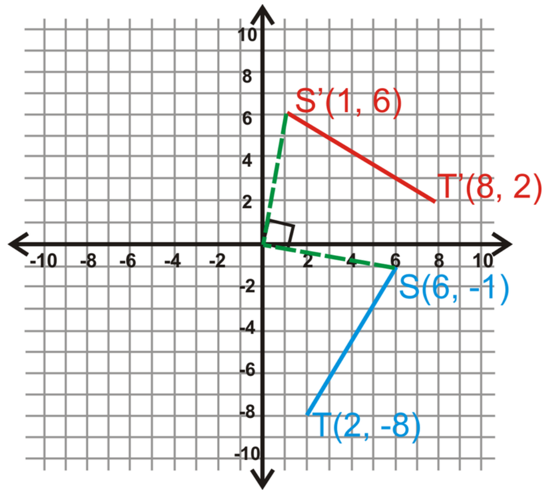
Similar to the 180° rotation, a 90° rotation (counterclockwise) is an isometry. Each image will be the same distance away from the origin as its preimage, but rotated 90°.

Example 2: Rotate $\overline{ST}$ 90°.



Solution: When we rotate something 90°, you can use Investigation 12-1. Draw lines from the origin to S and T. The line from each point to the origin is going to be ***perpendicular*** to the line from the origin to its image. Therefore, if S is 6 units to the ***right*** of the origin and 1 unit ***down,*** S' will be 6 units ***up*** and 1 to the ***right.***

Using this pattern, T' is (8, 2).



If you were to write the slope of each point to the origin, S would be $\frac{-1}{6} \rightarrow \frac{y}{x}$, and S' must be $\frac{6}{1} \rightarrow \frac{y'}{x'}$. Again, they are perpendicular slopes, following along with the 90° rotation. Therefore, the x and the y values switch and the new $x-$value is the opposite sign of the original $y-$value.

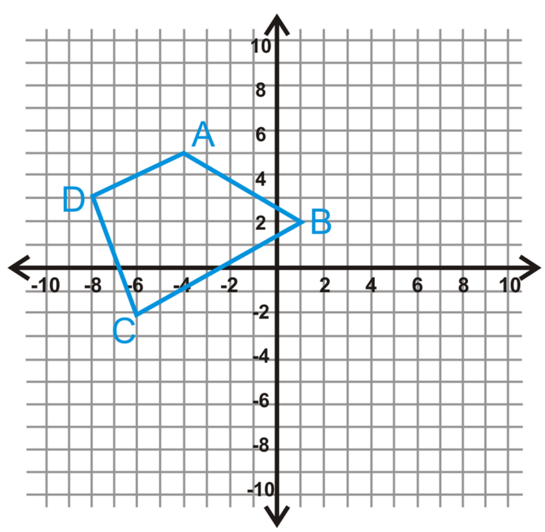
Rotation of 90°**:** If (x,y) is rotated 90° around the origin, then the image will be $(-y,x)$.

Rotation of 270°

A rotation of 270° counterclockwise would be the same as a clockwise rotation of 90°. We also know that a 90° rotation and a 270° rotation are 180° apart. We know that for every 180° rotation, the x and y values are negated. So, if the values of a 90° rotation are $(-y,x)$, then a 270° rotation would be the opposite sign of each, or $(y,-x)$.

Rotation of 270°**:** If (x,y) is rotated 270° around the origin, then the image will be $(y,-x)$.

Example 3: Find the coordinates of $ABCD$ after a 270° rotation.



Solution: Using the rule, we have:

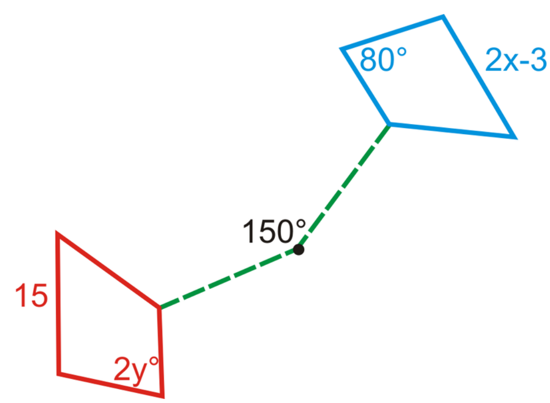
$$
\begin{aligned}
(x,y) &\rightarrow (y,-x)\\
A(-4,5) &\rightarrow A'(5,4)\\
B(1,2) &\rightarrow B'(2,-1)\\
C(-6,-2) &\rightarrow C'(-2,6)\\
D(-8,3) &\rightarrow D'(3,8)
\end{aligned}
$$

While we can rotate any image any amount of degrees, only $90^\circ, 180^\circ$ and 270° have special rules. To rotate a figure by an angle measure other than these three, you must use Investigation 12-1.

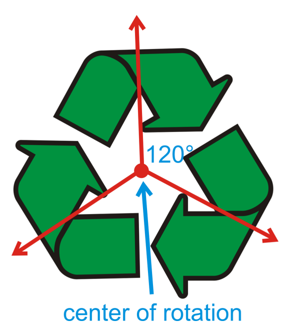
Example 4: ***Algebra Connection*** The rotation of a quadrilateral is shown below. What is the measure of x and y?

Solution: Because a rotation is an isometry, we can set up two equations to solve for x and y.

$$
\begin{aligned}
2y &= 80^\circ\\
y &= 40^\circ
\end{aligned}
\qquad\qquad
\begin{aligned}
2x-3 &= 15\\
2x &= 18\\
x &= 9
\end{aligned}
$$



Know What? Revisited The center of rotation is shown in the picture below. If we draw rays to the same point in each arrow, we see that the two images are a 120° rotation in either direction.

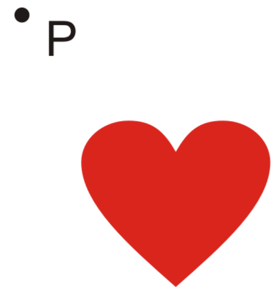


Review Questions

In the questions below, every rotation is *counterclockwise,* unless otherwise stated.

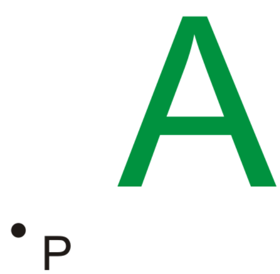
Using Investigation 12-1, rotate each figure around point P the given angle measure.

1. 50°



2. 120°

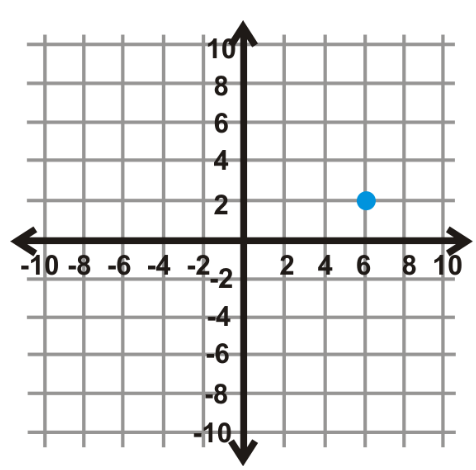
3. 200°



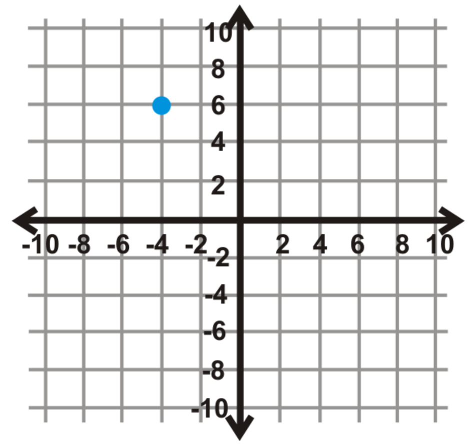
4. If you rotated the letter *p* 180° counterclockwise, what letter would you have?
5. If you rotated the letter *p* 180° *clockwise,* what letter would you have? Why do you think that is?
6. A 90° clockwise rotation is the same as what counterclockwise rotation?
7. A 270° clockwise rotation is the same as what counterclockwise rotation?
8. Rotating a figure 360° is the same as what other rotation?

Rotate each figure in the coordinate plane the given angle measure. The center of rotation is the origin.

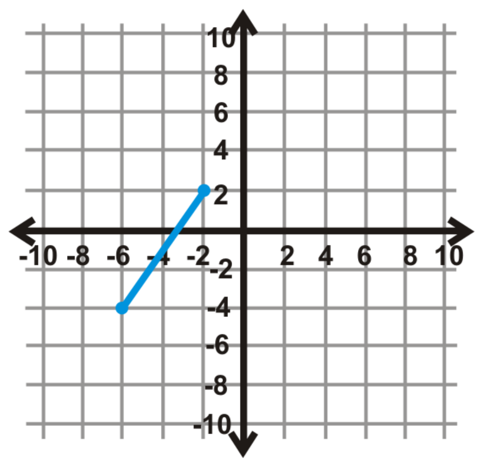
9. 180°



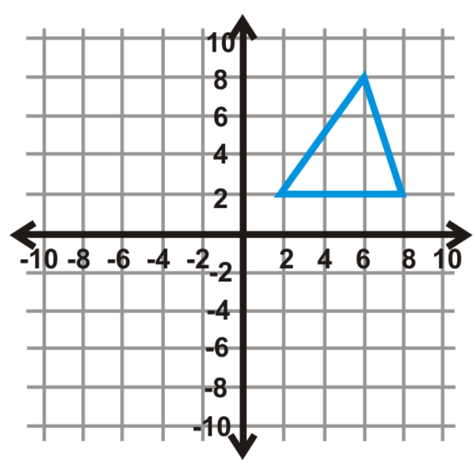
10. 90°



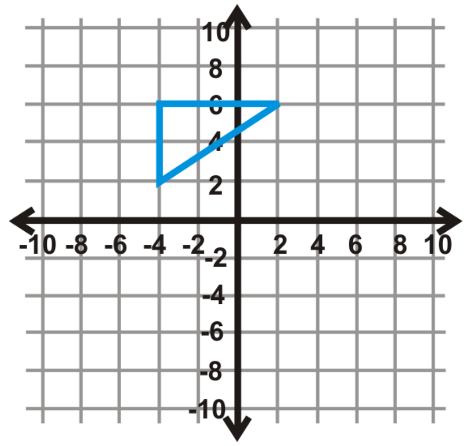
11. 180°



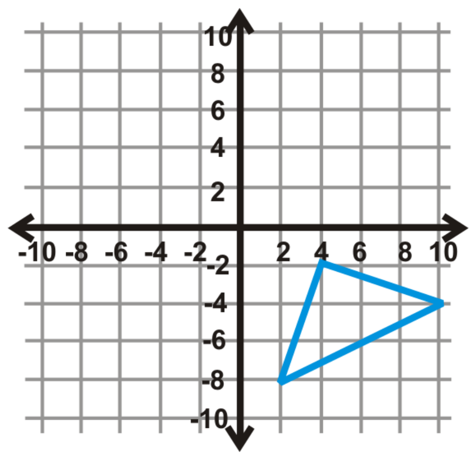
12. 270°



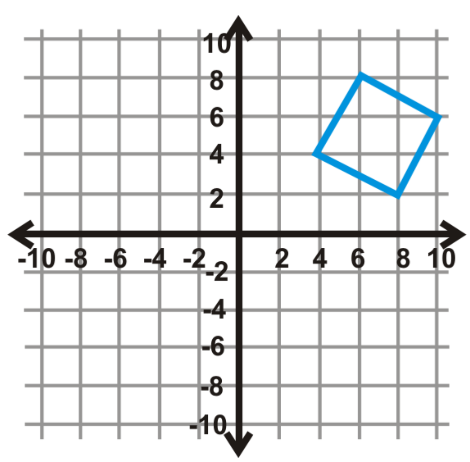
13. 90°



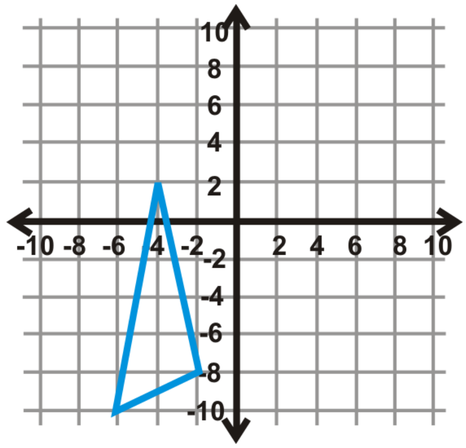
14. 270°



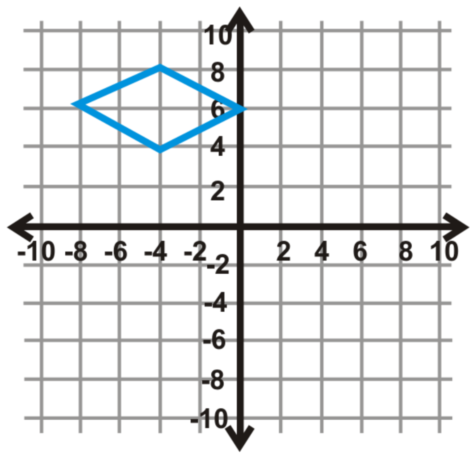
15. 180°



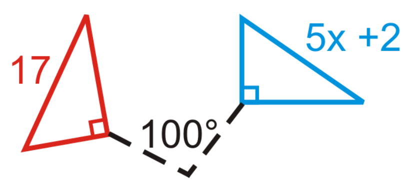
16. 270°

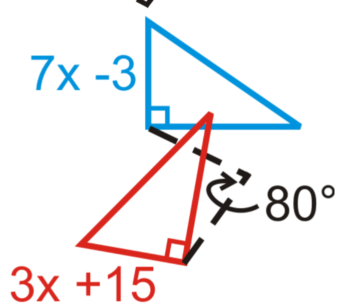


17. 90°



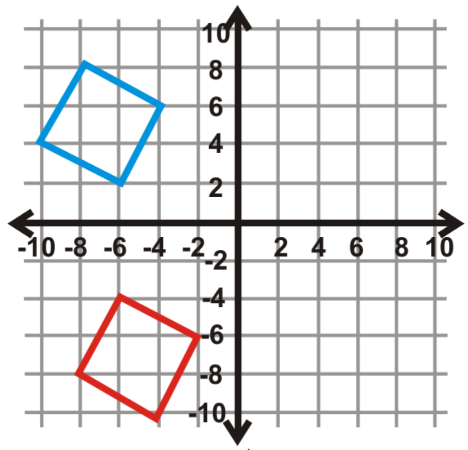
Algebra Connection Find the measure of *x* in the rotations below. The blue figure is the preimage.

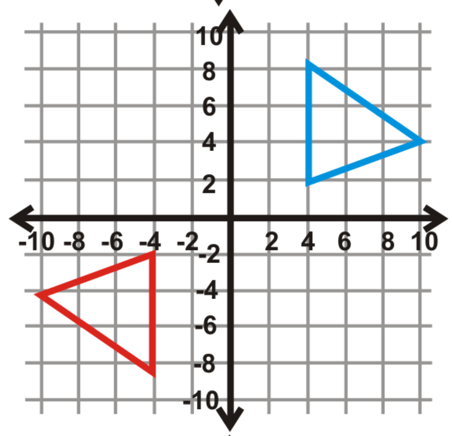
18.


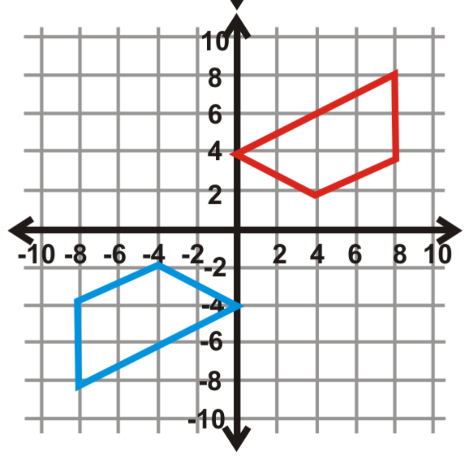
19.


20.


Find the angle of rotation for the graphs below. The center of rotation is the origin and the blue figure is the preimage.

21. 

22. 

23. 

Two Reflections The vertices of $\triangle GHI$ are $G(-2,2), H(8,2)$ and $I(6,8)$. Use this information to answer questions 24-27.

24. Plot $\triangle GHI$ on the coordinate plane.
25. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.
26. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.
27. What ***one*** transformation would be the same as this double reflection?

Multistep Construction Problem

28. Draw two lines that intersect, m and n, and $\triangle ABC$. Reflect $\triangle ABC$ over line m to make $\triangle A'B'C'$. Reflect $\triangle A'B'C'$ over line n to get $\triangle A''B''C''$. Make sure $\triangle ABC$ does not intersect either line.
29. Draw segments from the intersection point of lines m and n to A and A''. Measure the angle between these segments. This is the angle of rotation between $\triangle ABC$ and $\triangle A''B''C''$.

30. Measure the angle between lines m and n. Make sure it is the angle which contains $\triangle A'B'C'$ in the interior of the angle.
31. What is the relationship between the angle of rotation and the angle between the two lines of reflection?

Review Queue Answers

a. $X'(-9,2), Y'(-2,4), Z'(-7,8)$
b. $X''(-9,-2), Y''(-2,-4), Z''(-7,-8)$
c. $\triangle X''Y''Z''$ is the double negative of $\triangle XYZ; (x,y) \rightarrow (-x,-y)$

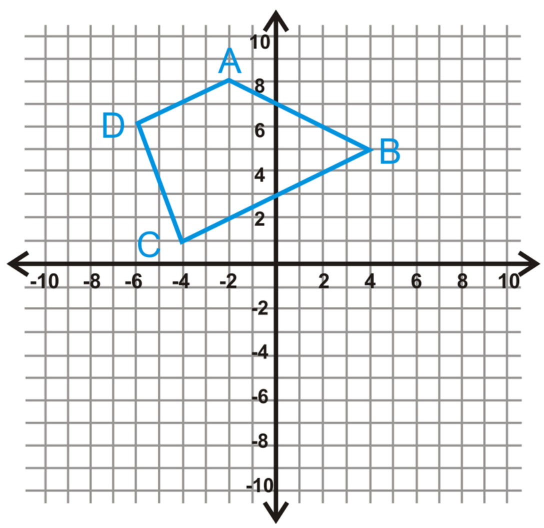
9.5 Composition of Transformations

Learning Objectives

- Perform a glide reflection.
- Perform a reflection over parallel lines and the axes.
- Perform a double rotation with the same center of rotation.
- Determine a single transformation that is equivalent to a composite of two transformations.

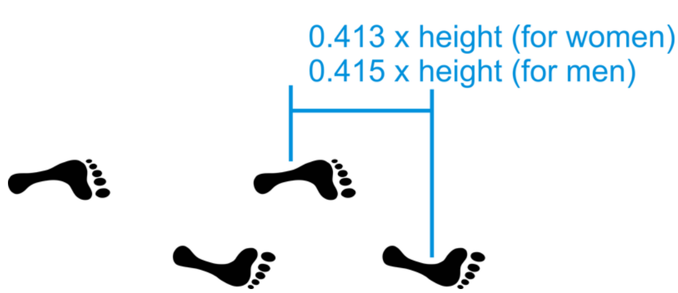
Review Queue

a. Reflect $ABCD$ over the $x-$axis. Find the coordinates of $A'B'C'D'$.



b. Translate $A'B'C'D'$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A''B''C''D''$.
c. Now, start over. Translate $ABCD$ such that $(x,y) \rightarrow (x+4,y)$. Find the coordinates of $A'B'C'D'$.
d. Reflect $A'B'C'D'$ from #3 over the $x-$axis. Find the coordinates of $A''B''C''D''$. Are they the same as #2?

Know What? An example of a glide reflection is your own footprint. The equations to find your average footprint are in the diagram below. Determine your average footprint and write the rule for one stride. You may assume your stride starts at (0, 0).



Glide Reflections

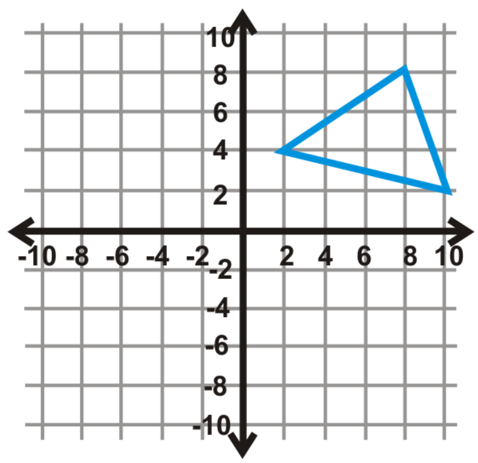
Now that we have learned all our rigid transformations, or isometries, we can perform more than one on the same figure. In your homework last night you actually performed a composition of two reflections. And, in the Review Queue above, you performed a composition of a reflection and a translation.

Composition (of transformations): To perform more than one rigid transformation on a figure.

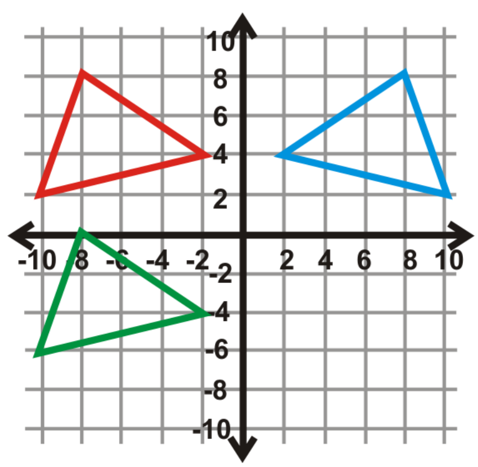
Glide Reflection: A composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

So, in the Review Queue above, you performed a glide reflection on $ABCD$. Hopefully, in #4, you noticed that ***the order in which you reflect or translate does not matter.*** It is important to note that the translation for any glide reflection will always be in one direction. So, if you reflect over a vertical line, the translation can be up or down, and if you reflect over a horizontal line, the translation will be to the left or right.

Example 1: Reflect $\triangle ABC$ over the $y-$axis and then translate the image 8 units down.



Solution: The green image below is the final answer.



$$A(8,8) \rightarrow A''(-8,0)$$
$$B(2,4) \rightarrow B''(-2,-4)$$
$$C(10,2) \rightarrow C''(-10,-6)$$

One of the interesting things about compositions is that they can always be written as one rule. What this means is, you don't necessarily have to perform one transformation followed by the next. You can write a rule and perform them at the same time.

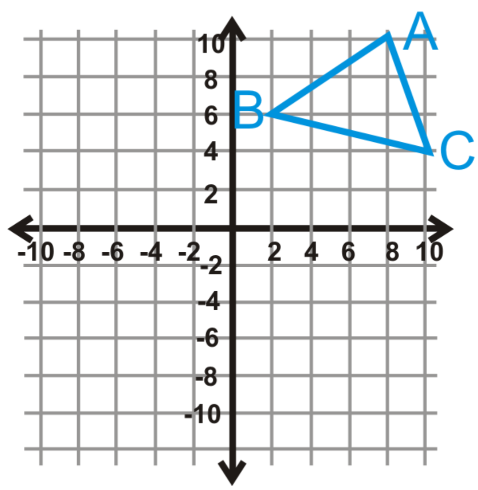
Example 2: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 1.

Solution: Looking at the coordinates of A to A'', the $x-$value is the opposite sign and the $y-$value is $y-8$. Therefore the rule would be $(x,y) \rightarrow (-x, y-8)$.

Notice that this follows the rules we have learned in previous sections about a reflection over the $y-$axis and translations.

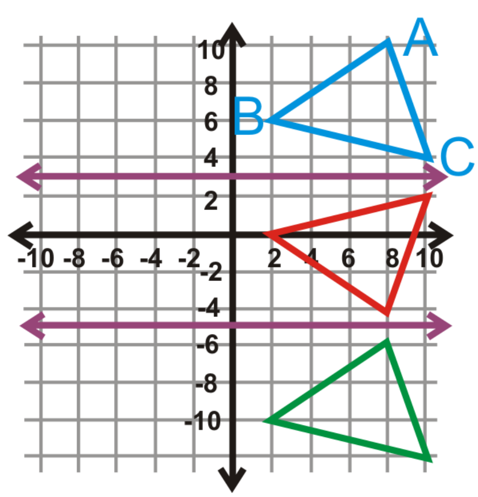
Reflections over Parallel Lines

The next composition we will discuss is a double reflection over parallel lines. For this composition, we will only use horizontal or vertical lines.



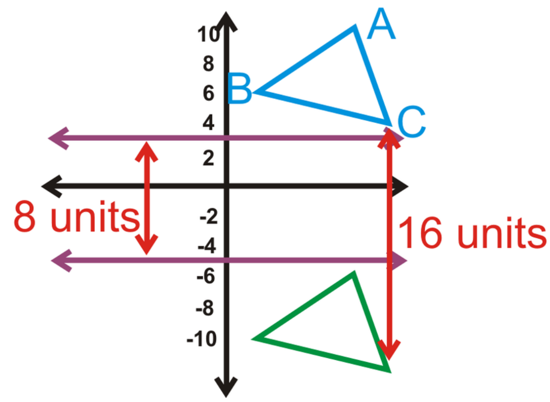
Example 3: Reflect $\triangle ABC$ over $y = 3$ and $y = -5$.

Solution: Unlike a glide reflection, order matters. Therefore, you would reflect over $y = 3$ first, followed by a reflection of this image (red triangle) over $y = -5$. Your answer would be the green triangle in the graph below.



Example 4: Write a single rule for $\triangle ABC$ to $\triangle A''B''C''$ from Example 3.

Solution: Looking at the graph below, we see that the two lines are 8 units apart and the figures are 16 units apart. Therefore, the double reflection is the same as a single translation that is double the distance between the two lines.



$$(x,y) \rightarrow (x,y-16)$$

Reflections over Parallel Lines Theorem: If you compose two reflections over parallel lines that are h units apart, it is the same as a single translation of $2h$ units.

Be careful with this theorem. Notice, it does not say which direction the translation is in. So, to apply this theorem, you would still need to visualize, or even do, the reflections to see in which direction the translation would be.

Example 5: $\triangle DEF$ has vertices $D(3,-1), E(8,-3)$, and $F(6,4)$. Reflect $\triangle DEF$ over $x=-5$ and $x=1$. This double reflection would be the same as which one translation?

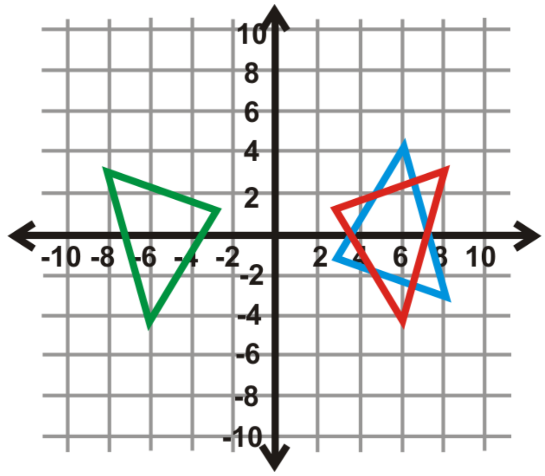
Solution: From the Reflections over Parallel Lines Theorem, we know that this double reflection is going to be the same as a single translation of $2(1-(-5))$ or 12 units. Now, we need to determine if it is to the right or to the left. Because we first reflect over a line that is further away from $\triangle DEF$, to the ***left***, $\triangle D''E''F''$ will be on the ***right*** of $\triangle DEF$. So, it would be the same as a translation of 12 units to the right. If the lines of reflection were switched and we reflected the triangle over $x=1$ followed by $x=-5$, then it would have been the same as a translation of 12 units to the ***left.***

Reflections over the x and y Axes

You can also reflect over intersecting lines. First, we will reflect over the x and y axes.

Example 6: Reflect $\triangle DEF$ from Example 5 over the $x-$axis, followed by the $y-$axis. Determine the coordinates of $\triangle D''E''F''$ and what one transformation this double reflection would be the same as.

Solution: $\triangle D''E''F''$ is the green triangle in the graph below. If we compare the coordinates of it to $\triangle DEF$, we have:



$$D(3,-1) \rightarrow D'(-3,1)$$
$$E(8,-3) \rightarrow E'(-8,3)$$
$$F(6,4) \rightarrow F'(-6,-4)$$

If you recall the rules of rotations from the previous section, this is the same as a rotation of 180°.

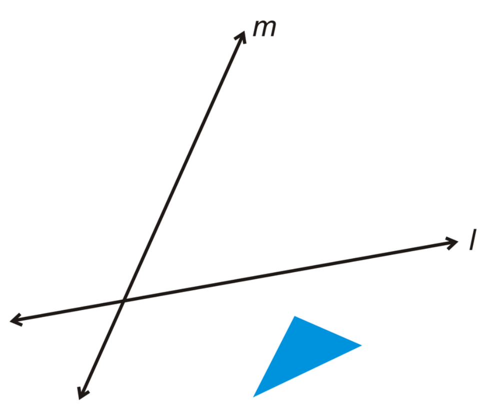
Reflection over the Axes Theorem: If you compose two reflections over each axis, then the final image is a rotation of 180° of the original.

With this particular composition, order does not matter. Let's look at the angle of intersection for these lines. We know that the axes are perpendicular, which means they intersect at a 90° angle. The final answer was a rotation of 180°, which is double 90°. Therefore, we could say that the composition of the reflections over each axis is a rotation of double their angle of intersection.

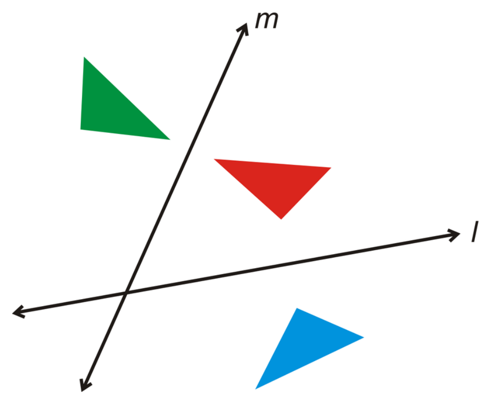
Reflections over Intersecting Lines

Now, we will take the concept we were just discussing and apply it to any pair of intersecting lines. For this composition, we are going to take it out of the coordinate plane. Then, we will apply the idea to a few lines in the coordinate plane, where the point of intersection will always be the origin.

Example 7: Copy the figure below and reflect it over l, followed by m.



Solution: The easiest way to reflect the triangle is to fold your paper on each line of reflection and draw the image. It should look like this:

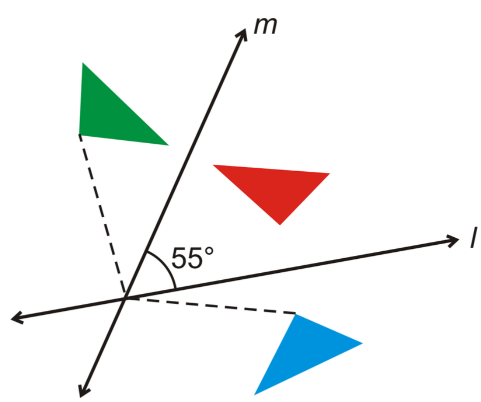


The green triangle would be the final answer.

Investigation 12-2: Double Reflection over Intersecting Lines

Tools Needed: Example 7, protractor, ruler, pencil

a. Take your answer from Example 7 and measure the angle of intersection for lines l and m. If you copied it exactly from the text, it should be about 55°.
b. Draw lines from two corresponding points on the blue triangle and the green triangle. These are the dotted lines in the diagram below.
c. Measure this angle using your protractor. How does it related to 55°?



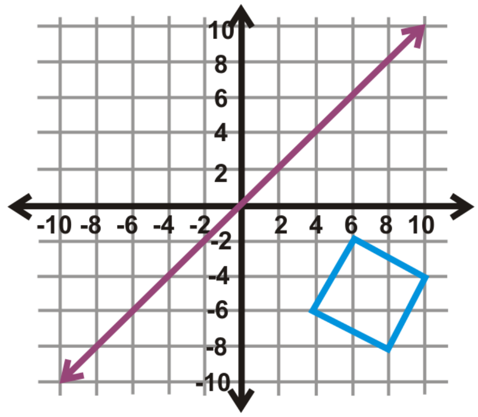
Again, if you copied the image exactly from the text, the angle should be 110°.

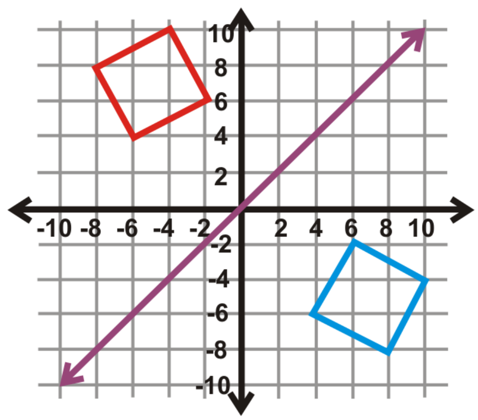
From this investigation, we see that the double reflection over two lines that intersect at a 55° angle is the same as a rotation of 110° counterclockwise, where the point of intersection is the center of rotation. Notice that order would matter in this composition. If we had reflected the blue triangle over m followed by l, then the green triangle would be rotated 110° clockwise.

Reflection over Intersecting Lines Theorem: If you compose two reflections over lines that intersect at x°, then the resulting image is a rotation of $2x^\circ$, where the center of rotation is the point of intersection.

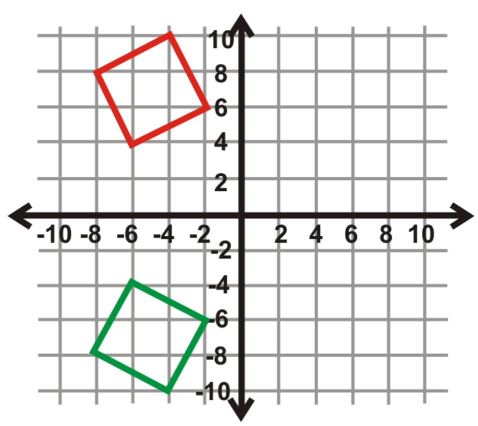
Notice that the Reflection over the Axes Theorem is a specific case of this one.

Example 8: Reflect the square over $y = x$, followed by a reflection over the $x-$axis.



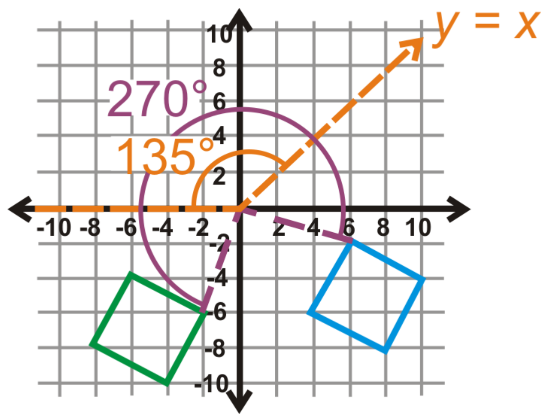


Solution: First, reflect the square over $y = x$. The answer is the red square in the graph above. Second, reflect the red square over the $x-$axis. The answer is the green square below.



Example 9: Determine the one rotation that is the same as the double reflection from Example 8.

Solution: Let's use the theorem above. First, we need to figure out what the angle of intersection is for $y = x$ and the $x-$axis. $y = x$ is halfway between the two axes, which are perpendicular, so is 45° from the $x-$axis. Therefore, the angle of rotation is 90° clockwise or 270° counterclockwise. The correct answer is 270° counterclockwise because we always measure angle of rotation in the coordinate plane in a counterclockwise direction. From the diagram, we could have also said the two lines are 135° apart, which is supplementary to 45°.

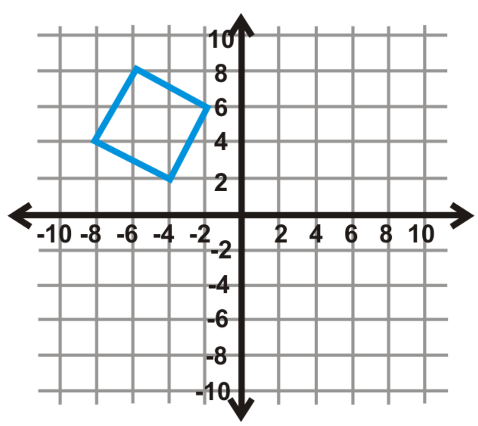


Know What? Revisited The average 6 foot tall man has a $0.415 \times 6 = 2.5$ foot stride. Therefore, the transformation rule for this person would be $(x, y) \rightarrow (-x, y + 2.5)$.

Review Questions

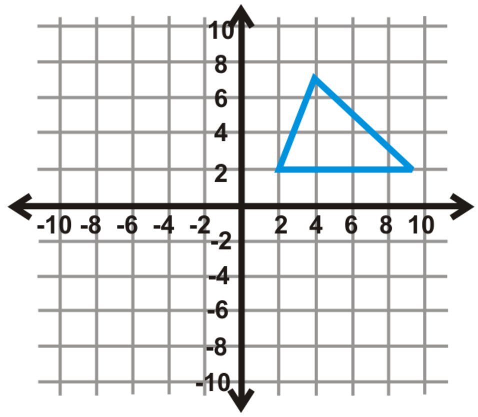
1. *Explain* why the composition of two or more isometries must also be an isometry.
2. What one transformation is equivalent to a reflection over two parallel lines?
3. What one transformation is equivalent to a reflection over two intersecting lines?

Use the graph of the square below to answer questions 4-7.



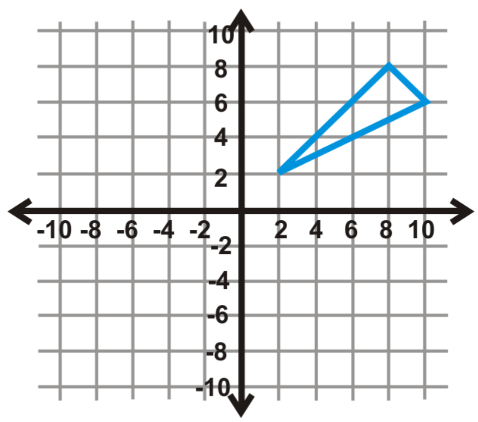
4. Perform a glide reflection over the $x-$axis and to the right 6 units. Write the new coordinates.
5. What is the rule for this glide reflection?
6. What glide reflection would move the image back to the preimage?
7. Start over. Would the coordinates of a glide reflection where you move the square 6 units to the right and then reflect over the $x-$axis be any different than #4? Why or why not?

Use the graph of the triangle below to answer questions 8-10.



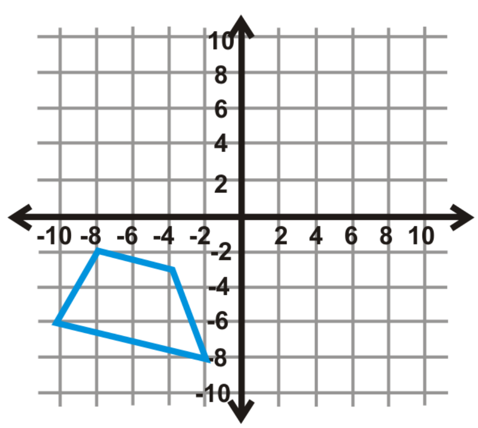
8. Perform a glide reflection over the $y-$axis and down 5 units. Write the new coordinates.
9. What is the rule for this glide reflection?
10. What glide reflection would move the image back to the preimage?

Use the graph of the triangle below to answer questions 11-15.



11. Reflect the preimage over $y = -1$ followed by $y = -7$. Write the new coordinates.
12. What one transformation is this double reflection the same as?
13. What one translation would move the image back to the preimage?
14. Start over. Reflect the preimage over $y = -7$, then $y = -1$. How is this different from #11?
15. Write the rules for #11 and #14. How do they differ?

Use the graph of the trapezoid below to answer questions 16-20.



16. Reflect the preimage over $y = -x$ then the $y-$axis. Write the new coordinates.
17. What one transformation is this double reflection the same as?
18. What one transformation would move the image back to the preimage?
19. Start over. Reflect the preimage over the $y-$axis, then $y = -x$. How is this different from #16?
20. Write the rules for #16 and #19. How do they differ?

Fill in the blanks or answer the questions below.

21. Two parallel lines are 7 units apart. If you reflect a figure over both how far apart with the preimage and final image be?
22. After a double reflection over parallel lines, a preimage and its image are 28 units apart. How far apart are the parallel lines?
23. A double reflection over the x and y axes is the same as a _________ of ________ $^\circ$.
24. What is the center of rotation for #23?
25. Two lines intersect at an 83° angle. If a figure is reflected over both lines, how far apart will the preimage and image be?
26. A preimage and its image are 244° apart. If the preimage was reflected over two intersected lines, at what angle did they intersect?
27. A rotation of 45° clockwise is the same as a rotation of ________ $^\circ$ counterclockwise.
28. After a double reflection over parallel lines, a preimage and its image are 62 units apart. How far apart are the parallel lines?
29. A figure is to the left of $x = a$. If it is reflected over $x = a$ followed by $x = b$ and $b > a$, then the preimage and image are _________ units apart and the image is to the _________ of the preimage.
30. A figure is to the left of $x = a$. If it is reflected over $x = b$ followed by $x = a$ and $b > a$, then the preimage and image are _________ units apart and the image is to the _________ of the preimage.

Review Queue Answers

a. $A'(-2,-8), B'(4,-5), C'(-4,-1), D'(-6,-6)$
b. $A''(2,-8), B''(8,-5), C''(0,-1), D''(-2,-6)$
c. $A'(2,8), B'(8,5), C''(0,1), D''(-2,6)$
d. The coordinates are the same as #2.

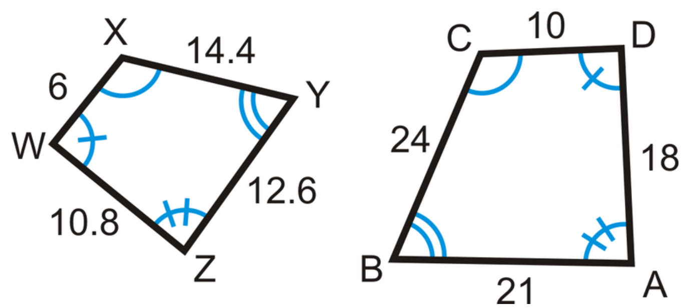
9.6 Dilations

Learning Objectives

- Draw a dilation of a given figure.
- Plot an image when given the center of dilation and scale factor.
- Determine if one figure is the dilation of another.

Review Queue

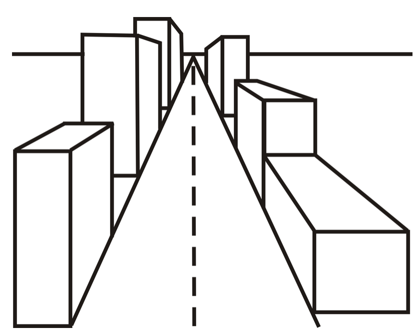
a. Are the two quadrilaterals similar? How do you know?



b. What is the scale factor from $XYZW$ to $CDAB$? *Leave as a fraction.*

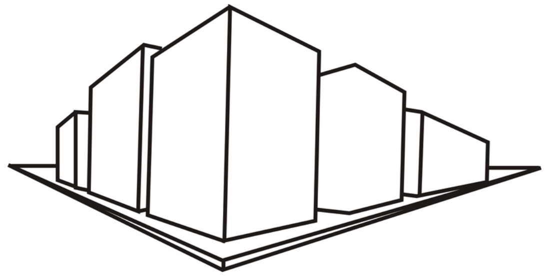
c. Quadrilateral $EFGH$ has vertices $E(-4,-2), F(2,8), G(6,2)$ and $H(0,-4)$. Quadrilateral $LMNO$ has vertices $L(-2,-1), M(1,4), N(3,1)$, and $O(0,-2)$. Determine if the two quadrilaterals are similar. Explain your reasoning.

Know What? One practical application of dilations is perspective drawings. These drawings use a *vanishing point* (the point where the road meets the horizon) to trick the eye into thinking the picture is three-dimensional. The picture to the right is a one-point perspective and is typically used to draw streets, train tracks, rivers or anything else that is linear.



There are also two-point perspective drawings, which are very often used to draw a street corner or a scale drawing of a building.

Both of these drawing are simple representations of one and two perspective drawings. Your task for this **Know What?** is to draw your own perspective drawing with either one or two vanishing points and at least 5 objects. Each object should have detail (windows, doors, sign, stairs, etc.)



Dilations

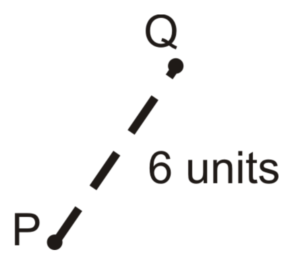
A dilation makes a figure larger or smaller, but has the same shape as the original. In other words, the dilation is similar to the original.

Transformation: An operation that moves, flips, or changes a figure to create a new figure. Transformations that preserve size are ***rigid*** and ones that do not are ***non-rigid.***

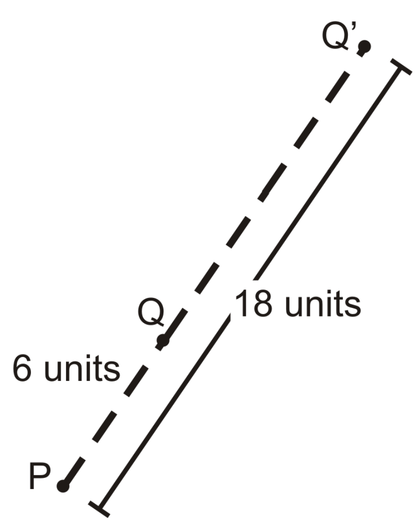
Dilation: A non-rigid transformation that preserves shape but not size.

All dilations have a **center** and a **scale factor.** The center is the point of reference for the dilation (like the vanishing point in a perspective drawing) and scale factor tells us how much the figure stretches or shrinks. A scale factor is typically labeled k and is always greater than zero. Also, if the original figure is labeled $\triangle ABC$, for example, the dilation would be $\triangle ABC$. The ' indicates that it is a copy. This tic mark is said "prime," so A is read "A prime." A second dilation would be A, read "A double-prime."

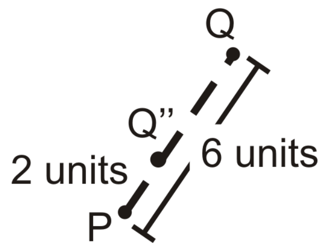
Example 1: The center of dilation is P and the scale factor is 3. Find Q.



Solution: If the scale factor is 3 and Q is 6 units away from P, then Q is going to be $6 \times 3 = 18$ units away from P. Because we are only dilating apoint, the dilation will be collinear with the original and center.

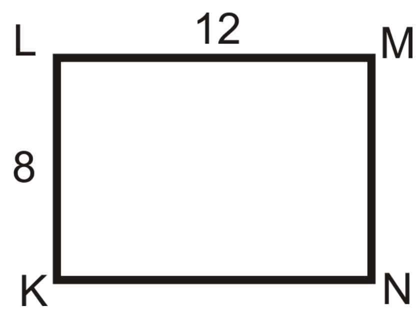


Example 2: Using the picture above, change the scale factor to $\frac{1}{3}$. Find Q.

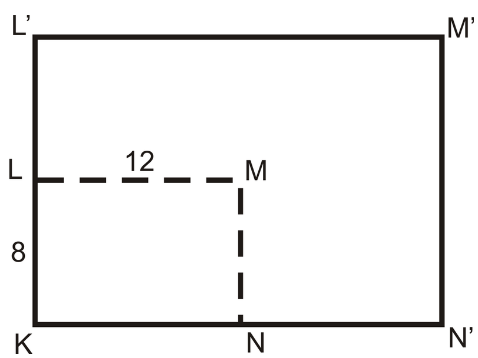


Solution: Now the scale factor is $\frac{1}{3}$, so Q is going to be $\frac{1}{3}$ the distance away from P as Q is. In other words, Q is going to be $6 \times \frac{1}{3} = 2$ units away from P. Q will also be collinear with Q and center.

Example 3: $KLMN$ is a rectangle with length 12 and width 8. If the center of dilation is K with a scale factor of 2, draw $KLMN$.



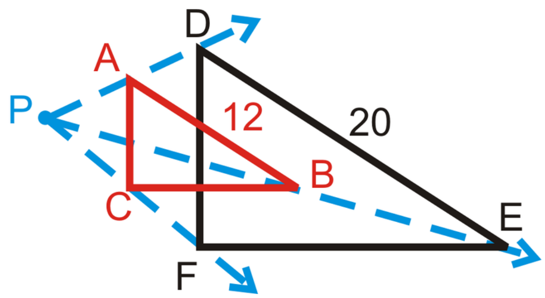
Solution: If K is the center of dilation, then K and K will be the same point. From there, L will be 8 units above L and N will be 12 units to the right of N.



Example 4: Find the perimeters of $KLMN$ and $KLMN$. Compare this to the scale factor.

Solution: The perimeter of $KLMN = 12 + 8 + 12 + 8 = 40$. The perimeter of $KLMN = 24 + 16 + 24 + 16 = 80$. The ratio of the perimeters is 80:40 or 2:1, which is the same as the scale factor.

Example 5: $\triangle ABC$ is a dilation of $\triangle DEF$. If P is the center of dilation, what is the scale factor?



Solution: Because $\triangle ABC$ is a dilation of $\triangle DEF$, we know that the triangles are similar. Therefore the scale factor is the ratio of the sides. Since $\triangle ABC$ is smaller than the original, $\triangle DEF$, the scale factor is going to be a fraction less than one, $\frac{12}{20} = \frac{3}{5}$.

If $\triangle DEF$ was the dilated image, the scale factor would have been $\frac{5}{3}$.

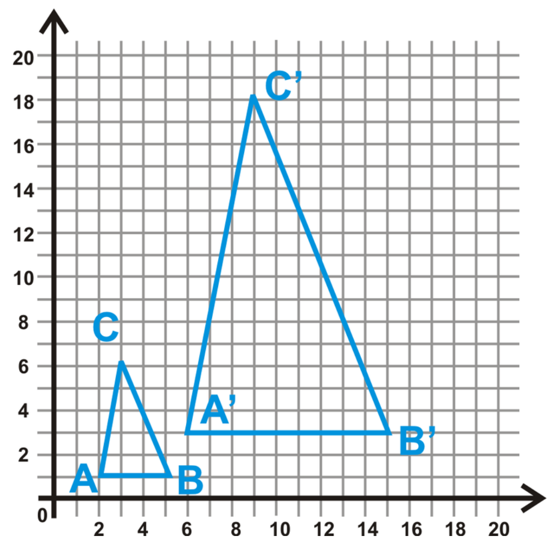
If the dilated image is smaller than the original, then the scale factor is $0 < k < 1$**.**

If the dilated image is larger than the original, then the scale factor is $k > 1$**.**

Dilations in the Coordinate Plane

In this text, the center of dilation will always be the origin, unless otherwise stated.

Example 6: Determine the coordinates of $\triangle ABC$ and $\triangle ABC$ and find the scale factor.



Solution: The coordinates of $\triangle ABC$ are $A(2,1), B(5,1)$ and $C(3,6)$. The coordinates of $\triangle ABC$ are $A(6,3), B(15,3)$ and $C(9,18)$. By looking at the corresponding coordinates, each is three times the original. That means $k = 3$.

Again, the center, original point, and dilated point are collinear. Therefore, you can draw a ray from the origin to C, B, and A such that the rays pass through C, B, and A, respectively.

Let's show that dilations are a similarity transformation (preserves shape). Using the distance formula, we will find the lengths of the sides of both triangles in Example 6 to demonstrate this.

$\underline{\triangle ABC}$	$\underline{\triangle A'B'C'}$
$AB = \sqrt{(2-5)^2+(1-1)^2} = \sqrt{9} = 3$	$A'B' = \sqrt{(6-15)^2+(3-3)^2} = \sqrt{81} = 9$
$AC = \sqrt{(2-3)^2+(1-6)^2} = \sqrt{26}$	$A'C' = \sqrt{(6-9)^2+(3-18)^2} = \sqrt{234} = 3\sqrt{26}$
$CB = \sqrt{(3-5)^2+(6-1)^2} = \sqrt{29}$	$C'B' = \sqrt{(9-15)^2+(18-3)^2} = \sqrt{261} = 3\sqrt{29}$

From this, we also see that all the sides of $\triangle A'B'C'$ are three times larger than $\triangle ABC$. Therefore, ***a dilation will always produce a similar shape to the original***.

In the coordinate plane, we say that A' is a "mapping" of A. So, if the scale factor is 3, then $A(2,1)$ is mapped to (usually drawn with an arrow) $A'(6,3)$. The entire mapping of $\triangle ABC$ can be written $(x,y) \rightarrow (3x,3y)$ because $k = 3$. ***For any dilation the mapping will be*** $(x,y) \rightarrow (kx,ky)$**.**

Know What? Revisited Answers to this project will vary depending on what you decide to draw. Make sure that you have at least five objects with some sort of detail. If you are having trouble getting started, go to the website: http://www.drawing-and-painting-techniques.com/drawing-perspective.html

Review Questions

Given A and the scale factor, determine the coordinates of the dilated point, A'. You may assume the center of dilation is the origin.

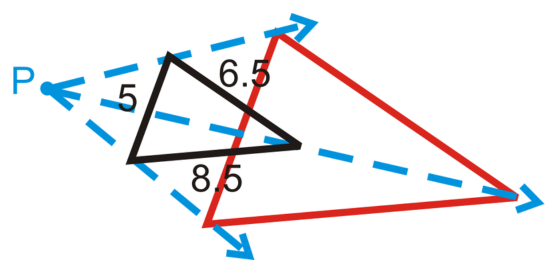
1. $A(3,9), k = \frac{2}{3}$
2. $A(-4,6), k = 2$
3. $A(9,-13), k = \frac{1}{2}$

Given A and A', find the scale factor. You may assume the center of dilation is the origin.

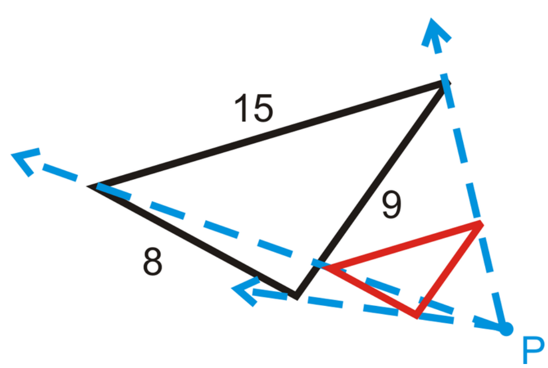
4. $A(8,2), A'(12,3)$
5. $A(-5,-9), A'(-45,-81)$
6. $A(22,-7), A(11,-3.5)$

In the two questions below, you are told the scale factor. Determine the dimensions of the dilation. In each diagram, the **black** figure is the original and P is the center of dilation.

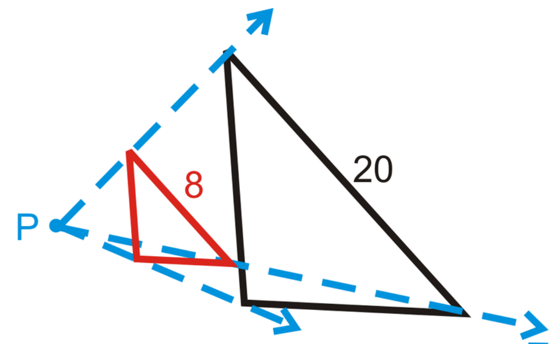
7. $k = 4$

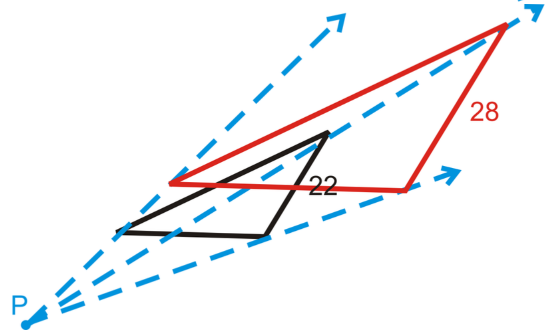


8. $k = \frac{1}{3}$



In the two questions below, find the scale factor, given the corresponding sides. In each diagram, the **black** figure is the original and P is the center of dilation.

9. 

10. 

11. Find the perimeter of both triangles in #7. What is the ratio of the perimeters?
12. ***Writing*** What happens if $k = 1$?

The origin is the center of dilation. Find the coordinates of the dilation of each figure, given the scale factor.

13. $A(2,4), B(-3,7), C(-1,-2); k = 3$
14. $A(12,8), B(-4,-16), C(0,10); k = \frac{3}{4}$

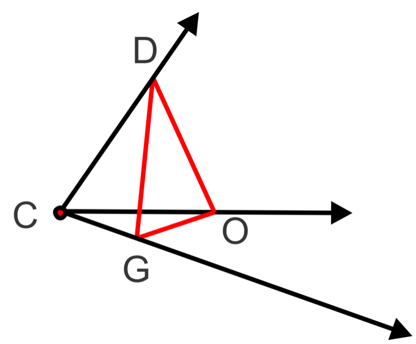
Multi-Step Problem Questions 15-21 build upon each other.

15. Plot $A(1,2), B(12,4), C(10,10)$. Connect to form a triangle.
16. Make the origin the center of dilation. Draw 4 rays from the origin to each point from #15. Then, plot $A'(2,4), B'(24,8), C'(20,20)$. What is the scale factor?
17. Use $k = 4$, to find $A''B''C''$. Plot these points.
18. What is the scale factor from $A'B'C'$ to $A''B''C''$?
19. Find ($O$is the origin):
    a. OA
    b. AA'
    c. AA''
    d. OA'
    e. OA''
20. Find:
    a. AB
    b. $A'B'$
    c. $A''B''$
21. Compare the ratios:
    a. $OA : OA'$ and $AB : A'B'$
    b. $OA : OA''$ and $AB : A''B''$

Algebra Connection For questions 22-27, use quadrilateral $ABCD$ with $A(1,5), B(2,6), C(3,3)$ and $D(1,3)$ and its transformation $A'B'C'D'$ with $A'(-3,1), B'(0,4), C'(3,-5)$ and $D'(-3,-5)$.

22. Plot the two quadrilaterals in the coordinate plane.
23. Find the equation of $\overleftrightarrow{CC'}$.
24. Find the equation of $\overleftrightarrow{DD'}$.
25. Find the intersection of these two lines algebraically or graphically.
26. What is the significance of this point?
27. What is the scale factor of the dilation?

Construction We can use a compass and straight edge to construct a dilation as well. Copy the diagram below.



28. Set your compass to be CG and use this setting to mark off a point 3 times as far from C as G is. Label this point G'. Repeat this process for CO and CD to find O' and D'.
29. Connect G', O' and D' to make $\triangle D'O'G'$. Find the ratios, $\frac{D'O'}{DO}$, $\frac{O'G'}{OG}$ and $\frac{G'D'}{GD}$.
30. What is the scale factor of this dilation?
31. Describe how you would dilate the figure by a scale factor of 4.
32. Describe how you would dilate the figure by a scale factor of $\frac{1}{2}$.

Review Queue Answers

a. Yes, all the angles are congruent and the corresponding sides are in the same ratio.
b. $\frac{5}{3}$
c. Yes, $LMNO \sim EFGH$ because $LMNO$ is exactly half of $EFGH$.

9.7 Tessellations

Learning Objectives

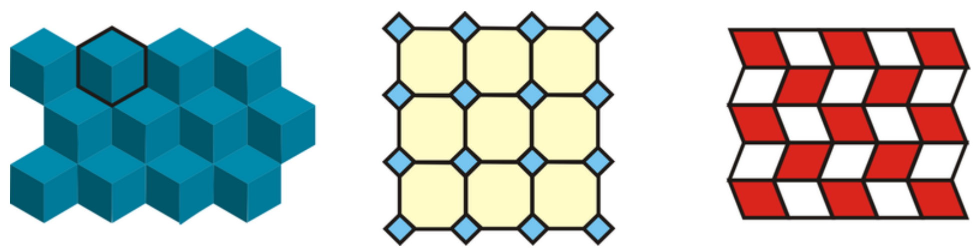
- Determine whether or not a given shape will tessellate.
- Draw your own tessellation.

What is a Tessellation?

You have probably seen tessellations before, even though you did not call them that. Examples of a tessellation are: a tile floor, a brick or block wall, a checker or chess board, and a fabric pattern.

Tessellation: A tiling over a plane with one or more figures such that the figures fill the plane with no overlaps and no gaps.

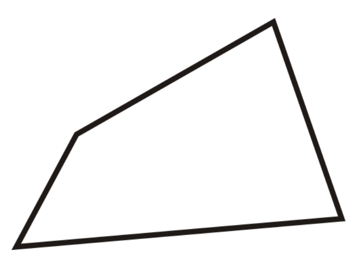
Here are a few examples.



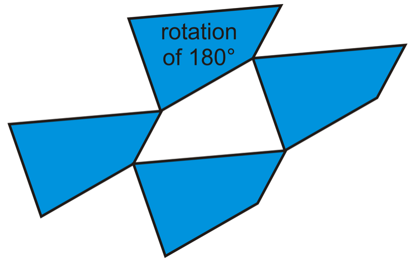
Notice the hexagon (cubes, first tessellation) and the quadrilaterals fit together perfectly. If we keep adding more, they will entirely cover the plane with no gaps or overlaps. The tessellation pattern could be colored creatively to make interesting and/or attractive patterns.

To tessellate a shape it must be able to exactly surround a point, or the sum of the angles around each point in a tessellation must be 360°. Therefore, every quadrilateral and hexagon will tessellate.

Example 1: Tessellate the quadrilateral below.

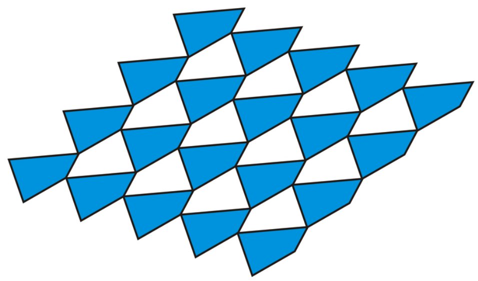


Solution: To tessellate any image you will need to reflect and rotate the image so that the sides all fit together. First, start by matching up each side with itself around the quadrilateral.



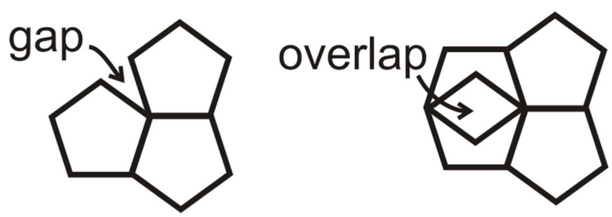
This is the final tessellation. You can continue to tessellate this shape forever.

Now, continue to fill in around the figures with either the original or the rotation.

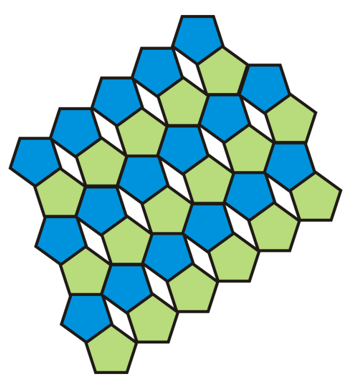


Example 2: Does a regular pentagon tessellate?

Solution: First, recall that there are $(5-2)180^\circ = 540^\circ$ in a pentagon and each angle is $540^\circ \div 5 = 108^\circ$. From this, we know that a regular pentagon will not tessellate by itself because $108^\circ \times 3 = 324^\circ$ and $108^\circ \times 4 = 432^\circ$.



For a shape to be tessellated, the angles around every point must add up to 360°. A regular pentagon does not tessellate by itself. But, if we add in another shape, a rhombus, for example, then the two shapes together will tessellate.

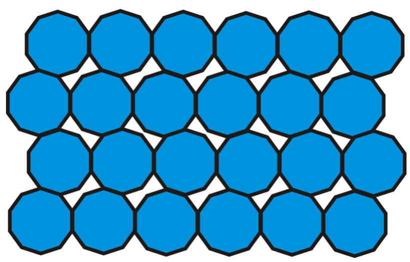


Tessellations can also be much more complicated. Here are a couple of examples.

Review Questions

Will the given shapes tessellate? If so, make a small drawing on grid paper to show the tessellation.

1. A square
2. A rectangle
3. A rhombus
4. A parallelogram
5. A trapezoid
6. A kite
7. A completely irregular quadrilateral
8. Which regular polygons will tessellate?
9. Use equilateral triangles and regular hexagons to draw a tessellation.
10. The blue shapes are regular octagons. Determine what type of polygon the white shapes are. Be as specific as you can.



11. Draw a tessellation using regular hexagons.
12. Draw a tessellation using octagons and squares.
13. Make a tessellation of an irregular quadrilateral using the directions from Example 1.

9.8 Chapter 9 Review

Keywords Theorems

Line of Symmetry
A line that passes through a figure such that it splits the figure into two congruent halves.

Line Symmetry
When a figure has one or more lines of symmetry.

Rotational Symmetry
When a figure can be rotated (less that 360°) and it looks the same way it did before the rotation.

Center of Rotation
The point at which the figure is rotated around such that the rotational symmetry holds. Typically, the center of rotation is the center of the figure.

angle of rotation
The angle of rotation, tells us how many degrees we can rotate a figure so that it still looks the same.

Transformation
An operation that moves, flips, or changes a figure to create a new figure.

Rigid Transformation
A transformation that preserves size and shape.

Translation
A transformation that moves every point in a figure the same distance in the same direction.

Vector
A quantity that has direction and size.

Reflection
A transformation that turns a figure into its mirror image by flipping it over a line.

Line of Reflection
The line that a figure is reflected over.

Reflection over the $y-$axis
If (x,y) is reflected over the $y-$axis, then the image is $(-x,y)$.

Reflection over the $x-$axis
If (x,y) is reflected over the $x-$axis, then the image is $(x,-y)$.

Reflection over $x=a$
If (x,y) is reflected over the vertical line $x=a$, then the image is $(2a-x,y)$.

Reflection over $y=b$
If (x,y) is reflected over the horizontal line $y=b$, then the image is $(x,2b-y)$.

Reflection over $y=x$
If (x,y) is reflected over the line $y=x$, then the image is (y,x).

Reflection over $y=-x$
If (x,y) is reflected over the line $y=-x$, then the image is $(-y,-x)$.

Rotation
A transformation by which a figure is turned around a fixed point to create an image.

Center of Rotation
The fixed point that a figure is rotated around.

Rotation of 180°
If (x,y) is rotated 180° around the origin, then the image will be $(-x,-y)$.

Rotation of 90°
If (x,y) is rotated 90° around the origin, then the image will be $(-y,x)$.

Rotation of 270°
If (x,y) is rotated 270° around the origin, then the image will be $(y,-x)$.

Composition (of transformations)
To perform more than one rigid transformation on a figure.

Glide Reflection
A composition of a reflection and a translation. The translation is in a direction parallel to the line of reflection.

Reflections over Parallel Lines Theorem
If you compose two reflections over parallel lines that are h units apart, it is the same as a single translation of $2h$ units.

Reflection over the Axes Theorem
If you compose two reflections over each axis, then the final image is a rotation of 180° of the original.

Reflection over Intersecting Lines Theorem
If you compose two reflections over lines that intersect at x°, then the resulting image is a rotation of $2x^\circ$, where the center of rotation is the point of intersection.

Tessellation
A tiling over a plane with one or more figures such that the figures fill the plane with no overlaps and no gaps.

Review Questions

Match the description with its rule.

1. Reflection over the $y-$axis - A. $(2a-x,y)$
2. Reflection over the $x-$axis - B. $(-y,-x)$
3. Reflection over $x=a$ - C. $(-x,y)$
4. Reflection over $y=b$ - D. $(-y,x)$
5. Reflection over $y=x$ - E. $(x,-y)$
6. Reflection over $y=-x$ - F. $(x,2b-y)$
7. Rotation of 180° - G. (x,y)
8. Rotation of 90° - H. $(-x,-y)$
9. Rotation of 270° - I. $(y,-x)$
10. Rotation of 360° - J. (y,x)

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9697 .

CHAPTER 10 Perimeter and Area

Chapter Outline

Now that we have explored triangles, quadrilaterals, polygons, and circles, we are going to learn how to find the perimeter and area of each. First we will derive each formula and then apply them to different types of polygons and circles. In addition, we will explore the properties of similar polygons, their perimeters and their areas.

10.1 Triangles and Parallelograms

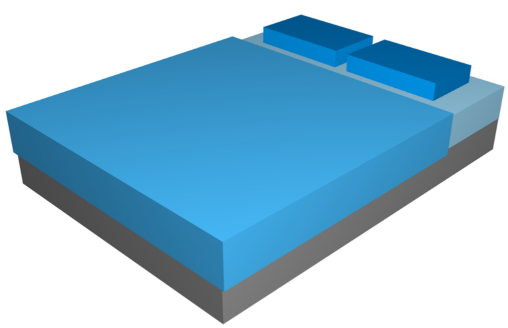
Learning Objectives

- Understand the basic concepts of area.
- Use formulas to find the area of triangles and parallelograms.

Review Queue

a. Define perimeter and area, in your own words.
b. Solve the equations below. Simplify any radicals.
   a. $x^2 = 121$
   b. $4x^2 = 80$
   c. $x^2 - 6x + 8 = 0$
c. If a rectangle has sides 4 and 7, what is the perimeter?

Know What? Ed's parents are getting him a new bed. He has decided that he would like a king bed. Upon further research, Ed discovered there are two types of king beds, an Eastern (or standard) King and a California King. The Eastern King has 76" $\times$ 80" dimensions, while the California King is 72" $\times$ 84" (both dimensions are *width* $\times$ *length*). Which bed has a larger area to lie on? Which one has a larger perimeter? If Ed is 6'4", which bed makes more sense for him to buy?



Areas and Perimeters of Squares and Rectangles

Perimeter: The distance around a shape. Or, the sum of all the edges of a two-dimensional figure.

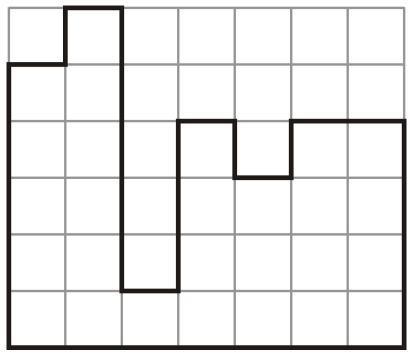
The perimeter of any figure must have a unit of measurement attached to it. If no specific units are given (feet, inches, centimeters, etc), write "units."

Example 1: Find the perimeter of the figure to the left.

Solution: First, notice there are no units, but the figure is on a grid. Here, we can use the grid as our units. Count around the figure to find the perimeter. We will start at the bottom left-hand corner and go around the figure clockwise.

$$5+1+1+1+5+1+3+1+1+1+1+2+4+7$$

The answer is 34 units.



You are probably familiar with the area of squares and rectangles from a previous math class. Recall that you must always establish a unit of measure for area. Area is always measured in square units, square feet $(ft.^2)$, square inches $(in.^2)$. square centimeters $(cm.^2)$, etc. Make sure that the length and width are in the same units before applying any area formula. If no specific units are given, write "$units^2$."

Example 2: Find the area of the figure from Example 1.

Solution: If the figure is not a standard shape, you can count the number of squares within the figure. If we start on the left and count each column, we would have:

$$5+6+1+4+3+4+4 = 27\ units^2$$

Area of a Rectangle: The area of a rectangle is the product of its base (width) and height (length) $A = bh$.

Example 3: Find the area and perimeter of a rectangle with sides 4 cm by 9 cm.

Solution: The perimeter is $4+9+4+9 = 36\ cm$. The area is $A = 9 \cdot 4 = 36\ cm^2$.

In this example we see that a formula can be generated for the perimeter of a rectangle.

Perimeter of a Rectangle: $P = 2b+2h$, where b is the base (or width) and h is the height (or length).

If a rectangle is a square, with sides of length s, the formulas are as follows:

Perimeter of a Square: $P_{square} = 2s + 2s = 4s$

Area of a Square: $A_{sqaure} = s \cdot s = s^2$

Example 4: The area of a square is 75 in^2. Find the perimeter.

Solution: To find the perimeter, we need to find the length of the sides.

$$A = s^2 = 75\ in^2$$
$$s = \sqrt{75} = 5\sqrt{3}\ in$$
$$\text{From this, } P = 4\left(5\sqrt{3}\right) = 20\sqrt{3}\ in.$$

Area Postulates

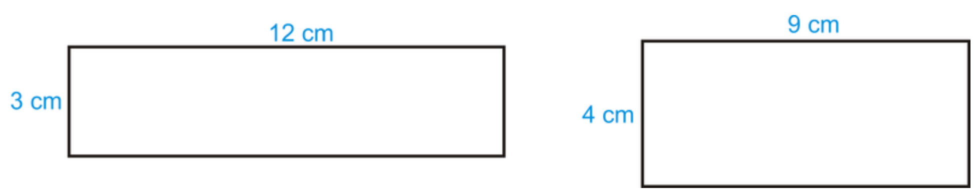
Congruent Areas Postulate: If two figures are congruent, they have the same area.

This postulate needs no proof because congruent figures have the same amount of space inside them. However, two figures with the same area are not necessarily congruent.

Example 5: Draw two different rectangles with an area of 36 cm^2.

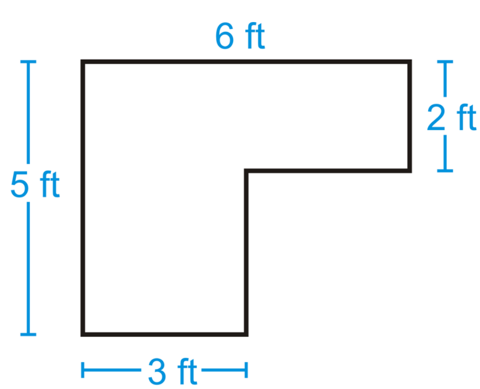
Solution: Think of all the different factors of 36. These can all be dimensions of the different rectangles.

Other possibilities could be $6 \times 6, 2 \times 18$, and 1×36.

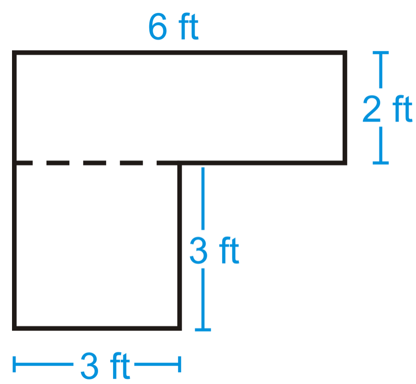


Area Addition Postulate: If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

Example 6: Find the area of the figure below. You may assume all sides are perpendicular.



Solution: Split the shape into two rectangles and find the area of each.

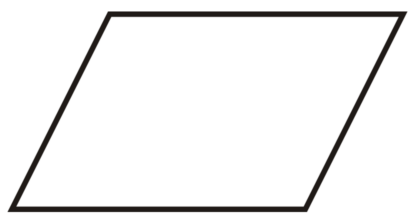


$$A_{top\ rectangle} = 6 \cdot 2 = 12\ ft^2$$
$$A_{bottom\ square} = 3 \cdot 3 = 9\ ft^2$$

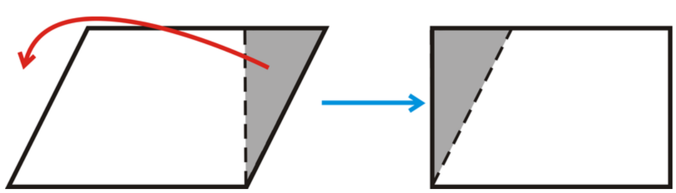
The total area is $12+9 = 21\ ft^2$.

Area of a Parallelogram

Recall that a parallelogram is a quadrilateral whose opposite sides are parallel.



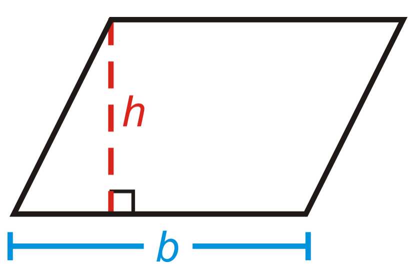
To find the area of a parallelogram, make it into a rectangle.



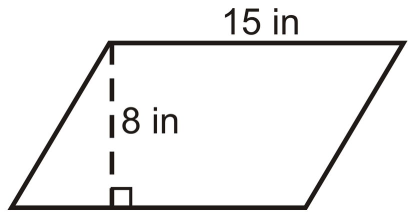
From this, we see that the area of a parallelogram is the same as the area of a rectangle.

Area of a Parallelogram: The area of a parallelogram is $A = bh$.

Be careful! The height of a parallelogram is always perpendicular to the base. This means that the sides are ***not*** the height.



Example 7: Find the area of the parallelogram.



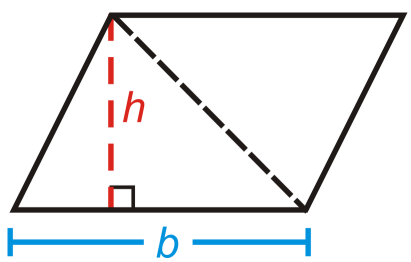
Solution: $A = 15 \cdot 8 = 120\ in^2$

Example 8: If the area of a parallelogram is 56 $units^2$ and the base is 4 units, what is the height?

Solution: Plug in what we know to the area formula and solve for the height.

$$\begin{aligned}56 &= 4h\\14 &= h\end{aligned}$$

Area of a Triangle

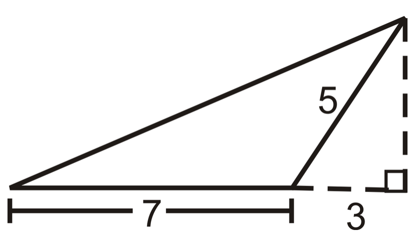


If we take parallelogram and cut it in half, along a diagonal, we would have two congruent triangles. Therefore, the formula for the area of a triangle is the same as the formula for area of a parallelogram, but cut in half.

Area of a Triangle: $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$.

In the case that the triangle is a right triangle, then the height and base would be the legs of the right triangle. If the triangle is an obtuse triangle, the altitude, or height, could be outside of the triangle.

Example 9: Find the area and perimeter of the triangle.

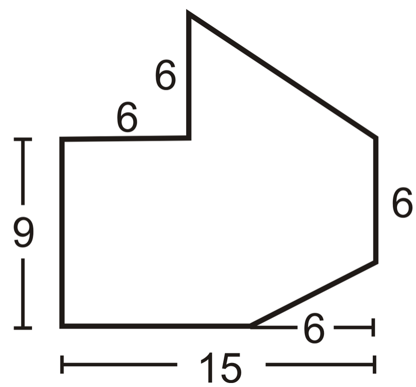


Solution: This is an obtuse triangle. First, to find the area, we need to find the height of the triangle. We are given the two sides of the small right triangle, where the hypotenuse is also the short side of the obtuse triangle. From these values, we see that the height is 4 because this is a 3-4-5 right triangle. The area is $A = \frac{1}{2}(4)(7) = 14\ units^2$.

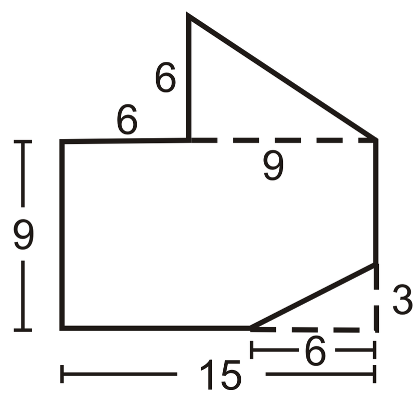
To find the perimeter, we would need to find the longest side of the obtuse triangle. If we used the dotted lines in the picture, we would see that the longest side is also the hypotenuse of the right triangle with legs 4 and 10. Use the Pythagorean Theorem.

$$\begin{aligned}4^2+10^2&=c^2\\16+100&=c^2\\c&=\sqrt{116}\approx 10.77 \qquad \text{The perimeter is } 7+5+10.77=22.77 \textit{ units}\end{aligned}$$

Example 10: Find the area of the figure below.



Solution: Divide the figure into a triangle and a rectangle with a small rectangle cut out of the lower right-hand corner.



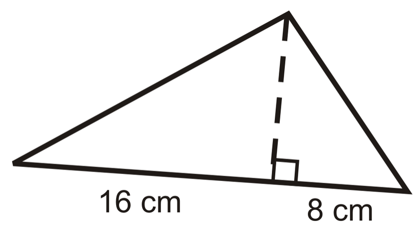
$$\begin{aligned}A&=A_{top\ triangle}+A_{rectangle}-A_{small\ triangle}\\A&=\left(\frac{1}{2}\cdot 6\cdot 9\right)+(9\cdot 15)+\left(\frac{1}{2}\cdot 3\cdot 6\right)\\A&=27+135+9\\A&=171\ units^2\end{aligned}$$

Know What? Revisited The area of an Eastern King is 6080 in^2 and the California King is 6048 in^2. The perimeter of both beds is 312 in. Because Ed is 6'4", he should probably get the California King because it is 4 inches longer.

Review Questions

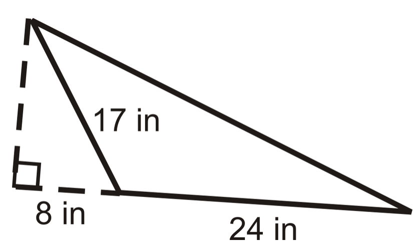
1. Find the area and perimeter of a square with sides of length 12 in.
2. Find the area and perimeter of a rectangle with height of 9 cm and base of 16 cm.
3. Find the area of a parallelogram with height of 20 m and base of 18 m.
4. Find the area and perimeter of a rectangle if the height is 8 and the base is 14.
5. Find the area and perimeter of a square if the sides are 18 ft.
6. If the area of a square is 81 ft^2, find the perimeter.
7. If the perimeter of a square is 24 in, find the area.
8. Find the area of a triangle with base of length 28 cm and height of 15 cm.
9. What is the height of a triangle with area 144 m^2 and a base of 24 m?
10. The perimeter of a rectangle is 32. Find two different dimensions that the rectangle could be.
11. Draw two different rectangles that haven an area of 90 mm^2.
12. Write the converse of the Congruent Areas Postulate. Determine if it is a true statement. If not, write a counterexample. If it is true, explain why.

Use the triangle to answer the following questions.



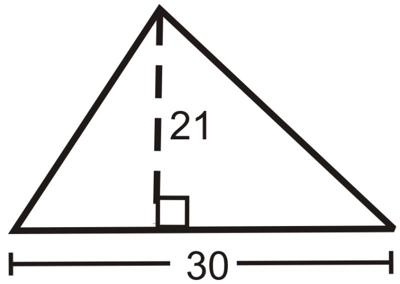
13. Find the height of the triangle by using the geometric mean.
14. Find the perimeter.
15. Find the area.

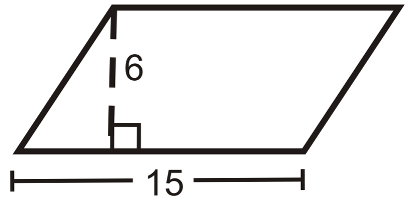
Use the triangle to answer the following questions.

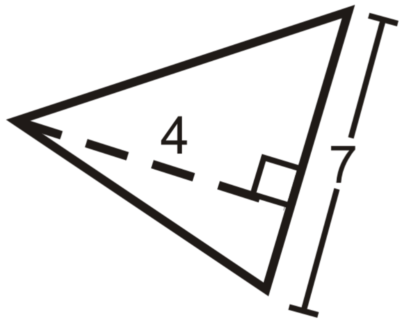


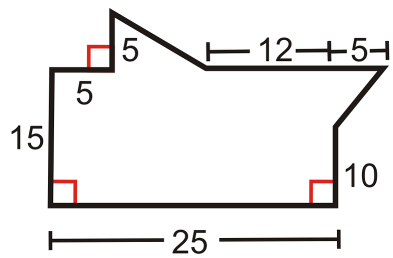
16. Find the height of the triangle.
17. Find the perimeter.
18. Find the area.

Find the area of the following shapes.

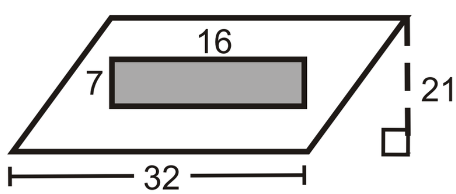
19. 

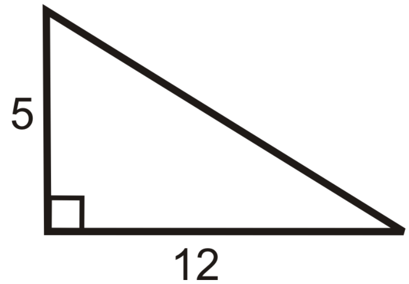
20. 

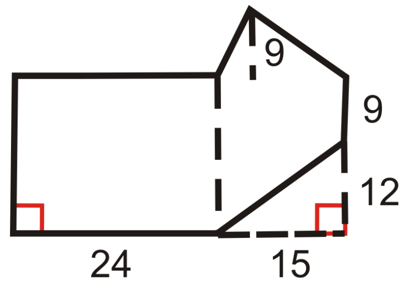
21. 

22. 

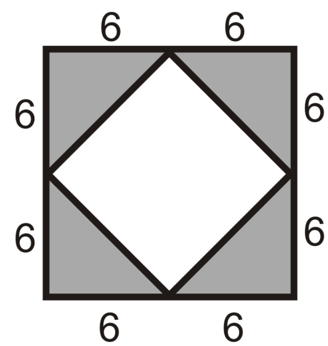
23. Find the area of the unshaded region.



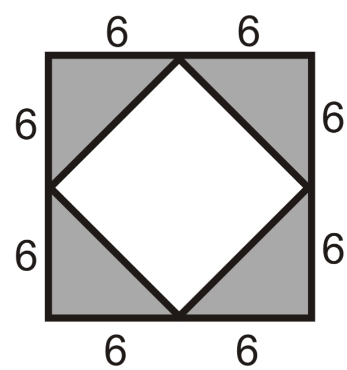
24. 

25. 

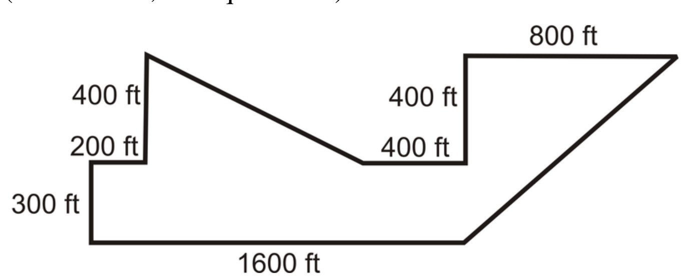
26. Find the area of the shaded region.



27. Find the area of the unshaded region.



28. Lin bought a tract of land for a new apartment complex. The drawing below shows the measurements of the sides of the tract. Approximately how many acres of land did Lin buy? You may assume any angles that look like right angles are 90°. (1 acre ≈ 40,000 square feet)



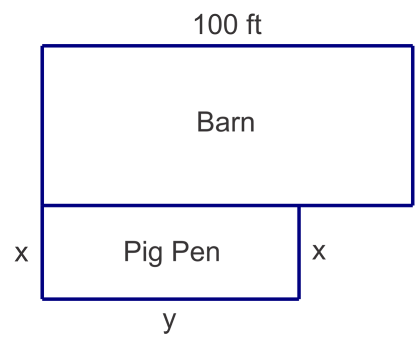
Challenge Problems

For problems 29 and 30 find the dimensions of the rectangles with the given information.

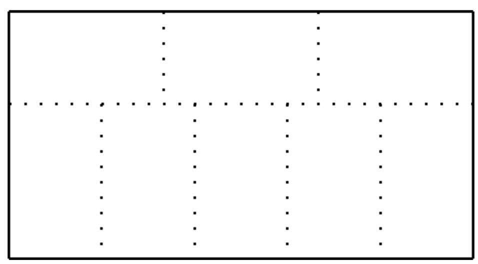
29. A rectangle with a perimeter of 20 units and an area of 24 $units^2$.
30. A rectangle with a perimeter of 72 units and an area of 288 $units^2$.

For problems 31 and 32 find the height and area of the equilateral triangle with the given perimeter.

31. Perimeter 18 units.
32. Perimeter 30 units.
33. Generalize your results from problems 31 and 32 into a formula to find the height and area of an equilateral triangle with side length x.
34. Linus has 100 ft of fencing to use in order to enclose a 1200 square foot rectangular pig pen. The pig pen is adjacent to the barn so he only needs to form three sides of the rectangular area as shown below. What dimensions should the pen be?



35. A rectangle with perimeter 138 units is divided into 8 congruent rectangles as shown in the diagram below. Find the perimeter and area of one of the 8 congruent rectangles.



Review Queue Answers

1. *Possible* Answers

Perimeter: The distance around a shape.

Area: The space inside a shape.

2. (a) $x = \pm 11$

(b) $x = \pm 2\sqrt{5}$

(c) $x = 4, 2$

3. $4 + 4 + 7 + 7 = 22$

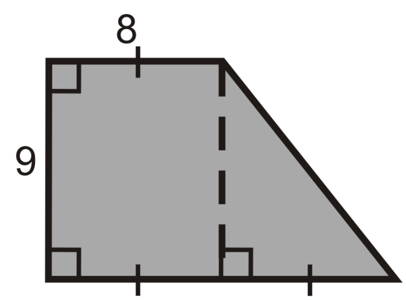
10.2 Trapezoids, Rhombi, and Kites

Learning Objectives

- Derive and use the area formulas for trapezoids, rhombi, and kites.

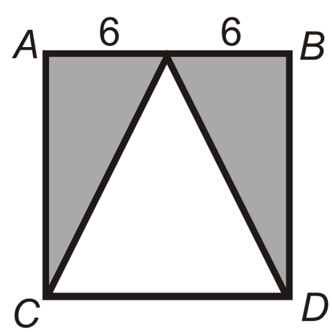
Review Queue

Find the area the *shaded* regions in the figures below.

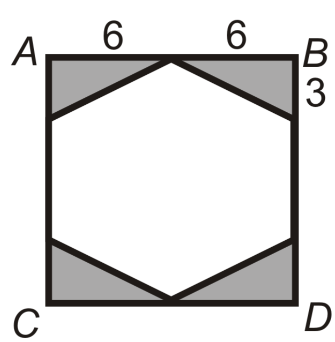


a.

b. *ABCD* is a square.



c. *ABCD* is a square.



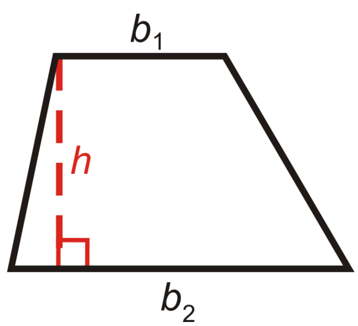
d. Find the area of #1 using a different method.

Know What? The Brazilian flag is to the right. The flag has dimensions of 20×14 (units vary depending on the size, so we will not use any here). The vertices of the yellow rhombus in the middle are 1.7 units from the midpoint of each side.

Find the total area of the flag and the area of the rhombus (including the circle). *Do not round your answers.*

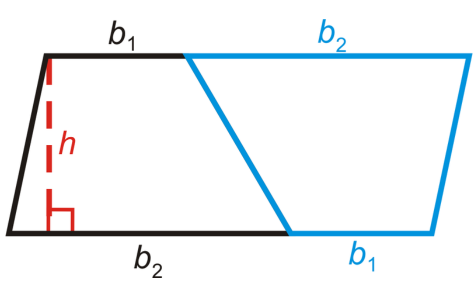
Area of a Trapezoid

Recall that a trapezoid is a quadrilateral with one pair of parallel sides. The lengths of the parallel sides are the bases. The perpendicular distance between the parallel sides is the height, or altitude, of the trapezoid.



To find the area of the trapezoid, let's turn it into a parallelogram. To do this, make a copy of the trapezoid and then rotate the copy 180°.

Now, this is a parallelogram with height h and base $b_1 + b_2$. Let's find the area of this shape.



$$A = h(b_1 + b_2)$$

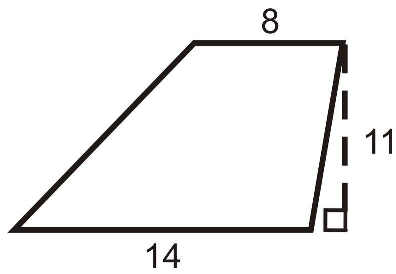
Because the area of this parallelogram is made up of two congruent trapezoids, the area of one trapezoid would be $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Trapezoid: The area of a trapezoid with height h and bases b_1 and b_2 is $A = \frac{1}{2}h(b_1 + b_2)$.

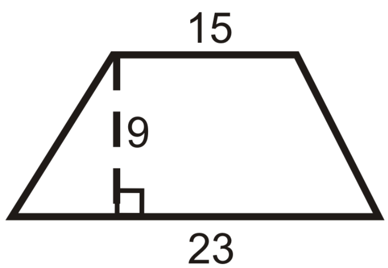
The formula for the area of a trapezoid could also be written as *the average of the bases time the height.*

Example 1: Find the area of the trapezoids below.

a)



b)

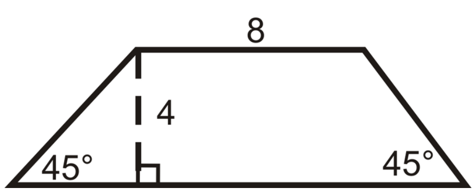


Solution:

a) $A = \frac{1}{2}(11)(14+8)$
$A = \frac{1}{2}(11)(22)$
$A = 121 \ units^2$

b) $A = \frac{1}{2}(9)(15+23)$
$A = \frac{1}{2}(9)(38)$
$A = 171 \ units^2$

Example 2: Find the perimeter and area of the trapezoid.



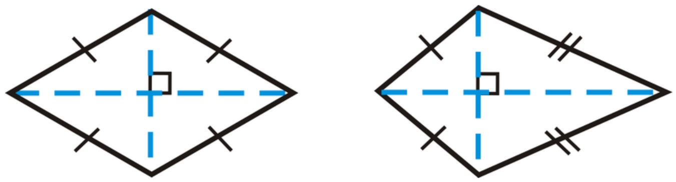
Solution: Even though we are not told the length of the second base, we can find it using special right triangles. Both triangles at the ends of this trapezoid are isosceles right triangles, so the hypotenuses are $4\sqrt{2}$ and the other legs are of length 4.

$$P = 8 + 4\sqrt{2} + 16 + 4\sqrt{2} \qquad A = \frac{1}{2}(4)(8+16)$$
$$P = 24 + 8\sqrt{2} \approx 35.3 \ units \qquad A = 48 \ units^2$$

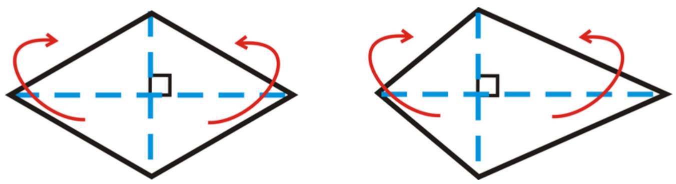
Area of a Rhombus and Kite

Recall that a rhombus is an equilateral quadrilateral and a kite has adjacent congruent sides.

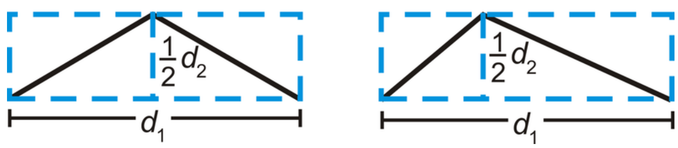
Both of these quadrilaterals have perpendicular diagonals, which is how we are going to find their areas.



Notice that the diagonals divide each quadrilateral into 4 triangles. In the rhombus, all 4 triangles are congruent and in the kite there are two sets of congruent triangles. If we move the two triangles on the bottom of each quadrilateral so that they match up with the triangles above the horizontal diagonal, we would have two rectangles.



So, the height of these rectangles is half of one of the diagonals and the base is the length of the other diagonal.



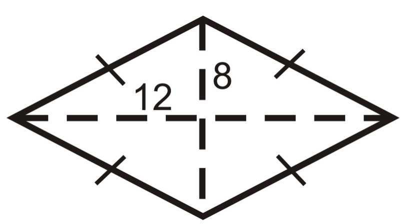
Area of a Rhombus: If the diagonals of a rhombus are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

Area of a Kite: If the diagonals of a kite are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

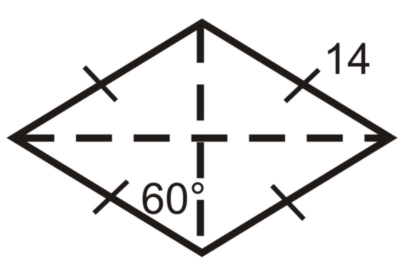
You could also say that the area of a kite and rhombus are *half the product of the diagonals.*

Example 3: Find the perimeter and area of the rhombi below.

a)



b)



Solution: In a rhombus, all four triangles created by the diagonals are congruent.

a) To find the perimeter, you must find the length of each side, which would be the hypotenuse of one of the four triangles. Use the Pythagorean Theorem.

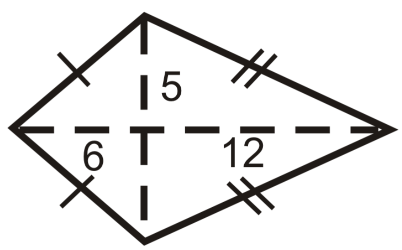
$$\begin{aligned} 12^2+8^2 &= side^2 \\ 144+64 &= side^2 \\ side &= \sqrt{208} = 4\sqrt{13} \\ P &= 4\left(4\sqrt{13}\right) = 16\sqrt{13} \end{aligned} \qquad \begin{aligned} A &= \frac{1}{2}\cdot 16\cdot 24 \\ A &= 192 \end{aligned}$$

b) Here, each triangle is a 30-60-90 triangle with a hypotenuse of 14. From the special right triangle ratios the short leg is 7 and the long leg is $7\sqrt{3}$.

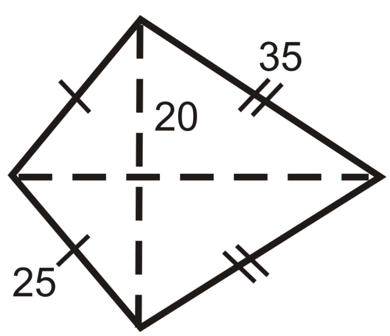
$$P = 4\cdot 14 = 56 \qquad A = \frac{1}{2}\cdot 7\cdot 7\sqrt{3} = \frac{49\sqrt{3}}{2} \approx 42.44$$

Example 4: Find the perimeter and area of the kites below.

a)



b)



Solution: In a kite, there are two pairs of congruent triangles. You will need to use the Pythagorean Theorem in both problems to find the length of sides or diagonals.

a)

Shorter sides of kite

$$6^2+5^2=s_1^2$$
$$36+25=s_1^2$$
$$s_1=\sqrt{61}$$
$$P=2\left(\sqrt{61}\right)+2(13)=2\sqrt{61}+26\approx 41.6$$
$$A=\frac{1}{2}(10)(18)=90$$

Longer sides of kite

$$12^2+5^2=s_2^2$$
$$144+25=s_2^2$$
$$s_2=\sqrt{169}=13$$

b)

Smaller diagonal portion

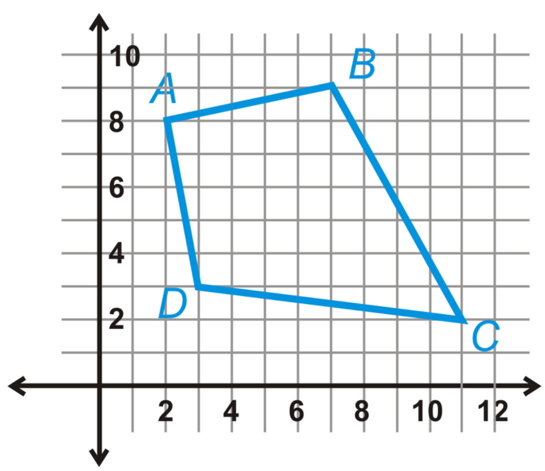
$$20^2+d_s^2=25^2$$
$$d_s^2=225$$
$$d_s=15$$
$$P=2(25)+2(35)=120$$
$$A=\frac{1}{2}\left(15+5\sqrt{33}\right)(40)\approx 874.5$$

Larger diagonal portion

$$20^2+d_l^2=35^2$$
$$d_l^2=825$$
$$d_l=5\sqrt{33}$$

Example 5: The vertices of a quadrilateral are $A(2,8), B(7,9), C(11,2)$, and $D(3,3)$. Determine the type of quadrilateral and find its area.

Solution: For this problem, it might be helpful to plot the points. From the graph we can see this is probably a kite. Upon further review of the sides, $AB=AD$ and $BC=DC$ (you can do the distance formula to verify). Let's see if the diagonals are perpendicular by calculating their slopes.



$$m_{AC}=\frac{2-8}{11-2}=-\frac{6}{9}=-\frac{2}{3}$$
$$m_{BD}=\frac{9-3}{7-3}=\frac{6}{4}=\frac{3}{2}$$

Yes, the diagonals are perpendicular because the slopes are opposite signs and reciprocals. $ABCD$ is a kite. To find the area, we need to find the length of the diagonals. Use the distance formula.

$$\begin{aligned} d_1 &= \sqrt{(2-11)^2+(8-2)^2} \\ &= \sqrt{(-9)^2+6^2} \\ &= \sqrt{81+36} = \sqrt{117} = 3\sqrt{13} \end{aligned} \qquad \begin{aligned} d_2 &= \sqrt{(7-3)^2+(9-3)^2} \\ &= \sqrt{4^2+6^2} \\ &= \sqrt{16+36} = \sqrt{52} = 2\sqrt{13} \end{aligned}$$

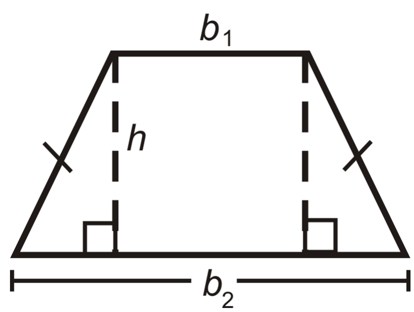
Now, plug these lengths into the area formula for a kite.

$$A = \frac{1}{2}\left(3\sqrt{13}\right)\left(2\sqrt{13}\right) = 39\ units^2$$

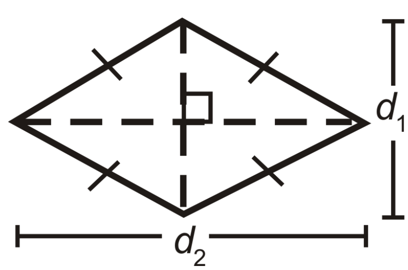
Know What? Revisited The total area of the Brazilian flag is $A = 14 \cdot 20 = 280\ units^2$. To find the area of the rhombus, we need to find the length of the diagonals. One diagonal is $20-1.7-1.7 = 16.6\ units$ and the other is $14-1.7-1.7 = 10.6\ units$. The area is $A = \frac{1}{2}(16.6)(10.6) = 87.98\ units^2$.

Review Questions

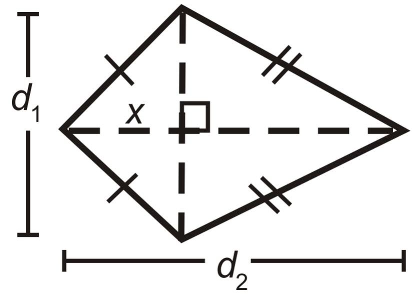
1. Do you think all rhombi and kites with the same diagonal lengths have the same area? *Explain* your answer.
2. Use the isosceles trapezoid to show that the area of this trapezoid can also be written as the sum of the area of the two triangles and the rectangle in the middle. Write the formula and then reduce it to equal $\frac{1}{2}h(b_1+b_2)$ or $\frac{h}{2}(b_1+b_2)$.



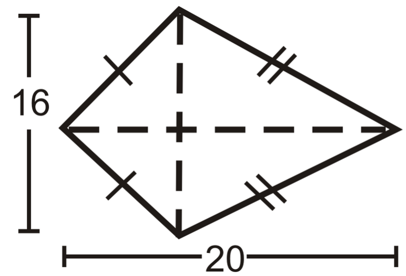
3. Use this picture of a rhombus to show that the area of a rhombus is equal to the sum of the areas of the four congruent triangles. Write a formula and reduce it to equal $\frac{1}{2}d_1d_2$.

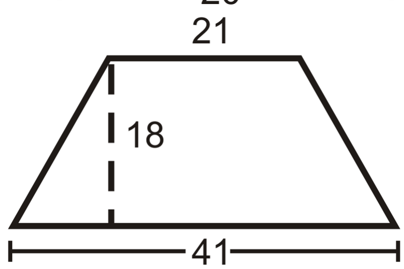


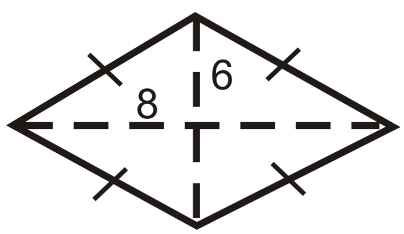
4. Use this picture of a kite to show that the area of a kite is equal to the sum of the areas of the two pairs of congruent triangles. Recall that d_1 is bisected by d_2. Write a formula and reduce it to equal $\frac{1}{2}d_1d_2$.

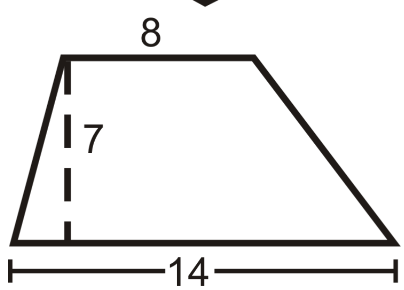


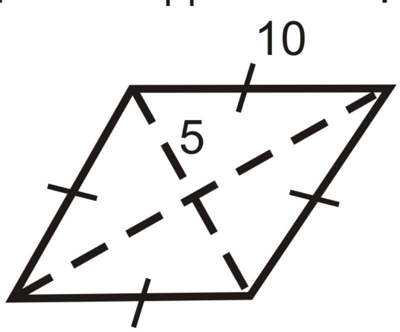
Find the area of the following shapes. *Leave answers in simplest radical form.*

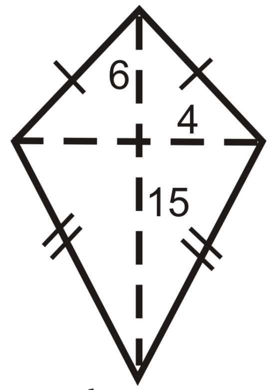
5. 

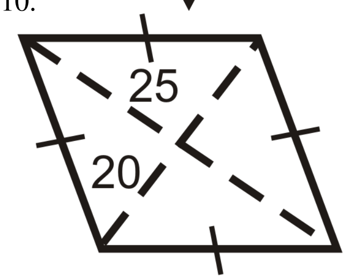
6. 

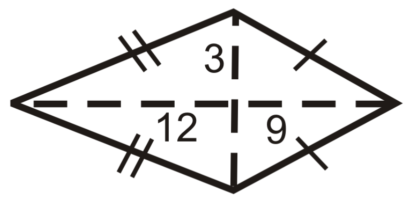
7. 

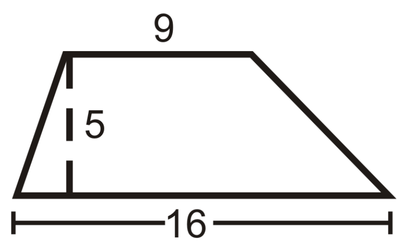
8. 

9. 

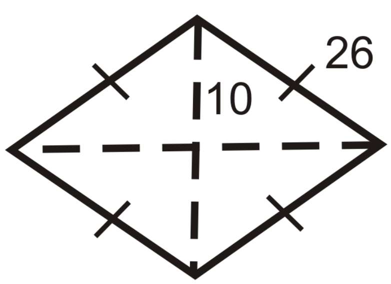
10. 

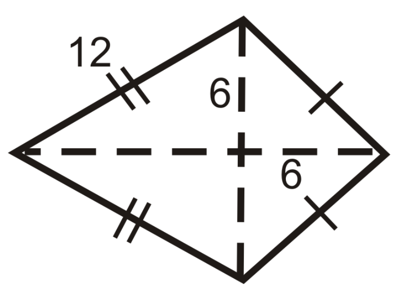
11. 

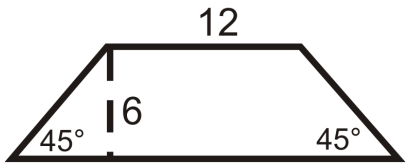
12. 

13. 

Find the area <u>and</u> perimeter of the following shapes. *Leave answers in simplest radical form.*

14. 

15. 



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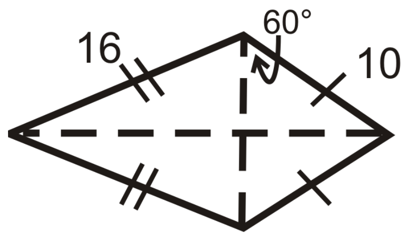

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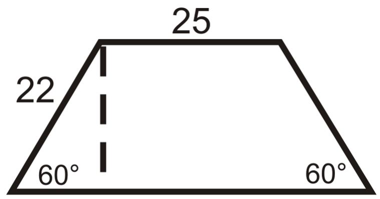

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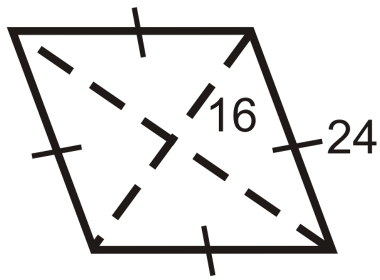
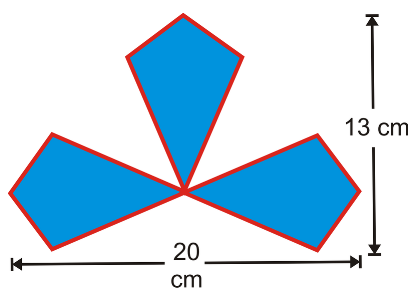


19.

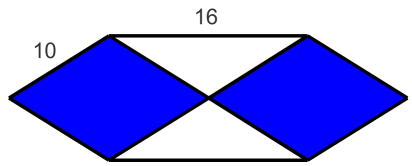
20. Quadrilateral $ABCD$ has vertices $A(-2,0), B(0,2), C(4,2)$, and $D(0,-2)$. Show that $ABCD$ is a trapezoid and find its area. *Leave your answer in simplest radical form.*
21. Quadrilateral $EFGH$ has vertices $E(2,-1), F(6,-4), G(2,-7)$, and $H(-2,-4)$. Show that $EFGH$ is a rhombus and find its area.
22. The area of a rhombus is 32 $units^2$. What are two possibilities for the lengths of the diagonals?
23. The area of a kite is 54 $units^2$. What are two possibilities for the lengths of the diagonals?
24. Sherry designed the logo for a new company. She used three congruent kites. What is the area of the entire logo?



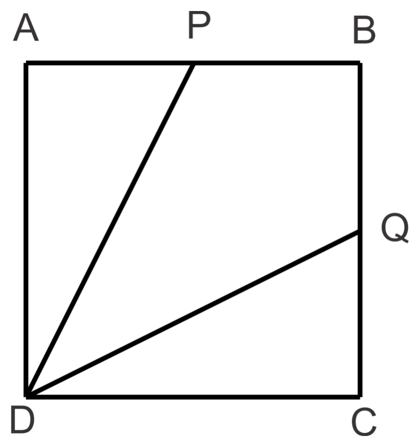
For problems 25-27, determine what kind of quadrilateral $ABCD$ is and find its area.

25. $A(-2,3), B(2,3), C(4,-3), D(-2,-1)$
26. $A(0,1), B(2,6), C(8,6), D(13,1)$
27. $A(-2,2), B(5,6), C(6,-2), D(-1,-6)$
28. Given that the lengths of the diagonals of a kite are in the ratio 4:7 and the area of the kite is 56 square units, find the lengths of the diagonals.
29. Given that the lengths of the diagonals of a rhombus are in the ratio 3:4 and the area of the rhombus is 54 square units, find the lengths of the diagonals.

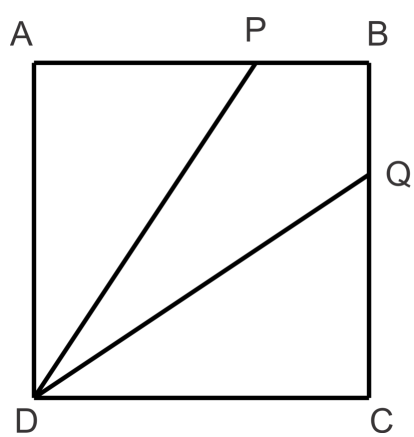
30. Sasha drew this plan for a wood inlay he is making. 10 is the length of the slanted side and 16 is the length of the horizontal line segment as shown in the diagram. Each shaded section is a rhombus. What is the total area of the shaded sections?



31. In the figure to the right, $ABCD$ is a square. $AP = PB = BQ$ and $DC = 20\ ft$.
    a. What is the area of $PBQD$?
    b. What is the area of $ABCD$?
    c. What fractional part of the area of $ABCD$ is $PBQD$?



32. In the figure to the right, $ABCD$ is a square. $AP = 20\ ft$ and $PB = BQ = 10\ ft$.
    a. What is the area of $PBQD$?
    b. What is the area of $ABCD$?
    c. What fractional part of the area of $ABCD$ is $PBQD$?



Review Queue Answers

a. $A = 9(8) + \left[\frac{1}{2}(9)(8)\right] = 72 + 36 = 108\ units^2$
b. $A = \frac{1}{2}(6)(12)2 = 72\ units^2$
c. $A = 4\left[\frac{1}{2}(6)(3)\right] = 36\ units^2$
d. $A = 9(16) - \left[\frac{1}{2}(9)(8)\right] = 144 - 36 = 108\ units^2$

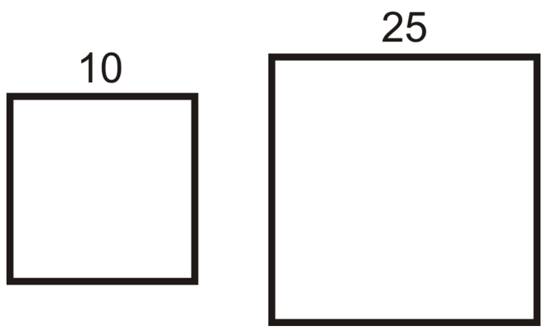
10.3 Areas of Similar Polygons

Learning Objectives

- Understand the relationship between the scale factor of similar polygons and their areas.
- Apply scale factors to solve problems about areas of similar polygons.

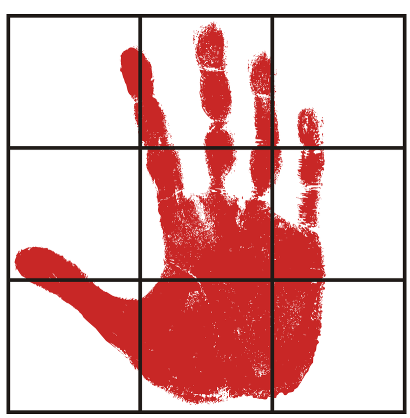
Review Queue

a. Are two squares similar? Are two rectangles?



b. Find the scale factor of the sides of the similar shapes. Both figures are squares.
c. Find the area of each square.
d. Find the ratio of the smaller square's area to the larger square's area. Reduce it. How does it relate to the scale factor?

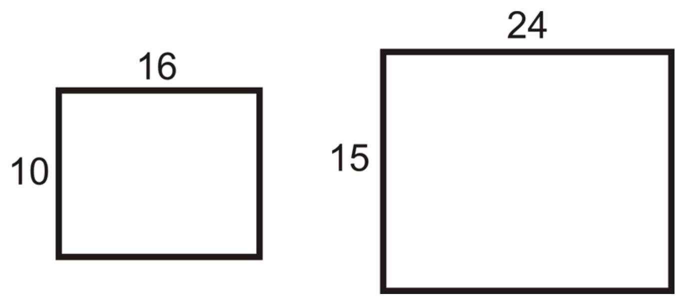
Know What? One use of scale factors and areas is scale drawings. This technique takes a small object, like the handprint to the right, divides it up into smaller squares and then blows up the individual squares. In this Know What? you are going to make a scale drawing of your own hand. Either trace your hand or stamp it on a piece of paper. Then, divide your hand into 9 squares, like the one to the right, probably $2\ in \times 2\ in$. Take a larger piece of paper and blow up each square to be $6\ in \times 6\ in$ (meaning you need at least an 18 in square piece of paper). Once you have your $6\ in \times 6\ in$ squares drawn, use the proportions and area to draw in your enlarged handprint.



Areas of Similar Polygons

In Chapter 7, we learned about similar polygons. Polygons are similar when the corresponding angles are equal and the corresponding sides are in the same proportion. In that chapter we also discussed the relationship of the perimeters of similar polygons. Namely, the scale factor for the sides of two similar polygons is the same as the ratio of the perimeters.

Example 1: The two rectangles below are similar. Find the scale factor and the ratio of the perimeters.



Solution: The scale factor is $\frac{16}{24}$, which reduces to $\frac{2}{3}$. The perimeter of the smaller rectangle is 52 units. The perimeter of the larger rectangle is 78 units. The ratio of the perimeters is $\frac{52}{78} = \frac{2}{3}$.

The ratio of the perimeters is the same as the scale factor. In fact, the ratio of any part of two similar shapes (diagonals, medians, midsegments, altitudes, etc.) is the same as the scale factor.

Example 2: Find the area of each rectangle from Example 1. Then, find the ratio of the areas.

Solution:

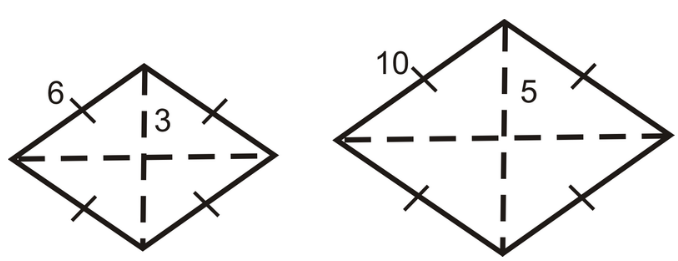
$$A_{small} = 10 \cdot 16 = 160 \ units^2$$
$$A_{large} = 15 \cdot 24 = 360 \ units^2$$

The ratio of the areas would be $\frac{160}{360} = \frac{4}{9}$.

The ratio of the sides, or scale factor was $\frac{2}{3}$ and the ratio of the areas is $\frac{4}{9}$. Notice that the ratio of the areas is the ***square*** of the scale factor. An easy way to remember this is to think about the units of area, which are always ***squared.*** Therefore, you would always ***square*** the scale factor to get the ratio of the areas.

Area of Similar Polygons Theorem: If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratio of the areas would be $\left(\frac{m}{n}\right)^2$.

Example 2: Find the ratio of the areas of the rhombi below. The rhombi are similar.



Solution: There are two ways to approach this problem. One way would be to use the Pythagorean Theorem to find the length of the 3^{rd} side in the triangle and then apply the area formulas and make a ratio. The second, and easier way, would be to find the ratio of the sides and then square that. $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$

Example 3: Two trapezoids are similar. If the scale factor is $\frac{3}{4}$ and the area of the smaller trapezoid is 81 cm^2, what is the area of the larger trapezoid?

Solution: First, the ratio of the areas would be $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. Now, we need the area of the larger trapezoid. To find this, we would multiply the area of the smaller trapezoid by the scale factor. However, we would need to flip the scale factor over to be $\frac{16}{9}$ because we want the larger area. This means we need to multiply by a scale factor that is larger than one. $A = \frac{16}{9} \cdot 81 = 144\ cm^2$.

Example 4: Two triangles are similar. The ratio of the areas is $\frac{25}{64}$. What is the scale factor?

Solution: The scale factor is $\sqrt{\frac{25}{64}} = \frac{5}{8}$.

Example 5: Using the ratios from Example 3, find the length of the base of the smaller triangle if the length of the base of the larger triangle is 24 units.

Solution: All you would need to do is multiply the scale factor we found in Example 3 by 24.

$$b = \frac{5}{8} \cdot 24 = 15\ units$$

Know What? Revisited You should end up with an 18 $in \times 18$ in drawing of your handprint.

Review Questions

Determine the ratio of the areas, given the ratio of the sides of a polygon.

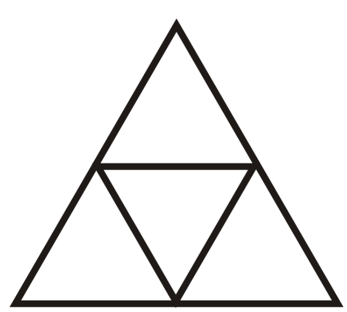
1. $\frac{3}{5}$
2. $\frac{1}{4}$
3. $\frac{7}{2}$
4. $\frac{6}{11}$

Determine the ratio of the sides of a polygon, given the ratio of the areas.

5. $\frac{1}{36}$
6. $\frac{4}{81}$
7. $\frac{49}{9}$
8. $\frac{25}{144}$

This is an equilateral triangle made up of 4 congruent equilateral triangles.

9. What is the ratio of the areas of the large triangle to one of the small triangles?



10. What is the scale factor of large to small triangle?
11. If the area of the large triangle is 20 $units^2$, what is the area of a small triangle?
12. If the length of the altitude of a small triangle is $2\sqrt{3}$, find the perimeter of the large triangle.
13. Carol drew two equilateral triangles. Each side of one triangle is 2.5 times as long as a side of the other triangle. The perimeter of the smaller triangle is 40 cm. What is the perimeter of the larger triangle?
14. If the area of the smaller triangle is 75 cm^2, what is the area of the larger triangle from #13?
15. Two rectangles are similar with a scale factor of $\frac{4}{7}$. If the area of the larger rectangle is 294 in^2, find the area of the smaller rectangle.
16. Two triangles are similar with a scale factor of $\frac{1}{3}$. If the area of the smaller triangle is 22 ft^2, find the area of the larger triangle.
17. The ratio of the areas of two similar squares is $\frac{16}{81}$. If the length of a side of the smaller square is 24 units, find the length of a side in the larger square.
18. The ratio of the areas of two right triangles is $\frac{2}{3}$. If the length of the hypotenuse of the larger triangle is 48 units, find the length of the smaller triangle's hypotenuse.

Questions 19-22 build off of each other. You may assume the problems are connected.

19. Two similar rhombi have areas of 72 $units^2$ and 162 $units^2$. Find the ratio of the areas.
20. Find the scale factor.
21. The diagonals in these rhombi are congruent. Find the length of the diagonals and the sides.
22. What type of rhombi are these quadrilaterals?
23. The area of one square on a game board is exactly twice the area of another square. Each side of the larger square is 50 mm long. How long is each side of the smaller square?
24. The distance from Charleston to Morgantown is 160 miles. The distance from Fairmont to Elkins is 75 miles. Charleston and Morgantown are 5 inches apart on a map. How far apart are Fairmont and Elkins on the same map.
25. Marlee is making models of historic locomotives (train engines). She uses the same scale for all of her models. The S1 locomotive was 140 ft long. The model is 8.75 inches long. The 520 Class locomotive was 87 feet long. What is the scale of Marlee's models? How long is the model of the 520 Class locomotive?
26. Tommy is drawing a floor plan for his dream home. On his drawing, 1cm represents 2 ft of the actual home. The actual dimensions of the dream home are 55 ft by 40 ft. What will the dimensions of his floor plan be? Will his scale drawing fit on a standard 8.5 in by 11 in piece of paper? Justify your answer.
27. Anne wants to purchase advertisement space in the school newspaper. Each square inch of advertisement space sells for $3.00. She wants to purchase a rectangular space with length and width in the ratio 3:2 and she has up to $50.00 to spend. What are the dimensions of the largest advertisement she can afford to purchase?
28. Aaron wants to enlarge a family photo from a 5 by 7 print to a print with an area of 140 inches. What are the dimensions of this new photo?
29. A popular pizza joint offers square pizzas: Baby Bella pizza with 10 inch sides, the Mama Mia pizza with 14 inch sides and the Big Daddy pizza with 18 inch sides. If the prices for these pizzas are $5.00, $9.00 and $15.00 respectively, find the price per square inch of each pizza. Which is the best deal?
30. Krista has a rectangular garden with dimensions 2 ft by 3 ft. She uses $\frac{2}{3}$ of a bottle of fertilizer to cover this area. Her friend, Hadleigh, has a garden with dimensions that are 1.5 times as long. How many bottles of fertilizer will she need?

Review Queue Answers

a. Two squares are always similar. Two rectangles can be similar as long as the sides are in the same proportion.
b. $\frac{10}{25} = \frac{2}{5}$
c. $A_{small} = 100, A_{large} = 625$

d. $\frac{100}{625} = \frac{4}{25}$, this is the square of the scale factor.

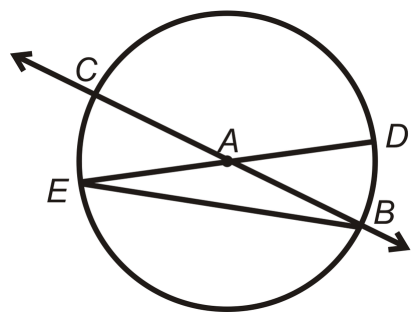
10.4 Circumference and Arc Length

Learning Objectives

- Find the circumference of a circle.
- Define the length of an arc and find arc length.

Review Queue

a. Find a central angle in that intercepts $\widehat{CE}$



b. Find an inscribed angle that intercepts $\widehat{CE}$.
c. How many degrees are in a circle? Find $m\widehat{ECD}$.
d. If $m\widehat{CE} = 26^\circ$, find $m\widehat{CD}$ and $m\angle CBE$.

Know What? A typical large pizza has a diameter of 14 inches and is cut into 8 or 10 pieces. Think of the crust as the circumference of the pizza. Find the *length* of the crust for the entire pizza. Then, find the length of the crust for one piece of pizza if the entire pizza is cut into a) 8 pieces or b) 10 pieces.

Circumference of a Circle

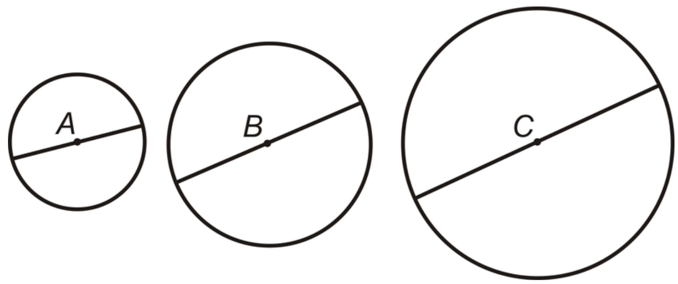
Circumference: The distance around a circle.

The circumference can also be called the perimeter of a circle. However, we use the term circumference for circles because they are round. The term perimeter is reserved for figures with straight sides. In order to find the formula for the circumference of a circle, we first need to determine the ratio between the circumference and diameter of a circle.

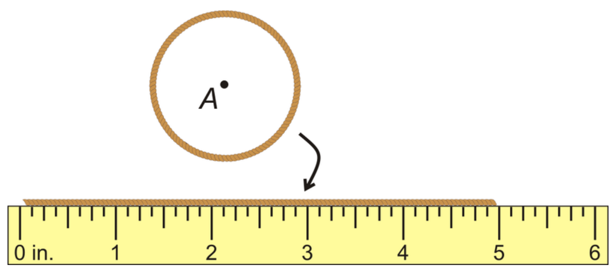
Investigation 10-1: Finding π (pi)

Tools Needed: paper, pencil, compass, ruler, string, and scissors

a. Draw three circles with radii of 2 in, 3 in, and 4 in. Label the centers of each A, B, and C.
b. Draw in the diameters and determine their lengths. Are all the diameter lengths the same in $\odot A$? $\odot B$? $\odot C$?



c. Take the string and outline each circle with it. The string represents the circumference of the circle. Cut the string so that it perfectly outlines the circle. Then, lay it out straight and measure, in inches. Round your answer to the nearest $\frac{1}{8}$-inch. Repeat this for the other two circles.



d. Find $\frac{circumference}{diameter}$ for each circle. Record your answers to the nearest thousandth. What do you notice?

From this investigation, you should see that $\frac{circumference}{diameter}$ approaches 3.14159... The bigger the diameter, the closer the ratio was to this number. We call this number π, the Greek letter "pi." It is an irrational number because the decimal never repeats itself. Pi has been calculated out to the millionth place and there is still no pattern in the sequence of numbers. When finding the circumference and area of circles, we must use π.

π, or **"pi":** The ratio of the circumference of a circle to its diameter. It is approximately equal to 3.14159265358979323846...

To see more digits of π, go to http://www.eveandersson.com/pi/digits/ .

You are probably familiar with the formula for circumference. From Investigation 10-1, we found that $\frac{circumference}{diameter} = \pi$. Let's shorten this up and solve for the circumference.

$\frac{C}{d} = \pi$, multiplying both sides by d, we have $C = \pi d$. We can also say $C = 2\pi r$ because $d = 2r$.

Circumference Formula: If d is the diameter or r is the radius of a circle, then $C = \pi d$ or $C = 2\pi r$.

Example 1: Find the circumference of a circle with a radius of 7 cm.

Solution: Plug the radius into the formula.

$$C = 2\pi(7) = 14\pi \approx 44\ cm$$

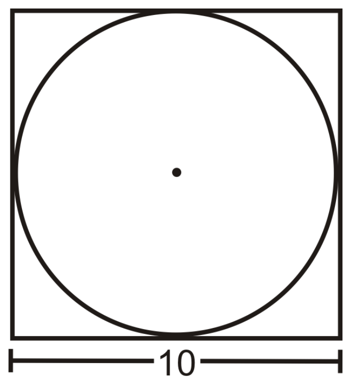
Depending on the directions in a given problem, you can either leave the answer in terms of π or multiply it out and get an approximation. Make sure you read the directions.

Example 2: The circumference of a circle is 64π. Find the diameter.

Solution: Again, you can plug in what you know into the circumference formula and solve for d.

$$64\pi = \pi d = 14\pi$$

Example 3: A circle is inscribed in a square with 10 in. sides. What is the circumference of the circle? Leave your answer in terms of π.



Solution: From the picture, we can see that the diameter of the circle is equal to the length of a side. Use the circumference formula.

$$C = 10\pi \ in.$$

Example 4: Find the perimeter of the square. Is it more or less than the circumference of the circle? Why?

Solution: The perimeter is $P = 4(10) = 40 \ in$. In order to compare the perimeter with the circumference we should change the circumference into a decimal.

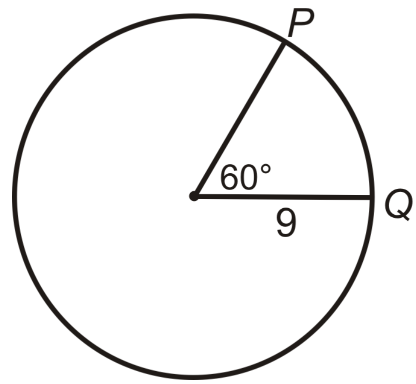
$C = 10\pi \approx 31.42 \ in$. This is less than the perimeter of the square, which makes sense because the circle is smaller than the square.

Arc Length

In Chapter 9, we measured arcs in degrees. This was called the "arc measure" or "degree measure." Arcs can also be measured in length, as a portion of the circumference.

Arc Length: The length of an arc or a portion of a circle's circumference.

The arc length is directly related to the degree arc measure. Let's look at an example.



Example 5: Find the length of $\widehat{PQ}$. Leave your answer in terms of π.

Solution: In the picture, the central angle that corresponds with $\widehat{PQ}$ is 60°. This means that $m\widehat{PQ} = 60^\circ$ as well. So, think of the arc length as a portion of the circumference. There are 360° in a circle, so 60° would be $\frac{1}{6}$ of that $\left(\frac{60^\circ}{360^\circ} = \frac{1}{6}\right)$. Therefore, the length of $\widehat{PQ}$ is $\frac{1}{6}$ of the circumference.

$$length\ of\ \widehat{PQ} = \frac{1}{6} \cdot 2\pi(9) = 3\pi$$

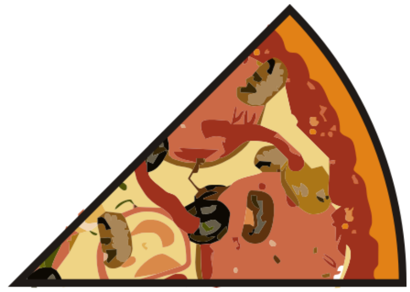
Arc Length Formula: If d is the diameter or r is the radius, the length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$.

Example 6: The arc length of $\widehat{AB} = 6\pi$ and is $\frac{1}{4}$ the circumference. Find the radius of the circle.

Solution: If 6π is $\frac{1}{4}$ the circumference, then the total circumference is $4(6\pi) = 24\pi$. To find the radius, plug this into the circumference formula and solve for r.

$$\begin{aligned}24\pi &= 2\pi r\\ 12 &= r\end{aligned}$$

Know What? Revisited The entire length of the crust, or the circumference of the pizza is $14\pi \approx 44$ *in*. In the picture to the right, the top piece of pizza is if it is cut into 8 pieces. Therefore, for $\frac{1}{8}$ of the pizza, one piece would have $\frac{44}{8} \approx 5.5$ *inches* of crust. The bottom piece of pizza is if the pizza is cut into 10 pieces. For $\frac{1}{10}$ of the crust, one piece would have $\frac{44}{10} \approx 4.4$ *inches* of crust.



Review Questions

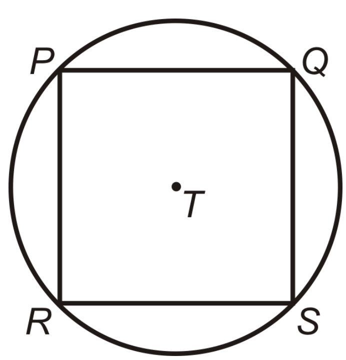
Fill in the following table. Leave all answers in terms of π.

TABLE 10.1:

	diameter	*radius*	*circumference*
1.	15		
2.		4	
3.	6		
4.			84π
5.		9	
6.			25π
7.			2π
8.	36		

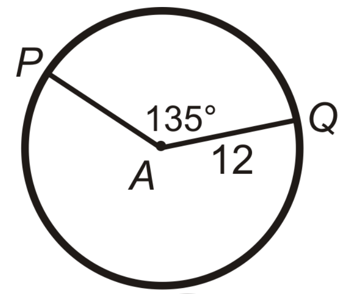
9. Find the radius of circle with circumference 88 in.
10. Find the circumference of a circle with $d = \frac{20}{\pi} cm$.

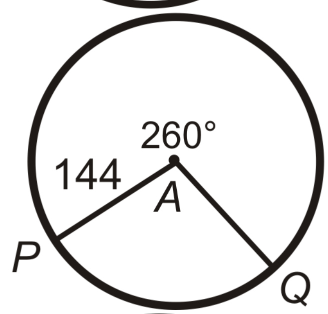
Square $PQSR$ is inscribed in $\odot T$. $RS = 8\sqrt{2}$.

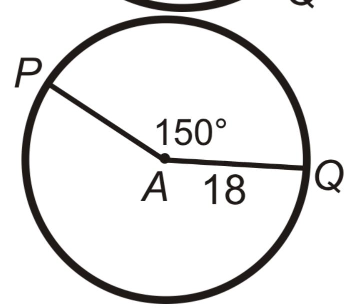


11. Find the length of the diameter of $\odot T$.
12. How does the diameter relate to $PQSR$?
13. Find the perimeter of $PQSR$.
14. Find the circumference of $\odot T$.

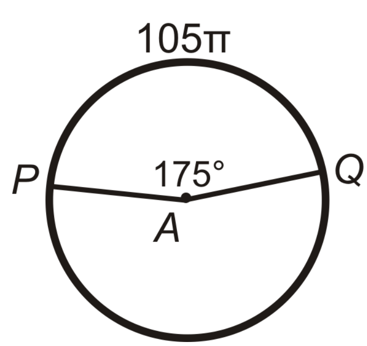
Find the arc length of $\widehat{PQ}$ in $\odot A$. Leave your answers in terms of π.

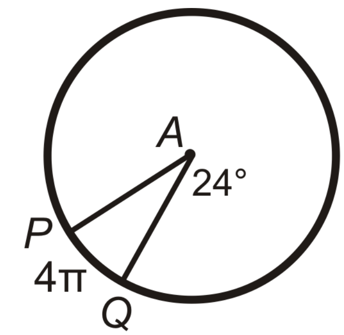
15. 

16. 

17. 

Find PA (the radius) in $\odot A$. Leave your answer in terms of π.

18. 



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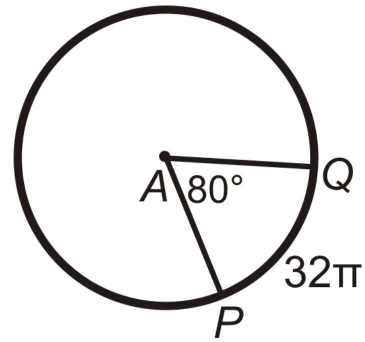
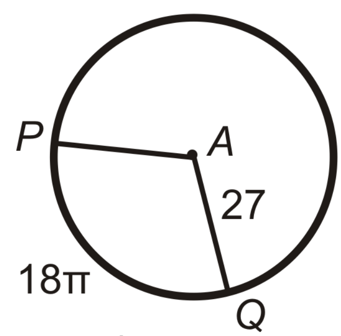


20.

Find the central angle or $m\widehat{PQ}$ in $\odot A$. Round any decimal answers to the nearest tenth.



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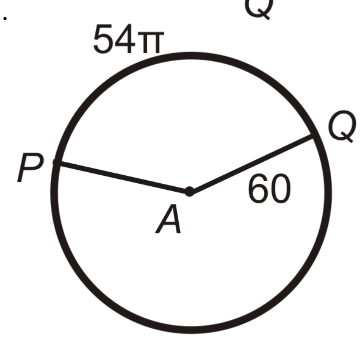

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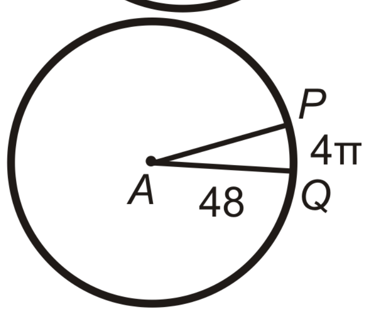
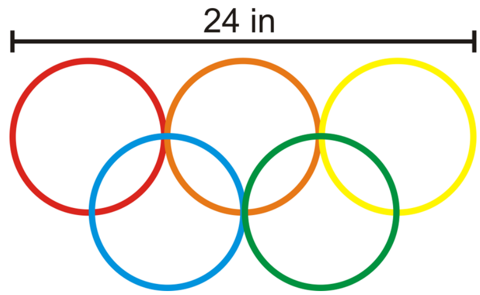


23.

24. The Olympics symbol is five congruent circles arranged as shown below. Assume the top three circles are tangent to each other. Brad is tracing the entire symbol for a poster. How far will his pen point travel?



25. A truck has tires with a 26 in diameter.
    a. How far does the truck travel every time a tire turns exactly once?
    b. How many times will the tire turn after the truck travels 1 mile? (1 mile = 5280 feet)
26. Mario's Pizza Palace offers a stuffed crust pizza in three sizes (diameter length) for the indicated prices: The Little Cheese, 8 in, \$7.00 The Big Cheese, 10 in, \$9.00 The Cheese Monster, 12 in, \$12.00 What is the crust (in) to price (\$) ratio for each of these pizzas? Michael thinks the cheesy crust is the best part of the pizza and wants to get the most crust for his money. Which pizza should he buy?
27. Jay is decorating a cake for a friend's birthday. They want to put gumdrops around the edge of the cake which has a 12 in diameter. Each gumdrop is has a diameter of 1.25 cm. To the nearest gumdrop, how many will they need?
28. A spedometer in a car measures the distance travelled by counting the rotations of the tires. The number of rotations required to travel one tenth of a mile in a particular vehicle is approximately 9.34. To the nearest inch, find the diameter of the wheel. (1 mile = 5280 feet)
29. Bob wants to put new weather stripping around a semicircular window above his door. The base of the window (diameter) is 36 inches. How much weather stripping does he need?
30. Each car on a Ferris wheel travels 942.5 ft during the 10 rotations of each ride. How high is each car at the highest point of each rotation?

Review Queue Answers

a. $\angle CAE$
b. $\angle CBE$
c. $360^\circ, 180^\circ$
d. $m\widehat{CD} = 180^\circ - 26^\circ = 154^\circ, m\angle CBE = 13^\circ$

10.5 Areas of Circles and Sectors

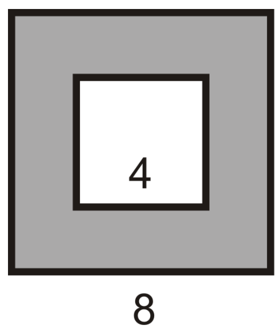
Learning Objectives

- Find the area of circles, sectors, and segments.

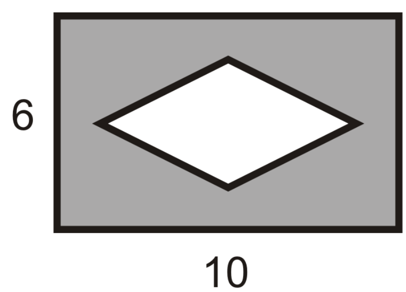
Review Queue

Find the area of the shaded region in the following figures.

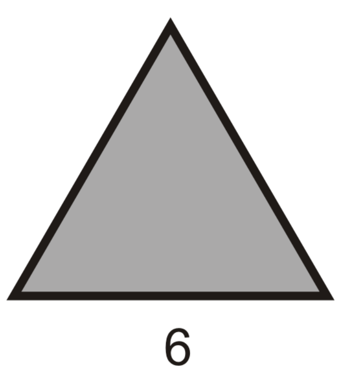
a. Both figures are squares.



b. Each vertex of the rhombus is 1.5 in from midpoints of the sides of the rectangle.



c. The figure is an equilateral triangle. (find the altitude)



d. Find the area of an equilateral triangle with side s.

Know What? Back to the pizza. In the previous section, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Leave your answers in terms of π and reduced improper fractions.

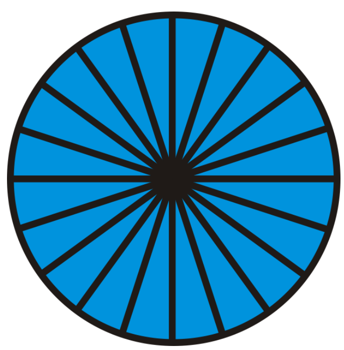
a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.

b) A thin crust pizza has $\frac{1}{2}$ - in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

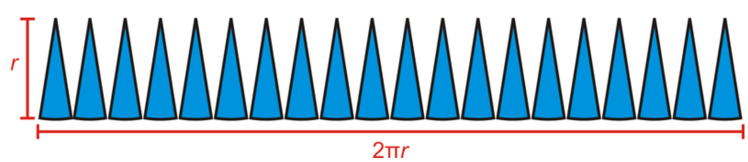
c) Which piece of pizza has more crust? A twelfth of the deep dish pizza or a fourth of the thin crust pizza?

Area of a Circle

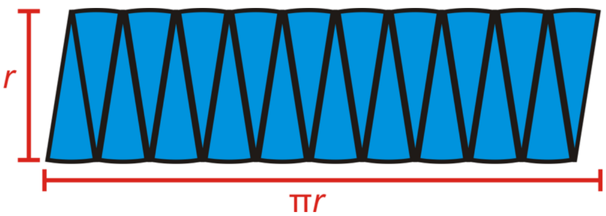
Recall in the previous section we derived π as the ratio between the circumference of a circle and its diameter. We are going to use the formula for circumference to derive the formula for area.



First, take a circle and divide it up into several wedges, or sectors. Then, unfold the wedges so they are all on one line, with the points at the top.



Notice that the height of the wedges is r, the radius, and the length is the circumference of the circle. Now, we need to take half of these wedges and flip them upside-down and place them in the other half so they all fit together.



Now our circle looks like a parallelogram. The area of this parallelogram is $A = bh = \pi r \cdot r = \pi r^2$.

To see an animation of this derivation, see http://www.rkm.com.au/ANIMATIONS/animation-Circle-Area-Derivation.html , by Russell Knightley.

Area of a Circle: If r is the radius of a circle, then $A = \pi r^2$.

Example 1: Find the area of a circle with a diameter of 12 cm.

Solution: If the diameter is 12 cm, then the radius is 6 cm. The area is $A = \pi(6^2) = 36\pi\ cm^2$.

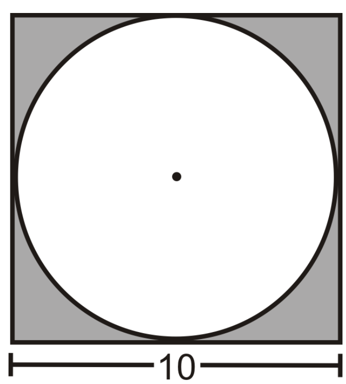
Example 2: If the area of a circle is 20π, what is the radius?

Solution: Work backwards on this problem. Plug in the area and solve for the radius.

$$\begin{aligned} 20\pi &= \pi r^2 \\ 20 &= r^2 \\ r &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Just like the circumference, we will leave our answers in terms of π, unless otherwise specified. In Example 2, the radius could be $\pm 2\sqrt{5}$, however the radius is always positive, so we do not need the negative answer.

Example 3: A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



Solution: The diameter of the circle is the same as the length of a side of the square. Therefore, the radius is half the length of the side, or 5 cm.

$$A = \pi 5^2 = 25\pi\ cm$$

Example 4: Find the area of the shaded region.

Solution: The area of the shaded region would be the area of the square minus the area of the circle.

$$A = 10^2 - 25\pi = 100 - 25\pi \approx 21.46\ cm^2$$

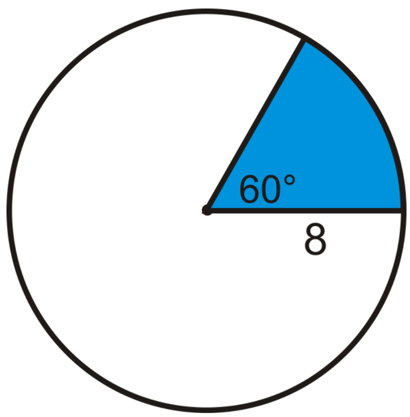
Area of a Sector

Sector of a Circle: The area bounded by two radii and the arc between the endpoints of the radii.

The area of a sector is a fractional part of the area of the circle, just like arc length is a fractional portion of the circumference.

Area of a Sector: If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Example 5: Find the area of the blue sector. Leave your answer in terms of π.



Solution: In the picture, the central angle that corresponds with the sector is 60°. 60° would be $\frac{1}{6}$ of 360°, so this sector is $\frac{1}{6}$ of the total area.

$$area\ of\ blue\ sector = \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$$

Another way to write the sector formula is $A = \frac{central\ angle}{360^\circ} \cdot \pi r^2$.

Example 6: The area of a sector is 8π and the radius of the circle is 12. What is the central angle?

Solution: Plug in what you know to the sector area formula and then solve for the central angle, we will call it x.

$$\begin{aligned} 8\pi &= \frac{x}{360^\circ} \cdot \pi 12^2 \\ 8\pi &= \frac{x}{360^\circ} \cdot 144\pi \\ 8 &= \frac{2x}{5^\circ} \\ x &= 8 \cdot \frac{5^\circ}{2} = 20^\circ \end{aligned}$$

Example 7: The area of a sector of circle is 50π and its arc length is 5π. Find the radius of the circle.

Solution: First plug in what you know to both the sector formula and the arc length formula. In both equations we will call the central angle, "CA."

$$\begin{aligned} 50\pi &= \frac{CA}{360}\pi r^2 \\ 50 \cdot 360 &= CA \cdot r^2 \\ 18000 &= CA \cdot r^2 \end{aligned} \qquad\qquad \begin{aligned} 5\pi &= \frac{CA}{360} 2\pi r \\ 5 \cdot 180 &= CA \cdot r \\ 900 &= CA \cdot r \end{aligned}$$

Now, we can use substitution to solve for either the central angle or the radius. Because the problem is asking for the radius we should solve the second equation for the central angle and substitute that into the first equation for the central angle. Then, we can solve for the radius. Solving the second equation for CA, we have: $CA = \frac{900}{r}$. Plug this into the first equation.

$$\begin{aligned} 18000 &= \frac{900}{r} \cdot r^2 \\ 18000 &= 900r \\ r &= 20 \end{aligned}$$

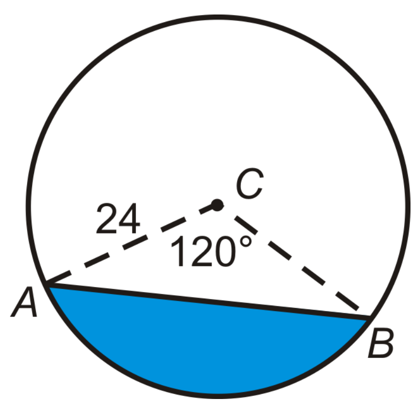
We could have also solved for the central angle in Example 7 once r was found. The central angle is $\frac{900}{20} = 45^\circ$.

Segments of a Circle

The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment.

Segment of a Circle: The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Example 8: Find the area of the blue segment below.



Solution: As you can see from the picture, the area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, we see that each half is a 30-60-90 triangle, where the radius is the hypotenuse. Therefore, the height of $\triangle ABC$ is 12 and the base would be $2\left(12\sqrt{3}\right) = 24\sqrt{3}$.

$$A_{sector} = \frac{120}{360}\pi \cdot 24^2 \qquad A_{\triangle} = \frac{1}{2}\left(24\sqrt{3}\right)(12)$$
$$= 192\pi \qquad = 144\sqrt{3}$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

In the review questions, make sure you know how the answer is wanted. If the directions say "leave in terms of π and simplest radical form," your answer would be the first one above. If it says "give an approximation," your answer would be the second. It is helpful to leave your answer in simplest radical form and in terms of π because that is the most accurate answer. However, it is also nice to see what the approximation of the answer is, to see how many square units something is.

Know What? Revisited The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$. One-twelfth of the deep dish pizza has $\frac{15}{12}\pi$ or $\frac{5}{4}\pi$ in^2 of crust. One-fourth of the thin crust pizza has $\frac{31}{16}\pi$ in^2. To compare the two measurements, it might be easier to put them both into decimals. $\frac{5}{4}\pi \approx 3.93$ in^2 and $\frac{31}{16}\pi \approx 6.09$ in^2. From this, we see that one-fourth of the thin-crust pizza has more crust than one-twelfth of the deep dish pizza.

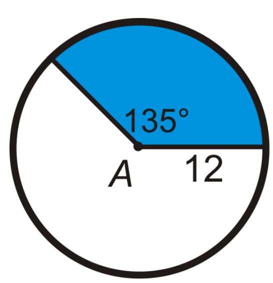
Review Questions

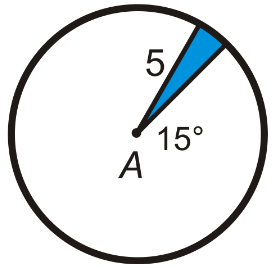
Fill in the following table. Leave all answers in terms of π.

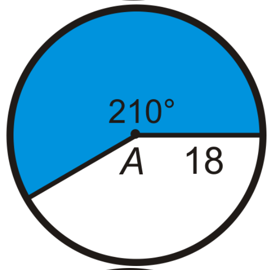
TABLE 10.2:

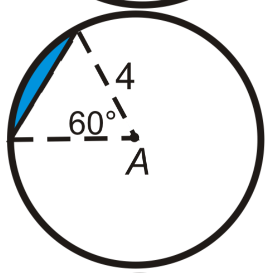
	radius	*Area*	*circumference*
1.	2		
2.		16π	
3.			10π
4.			24π
5.	9		
6.		90π	
7.			35π
8.	$\frac{7}{\pi}$		
9.			60
10.		36	

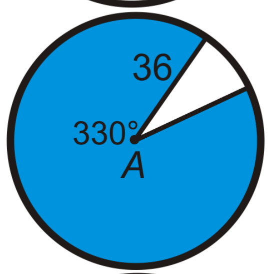
Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π. You may use decimals or fractions in your answers, but do not round.

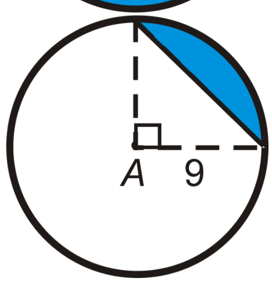
11. 

12. 

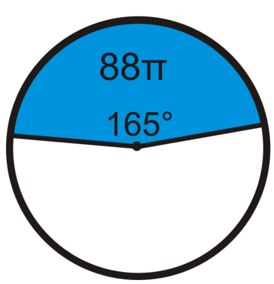
13. 

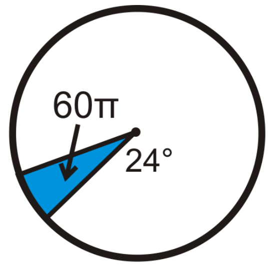
14. 

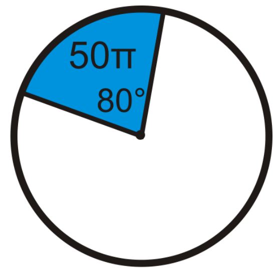
15. 

16. 

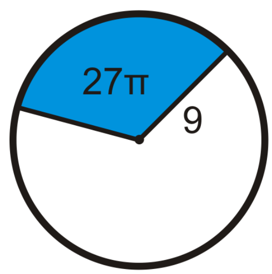
Find the radius of the circle. Leave your answer in simplest radical form.

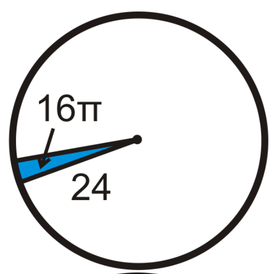
17. 

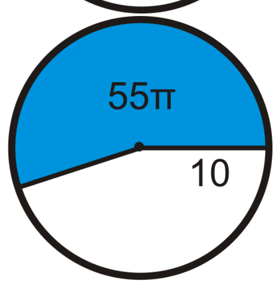
18. 

19. 

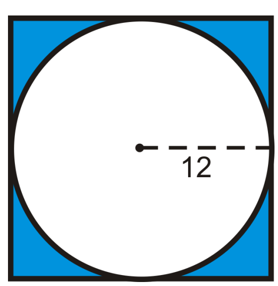
Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.

20. 

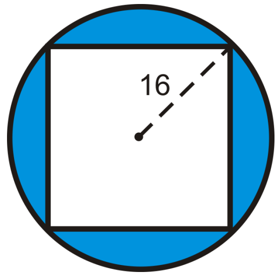
21. 

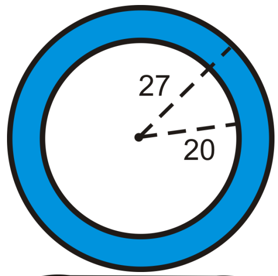
22. 

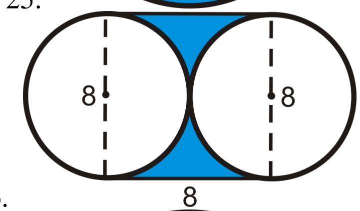
Find the area of the shaded region. Round your answer to the nearest hundredth.

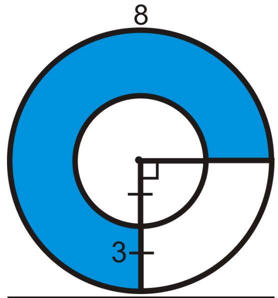
23. 

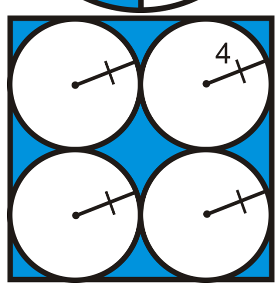
24. The quadrilateral is a square.



25. 

26. 

27. 

28. 

29. Carlos has 400 ft of fencing to completely enclose an area on his farm for an animal pen. He could make the area a square or a circle. If he uses the entire 400 ft of fencing, how much area is contained in the square and the circle? Which shape will yield the greatest area?
30. The area of a sector of a circle is 54π and its arc length is 6π. Find the radius of the circle.
31. The area of a sector of a circle is 2304π and its arc length is 32π. Find the central angle of the sector.

Review Queue Answers

a. $8^2 - 4^2 = 64 - 16 = 48$
b. $6(10) - \frac{1}{2}(7)(3) = 60 - 10.5 = 49.5$
c. $\frac{1}{2}(6)\left(3\sqrt{3}\right) = 9\sqrt{3}$

d. $\frac{1}{2}(s)\left(\frac{1}{2}s\sqrt{3}\right) = \frac{1}{4}s^2\sqrt{3}$

10.6 Area and Perimeter of Regular Polygons

Learning Objectives

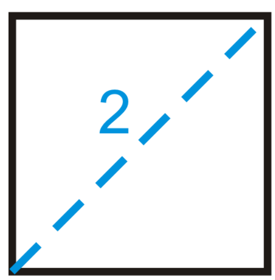
- Calculate the area and perimeter of a regular polygon.

Review Queue

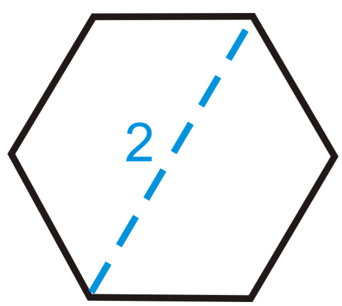
1. What is a regular polygon?

Find the area of the following regular polygons. For the hexagon and octagon, divide the figures into rectangles and/or triangles.

2.

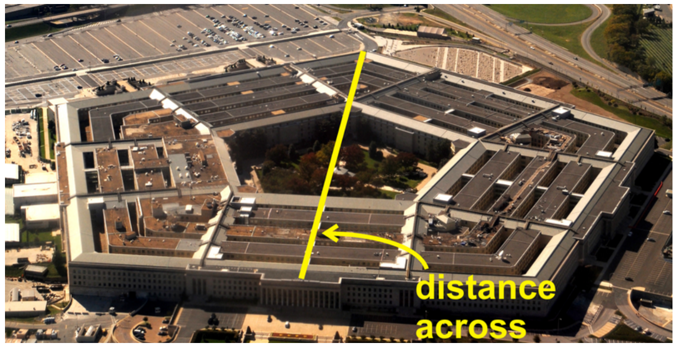


3.



4. Find the length of the sides in Problems 2 and 3.

Know What? The Pentagon in Arlington, VA houses the Department of Defense, is two regular pentagons with the same center. The entire area of the building is 29 acres (40,000 square feet in an acre), with an additional 5 acre courtyard in the center. The length of each outer wall is 921feet. What is the total distance across the pentagon? Round your answer to the nearest hundredth.



Perimeter of a Regular Polygon

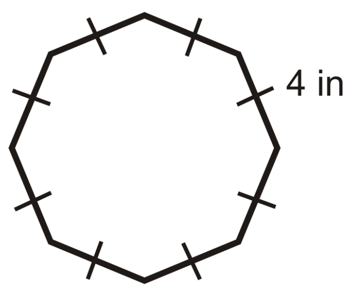
Recall that a regular polygon is a polygon with congruent sides and angles. In this section, we are only going to deal with regular polygons because they are the only polygons that have a consistent formula for area and perimeter. First, we will discuss the perimeter.

Recall that the perimeter of a square is 4 times the length of a side because each side is congruent. We can extend this concept to any regular polygon.

Perimeter of a Regular Polygon: If the length of a side is s and there are n sides in a regular polygon, then the perimeter is $P = ns$.

Example 1: What is the perimeter of a regular octagon with 4 inch sides?

Solution: If each side is 4 inches and there are 8 sides, that means the perimeter is 8(4 in) = 32 inches.

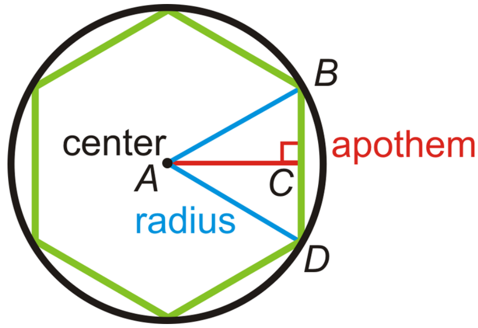


Example 2: The perimeter of a regular heptagon is 35 cm. What is the length of each side?

Solution: If $P = ns$, then 35 $cm = 7s$. Therefore, $s = 5$ cm.

Area of a Regular Polygon

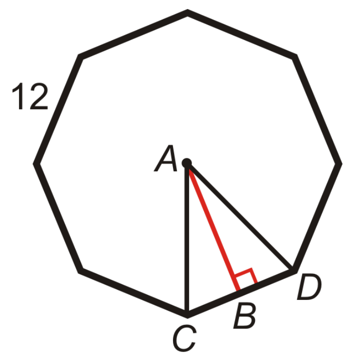
In order to find the area of a regular polygon, we need to define some new terminology. First, all regular polygons can be inscribed in a circle. So, ***regular polygons have a center and radius***, which are the center and radius of the circumscribed circle. Also like a circle, a regular polygon will have a central angle formed. In a regular polygon, however, ***the central angle is the angle formed by two radii drawn to consecutive vertices of the polygon.*** In the picture below, the central angle is $\angle BAD$. Also, notice that $\triangle BAD$ is an isosceles triangle. ***Every regular polygon with n sides is formed by n isosceles triangles.*** In a regular hexagon, the triangles are equilateral. The height of these isosceles triangles is called the ***apothem***.



Apothem: A line segment drawn from the center of a regular polygon to the midpoint of one of its sides.

We could have also said that the apothem is perpendicular to the side it is drawn to. By the Isosceles Triangle Theorem, the apothem is the perpendicular bisector of the side of the regular polygon. The apothem is also the height, or altitude of the isosceles triangles.

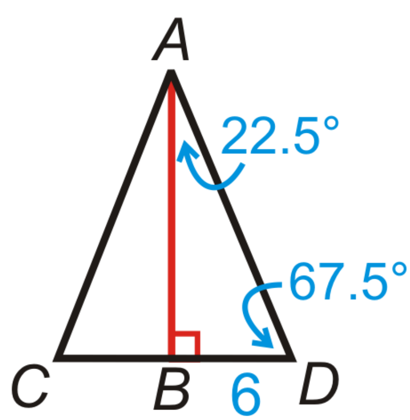
Example 3: Find the length of the apothem in the regular octagon. Round your answer to the nearest hundredth.



Solution: To find the length of the apothem, AB, you will need to use the trig ratios. First, find $m\angle CAD$. There are 360° around a point, so $m\angle CAD = \frac{360^\circ}{8} = 45^\circ$. Now, we can use this to find the other two angles in $\triangle CAD$. $m\angle ACB$ and $m\angle ADC$ are equal because $\triangle CAD$ is a right triangle.

$$\begin{aligned} m\angle CAD + m\angle ACB + m\angle ADC &= 180^\circ \\ 45^\circ + 2m\angle ACB &= 180^\circ \\ 2m\angle ACB &= 135^\circ \\ m\angle ACB &= 67.5^\circ \end{aligned}$$

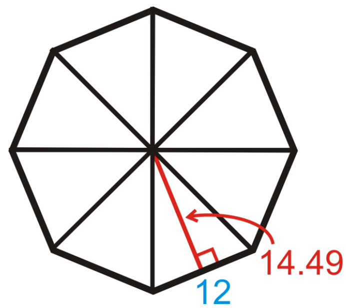
To find AB, we must use the tangent ratio. You can use either acute angle.



$$\tan 67.5^\circ = \frac{AB}{6}$$
$$AB = 6 \cdot \tan 67.5^\circ \approx 14.49$$

The apothem is used to find the area of a regular polygon. Let's continue with Example 3.

Example 4: Find the area of the regular octagon in Example 3.



Solution: The octagon can be split into 8 congruent triangles. So, if we find the area of one triangle and multiply it by 8, we will have the area of the entire octagon.

$$A_{octagon} = 8\left(\frac{1}{2} \cdot 12 \cdot 14.49\right) = 695.52\ units^2$$

From Examples 3 and 4, we can derive a formula for the area of a regular polygon.

The area of each triangle is: $A_\triangle = \frac{1}{2}bh = \frac{1}{2}sa$, where s is the length of a side and a is the apothem.

If there are n sides in the regular polygon, then it is made up of n congruent triangles.

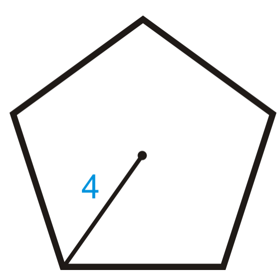
$$A = nA_\triangle = n\left(\frac{1}{2}sa\right) = \frac{1}{2}nsa$$

In this formula we can also substitute the perimeter formula, $P = ns$, for n and s.

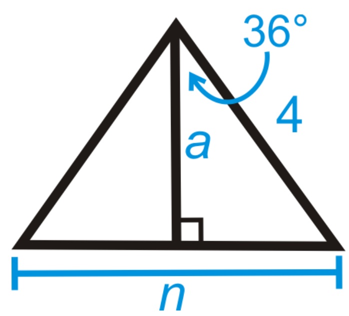
$$A = \frac{1}{2}nsa = \frac{1}{2}Pa$$

Area of a Regular Polygon: If there are n sides with length s in a regular polygon and a is the apothem, then $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$, where P is the perimeter.

Example 5: Find the area of the regular polygon with radius 4.



Solution: In this problem we need to find the apothem and the length of the side before we can find the area of the entire polygon. Each central angle for a regular pentagon is $\frac{360^\circ}{5} = 72^\circ$. So, half of that, to make a right triangle with the apothem, is 36°. We need to use sine and cosine.



$$\begin{aligned} \sin 36^\circ &= \frac{.5n}{4} \\ 4\sin 36^\circ &= \frac{1}{2}n \\ 8\sin 36^\circ &= n \\ n &\approx 4.7 \end{aligned} \qquad\qquad \begin{aligned} \cos 36^\circ &= \frac{a}{4} \\ 4\cos 36^\circ &= a \\ a &\approx 3.24 \end{aligned}$$

Using these two pieces of information, we can now find the area. $A = \frac{1}{2}(3.24)(5)(4.7) \approx 38.07$ $units^2$.

Example 6: The area of a regular hexagon is $54\sqrt{3}$ and the perimeter is 36. Find the length of the sides and the apothem.

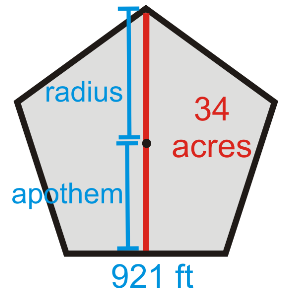
Solution: Plug in what you know into both the area and the perimeter formulas to solve for the length of a side and the apothem.

$$\begin{aligned} P &= sn \\ 36 &= 6s \\ s &= 6 \end{aligned} \qquad\qquad \begin{aligned} A &= \frac{1}{2}aP \\ 54\sqrt{3} &= \frac{1}{2}a(36) \\ 54\sqrt{3} &= 18a \\ 3\sqrt{3} &= a \end{aligned}$$

Know What? Revisited From the picture to the right, we can see that the total distance across the Pentagon is the length of the apothem plus the length of the radius. If the total area of the Pentagon is 34 acres, that is 2,720,000 square feet. Therefore, the area equation is $2720000 = \frac{1}{2}a(921)(5)$ and the apothem is 590.66 ft. To find the radius, we can either use the Pythagorean Theorem, with the apothem and half the length of a side or the sine ratio. Recall from Example 5, that each central angle in a pentagon is 72°, so we would use half of that for the right triangle.

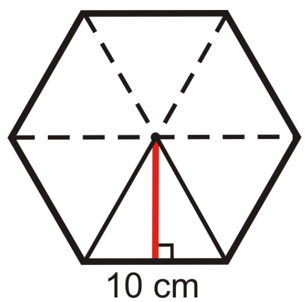
$$\sin 36^\circ = \frac{460.5}{r} \rightarrow r = \frac{460.5}{\sin 36^\circ} \approx 783.45\ ft.$$

Therefore, the total distance across is $590.66 + 783.45 = 1374.11\ ft$.



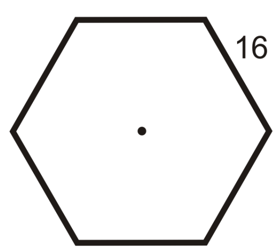
Review Questions

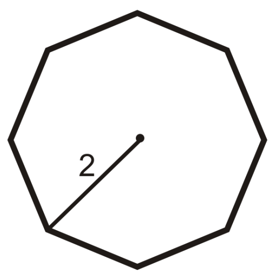
Use the regular hexagon below to answer the following questions. Each side is 10 cm long.

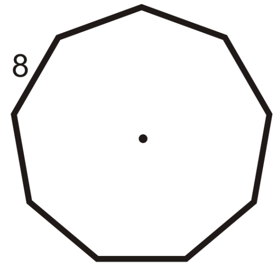


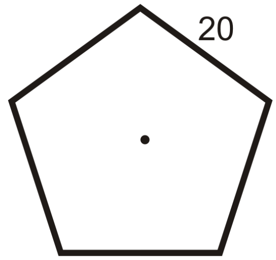
1. Each dashed line segment is $a(n)$ __________________.
2. The red line segment is $a(n)$ ____________________.
3. There are _____ congruent triangles in a regular hexagon.
4. In a regular hexagon, all the triangles are ___________________.
5. Find the radius of this hexagon.
6. Find the apothem.
7. Find the perimeter.
8. Find the area.

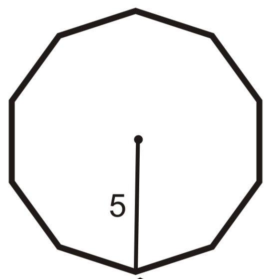
Find the area and perimeter of each of the following regular polygons. Round your answer to the nearest hundredth.

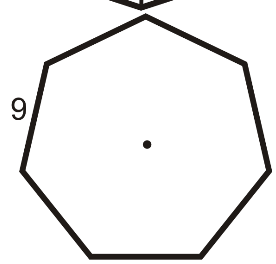
9. 

10. 

11. 

12. 

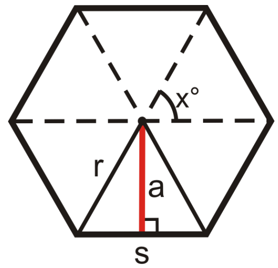
13. 

14. 

15. If the perimeter of a regular decagon is 65, what is the length of each side?
16. A regular polygon has a perimeter of 132 and the sides are 11 units long. How many sides does the polygon have?
17. The area of a regular pentagon is 440.44 in^2 and the perimeter is 80 in. Find the length of the pothem of the pentagon.
18. The area of a regular octagon is 695.3 cm^2 and the sides are 12 cm. What is the length of the apothem?

A regular 20-gon and a regular 40-gon are inscribed in a circle with a radius of 15 units.

19. Find the perimeter of both figures.
20. Find the circumference of the circle.
21. Which of the perimeters is closest to the circumference of the circle? Why do you think that is?
22. Find the area of both figures.
23. Find the area of the circle.
24. Which of the areas is closest to the area of the circle? Why do you think that is?
25. ***Challenge*** Derive a formula for the area of a regular hexagon with sides of length s. Your only variable will be s. HINT: Use 30-60-90 triangle ratios.
26. ***Challenge*** in the following steps you will derive an alternate formula for finding the area of a regular polygon with n sides.



We are going to start by thinking of a polygon with n sides as n congruent isosceles triangles. We will find the sum of the areas of these triangles using trigonometry. First, the area of a triangle is $\frac{1}{2}bh$. In the diagram to the right, this area formula would be $\frac{1}{2}sa$, where s is the length of a side and a is the length of the apothem. In the diagram, x represents the measure of the vertex angle of each isosceles triangle. a. The apothem, a, divides the triangle into two congruent right triangles. The top angle in each is $\frac{x^\circ}{2}$. Find $\sin\left(\frac{x^\circ}{2}\right)$ and $\cos\left(\frac{x^\circ}{2}\right)$. b. Solve your sin equation to find an expression for s in terms of r and x. c. Solve your cos equation to find an expression for a in terms of r and x. d. Substitute these expressions into the equation for the area of one of the triangles, $\frac{1}{2}sa$. e. Since there will be n triangles in an n-gon, you need to multiply your expression from part d by n to get the total area. f. How would you tell someone to find the value of x for a regular n-gon?

Use the formula you derived in problem 26 to find the area of the regular polygons described in problems 27-30. Round your answers to the nearest hundredth.

27. Decagon with radius 12 cm.
28. 20-gon with radius 5 in.
29. 15-gon with radius length 8 cm.
30. 45-gon with radius length 7 in.
31. What is the area of a regular polygon with 100 sides and radius of 9 in? What is the area of a circle with radius 9 in? How do these areas compare? Can you explain why?
32. How could you use the formula from problem 26 to find the area of a regular polygon given the number of sides and the length of a side? How can you find the radius?

Use your formula from problem 26 and the method you described to find r given the number of sides and the length of a side in problem 31 to find the area of the regular polygons below.

33. 30-gon with side length 15 cm.
34. Dodecagon with side length 20 in.

Review Queue Answers

a. A regular polygon is a polygon with congruent sides and angles.
b. $A = \left(\sqrt{2}\right)^2 = 2$
c. $A = 6\left(\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$
d. The sides of the square are $\sqrt{2}$ and the sides of the hexagon are 1 unit.

10.7 Chapter 10 Review

Keywords, Theorems and Formulas

Perimeter
The distance around a shape. Or, the sum of all the edges of a two-dimensional figure.

Area of a Rectangle
The area of a rectangle is the product of its base (width) and height (length) $A = bh$.

Perimeter of a Rectangle
$P = 2b + 2h$, where b is the base (or width) and h is the height (or length).

Perimeter of a Square
$P = 4s$

Area of a Square
$A = s^2$

Congruent Areas Postulate
If two figures are congruent, they have the same area.

Area Addition Postulate
If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

Area of a Parallelogram
$A = bh$.

Area of a Triangle
$A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$

Area of a Trapezoid
The area of a trapezoid with height h and bases b_1 and b_2 is $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Rhombus
If the diagonals of a rhombus are d_1 and d_2, then the area is $A = \frac{1}{2}d_1 d_2$.

Area of a Kite
If the diagonals of a kite are d_1 and d_2, then the area is $A = \frac{1}{2}d_1 d_2$.

Area of Similar Polygons Theorem
If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratio of the areas would be $\left(\frac{m}{n}\right)^2$.

π
The ratio of the circumference of a circle to its diameter.

Circumference
If d is the diameter or r is the radius of a circle, then $C = \pi d$ or $C = 2\pi r$.

Arc Length
The length of an arc or a portion of a circle's circumference.

Arc Length Formula
length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$

Area of a Circle
If r is the radius of a circle, then $A = \pi r^2$.

Sector of a Circle
The area bounded by two radii and the arc between the endpoints of the radii.

Area of a Sector
If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Segment of a Circle
The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Perimeter of a Regular Polygon
If the length of a side is s and there are n sides in a regular polygon, then the perimeter is $P = ns$.

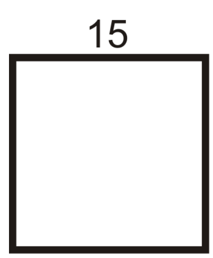
Apothem
A line segment drawn from the center of a regular polygon to the midpoint of one of its sides.

Area of a Regular Polygon
If there are n sides with length s in a regular polygon and a is the apothem, then $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$, where P is the perimeter.

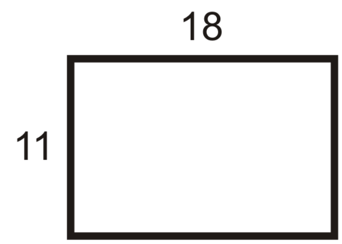
Review Questions

Find the area and perimeter of the following figures. Round your answers to the nearest hundredth.

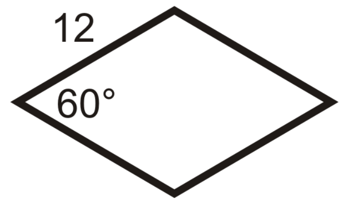
1. square



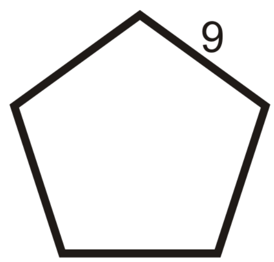
2. rectangle



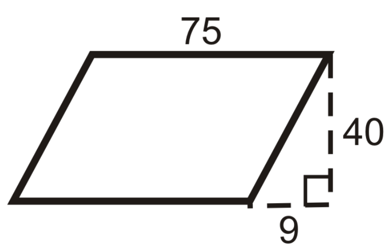
3. rhombus



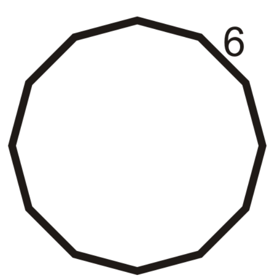
4. regular pentagon



5. parallelogram

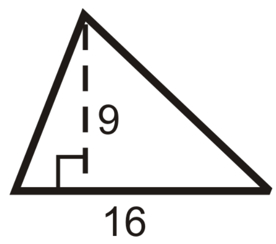


6. regular dodecagon

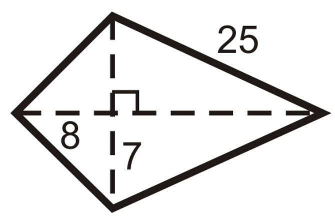


Find the area of the following figures. Leave your answers in simplest radical form.

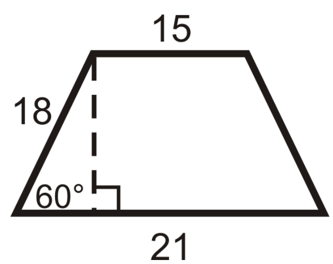
7. triangle



8. kite

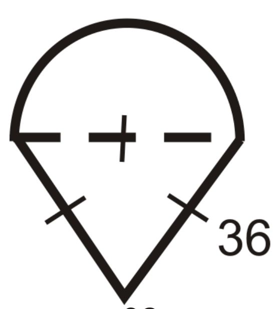


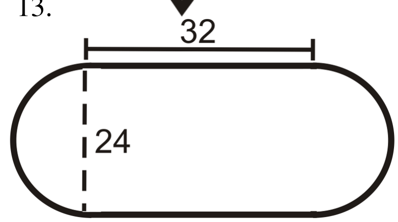
9. isosceles trapezoid



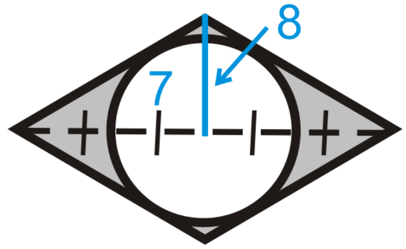
10. Find the area and circumference of a circle with radius 17.
11. Find the area and circumference of a circle with diameter 30.
12. Two similar rectangles have a scale factor $\frac{4}{3}$. If the area of the larger rectangle is 96 $units^2$, find the area of the smaller rectangle.

Find the area of the following figures. Round your answers to the nearest hundredth.

13. 

14. 

15. find the shaded area (figure is a rhombus)



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9695 .

CHAPTER 11 Surface Area and Volume

Chapter Outline

In this chapter we extend what we know about two-dimensional figures to three-dimensional shapes. First, we will determine the parts and different types of 3D shapes. Then, we will find the surface area and volume of prisms, cylinders, pyramids, cones, and spheres. Lastly, we will expand what we know about similar shapes and their areas to similar solids and their volumes.

11.1 Exploring Solids

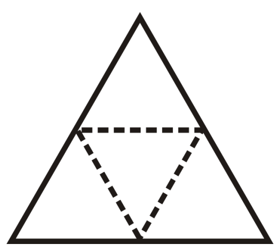
Learning Objectives

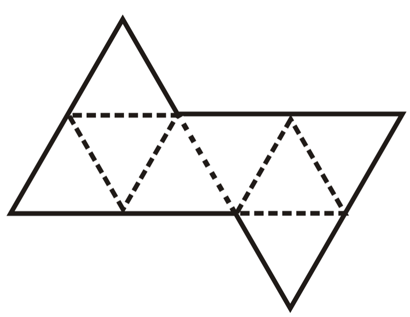
- Identify different types of solids and their parts.
- Use Euler's formula to solve problems.
- Draw and identify different views of solids.
- Draw and identify nets.

Review Queue

a. Draw an octagon and identify the edges and vertices of the octagon. How many of each are there?
b. Find the area of a square with 5 cm sides.
c. Find the area of an equilateral triangle with 10 in sides.
d. Draw the following polygons.
   a. A convex pentagon.
   b. A concave nonagon.

Know What? Until now, we have only talked about two-dimensional, or flat, shapes. In this chapter we are going to expand to 3D. Copy the equilateral triangle to the right onto a piece of paper and cut it out. Fold on the dotted lines. What shape do these four equilateral triangles make? If we place two of these equilateral triangles next to each other (like in the far right) what shape do these 8 equilateral triangles make?

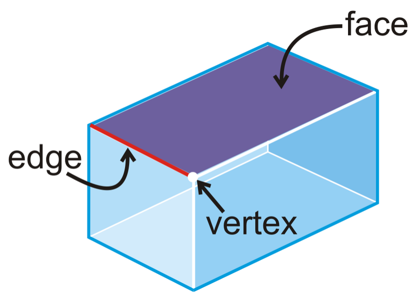




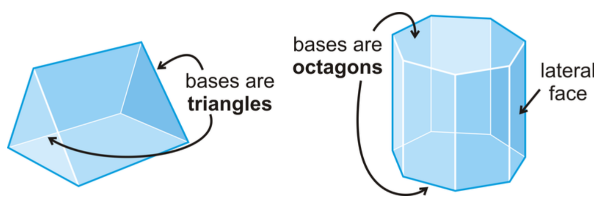
Polyhedrons

Polyhedron: A 3-dimensional figure that is formed by polygons that enclose a region in space.

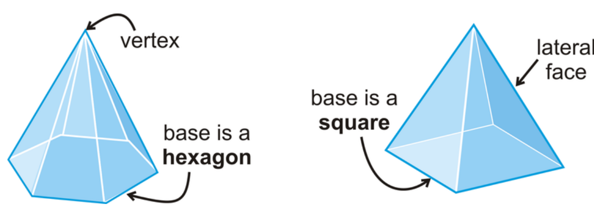
Each polygon in a polyhedron is called a ***face***. The line segment where two faces intersect is called an ***edge*** and the point of intersection of two edges is a ***vertex***. There are no gaps between the edges or vertices in a polyhedron. Examples of polyhedrons include a cube, prism, or pyramid. Non-polyhedrons are cones, spheres, and cylinders because they have sides that are not polygons.



Prism: A polyhedron with two congruent bases, in parallel planes, and the lateral sides are rectangles.



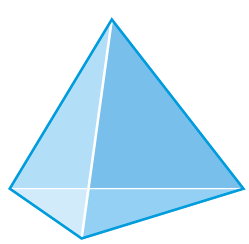
Pyramid: A polyhedron with one base and all the lateral sides meet at a common vertex. The lateral sides are triangles.



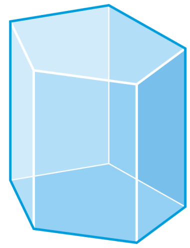
All prisms and pyramids are named by their bases. So, the first prism would be a triangular prism and the second would be an octagonal prism. The first pyramid would be a hexagonal pyramid and the second would be a square pyramid. The lateral faces of a pyramid are always triangles.

Example 1: Determine if the following solids are polyhedrons. If the solid is a polyhedron, name it and determine the number of faces, edges and vertices each has.

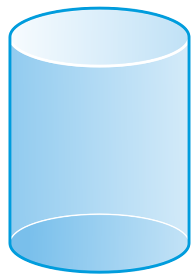
a)



b)



c)



Solution:

a) The base is a triangle and all the sides are triangles, so this is a polyhedron, a triangular pyramid. There are 4 faces, 6 edges and 4 vertices.

b) This solid is also a polyhedron because all the faces are polygons. The bases are both pentagons, so it is a pentagonal prism. There are 7 faces, 15 edges, and 10 vertices.

c) This is a cylinder and has bases that are circles. Circles are not polygons, so it is not a polyhedron.

Euler's Theorem

Let's put our results from Example 1 into a table.

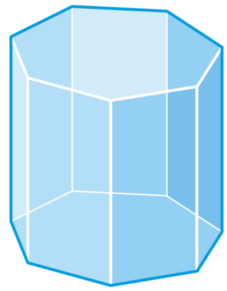
TABLE 11.1:

	Faces	***Vertices***	***Edges***
Triangular Pyramid	4	4	6
Pentagonal Prism	7	10	15

Notice that the sum of the faces + vertices is two more that the number of edges. This is called Euler's Theorem, after the Swiss mathematician Leonhard Euler.

Euler's Theorem: The number of faces (F), vertices (V), and edges (E) of a polyhedron can be related such that $F+V=E+2$.

Example 2: Find the number of faces, vertices, and edges in the octagonal prism.



Solution: Because this is a polyhedron, we can use Euler's Theorem to find either the number of faces, vertices or edges. It is easiest to count the faces, there are 10 faces. If we count the vertices, there are 16. Using this, we can solve for E in Euler's Theorem.

$$\begin{aligned} F+V &= E+2 \\ 10+16 &= E+2 \\ 24 &= E \qquad \text{There are 24 edges.} \end{aligned}$$

Example 3: In a six-faced polyhedron, there are 10 edges. How many vertices does the polyhedron have?

Solution: Solve for V in Euler's Theorem.

$$\begin{aligned} F+V &= E+2 \\ 6+V &= 10+2 \\ V &= 6 \qquad \text{There are 6 vertices.} \end{aligned}$$

Example 4: A three-dimensional figure has 10 vertices, 5 faces, and 12 edges. Is it a polyhedron?

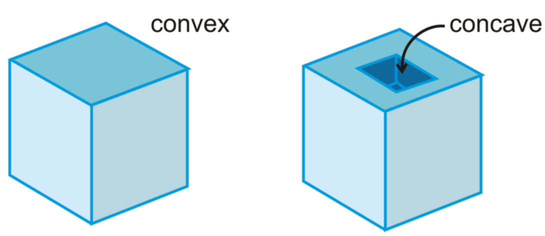
Solution: Plug in all three numbers into Euler's Theorem.

$$\begin{aligned} F+V &= E+2 \\ 5+10 &= 12+2 \\ 15 &\neq 14 \end{aligned}$$

Because the two sides are not equal, this figure is not a polyhedron.

Regular Polyhedra

Regular Polyhedron: A polyhedron where all the faces are congruent regular polygons.



Polyhedrons, just like polygons, can be ***convex*** or ***concave*** (also called non-convex). All regular polyhedron are convex. A concave polyhedron is similar to a concave polygon. The polyhedron "caves in," so that two non-adjacent vertices can be connected by a line segement that is outside the polyhedron.

There are five regular polyhedra called the Platonic solids, after the Greek philosopher Plato. These five solids are significant because they are the only five regular polyhedra. There are only five because the sum of the measures of the angles that meet at each vertex must be less than 360°. Therefore the only combinations are 3, 4 or 5 triangles at each vertex, 3 squares at each vertex or 3 pentagons. Each of these polyhedra have a name based on the number of sides, except the cube.

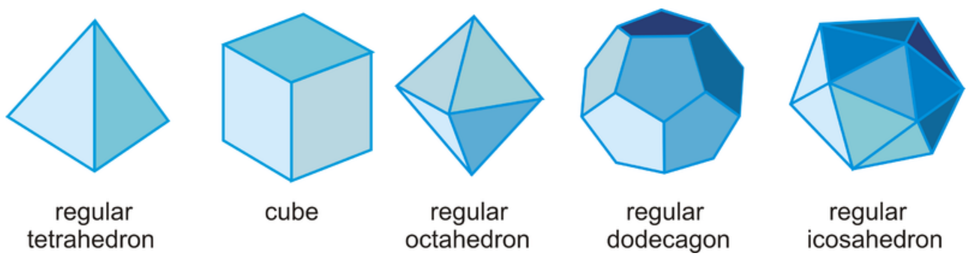
Regular Tetrahedron: A 4-faced polyhedron where all the faces are equilateral triangles.

Cube: A 6-faced polyhedron where all the faces are squares.

Regular Octahedron: An 8-faced polyhedron where all the faces are equilateral triangles.

Regular Dodecahedron: A 12-faced polyhedron where all the faces are regular pentagons.

Regular Icosahedron: A 20-faced polyhedron where all the faces are equilateral triangles.



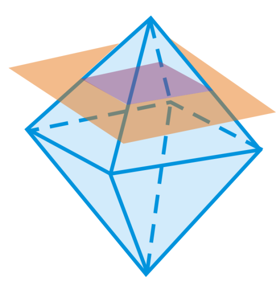
Cross-Sections

One way to "view" a three-dimensional figure in a two-dimensional plane, like this text, is to use cross-sections.

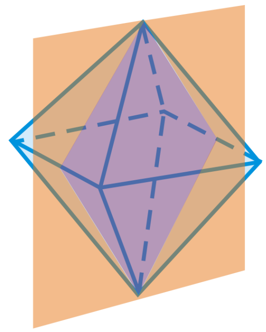
Cross-Section: The intersection of a plane with a solid.

Example 5: Describe the shape formed by the intersection of the plane and the regular octahedron.

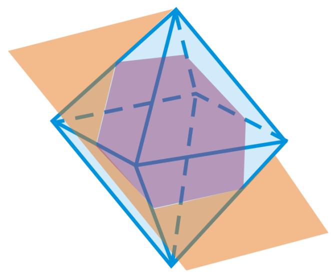
a)



b)



c)



Solution:

a) Square

b) Rhombus

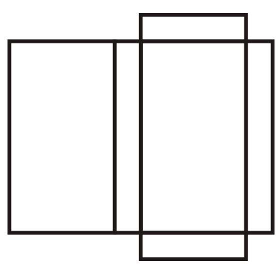
c) Hexagon

Nets

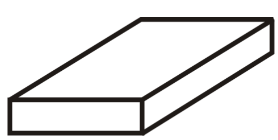
Another way to represent a three-dimensional figure in a two dimensional plane is to use a net.

Net: An unfolded, flat representation of the sides of a three-dimensional shape.

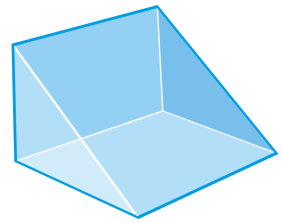
Example 6: What kind of figure does this net create?



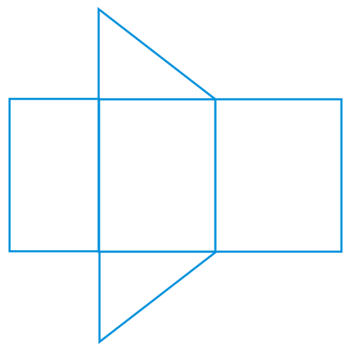
Solution: The net creates a rectangular prism.



Example 7: Draw a net of the right triangular prism below.



Solution: This net will have two triangles and three rectangles. The rectangles are all different sizes and the two triangles are congruent.



Notice that there could be a couple different interpretations of this, or any, net. For example, this net could have the triangles anywhere along the top or bottom of the three rectangles. Most prisms have multiple nets.

See the site http://www.cs.mcgill.ca/~sqrt/unfold/unfolding.html if you would like to see a few animations of other nets, including the Platonic solids.

Know What? Revisited The net of the first shape is a regular tetrahedron and the second is the net of a regular octahedron.

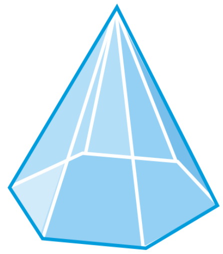
Review Questions

Complete the table using Euler's Theorem.

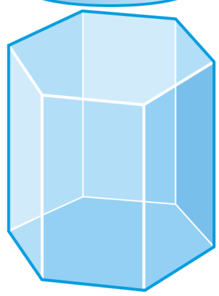
TABLE 11.2:

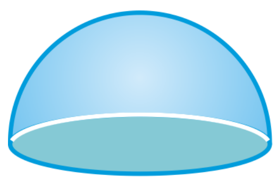
	Name	*Faces*	*Edges*	*Vertices*
1.	Rectangular Prism	6	12	
2.	Octagonal Pyramid		16	9
3.	Regular Icosahedron	20		12
4.	Cube		12	8
5.	Triangular Pyramid	4		4
6.	Octahedron	8	12	
7.	Heptagonal Prism		21	14
8.	Triangular Prism	5	9	

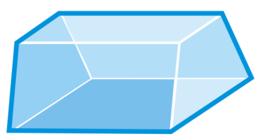
Determine if the following figures are polyhedra. If so, name the figure and find the number of faces, edges, and vertices.

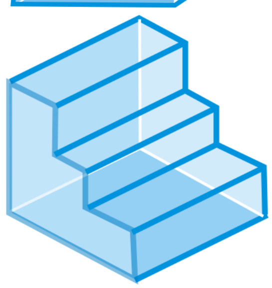
9. 

10. 

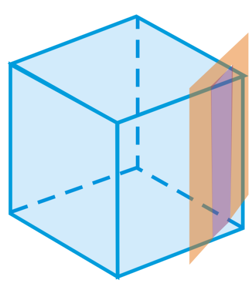
11. 

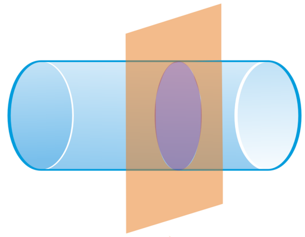
12. 

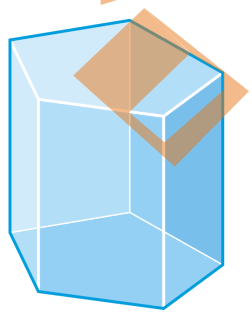
13. 

14. 

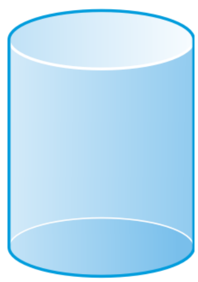
Describe the cross section formed by the intersection of the plane and the solid.

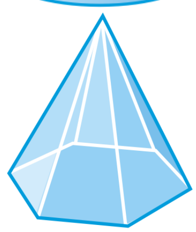
15. 

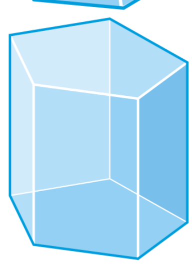
16. 

17. 

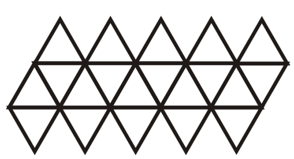
Draw the net for the following solids.

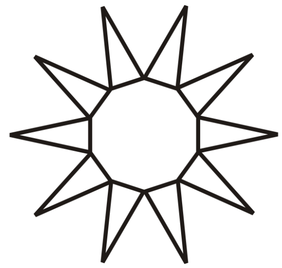
18. 

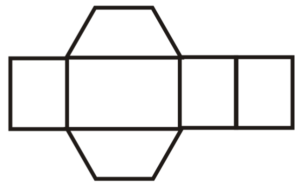
19. 

20. 

Determine what shape is formed by the following nets.

21. 

22. 

23. 

24. A cube has 11 unique nets. Draw 5 different nets of a cube.
25. A truncated icosahedron is a polyhedron with 12 regular pentagonal faces and 20 regular hexagonal faces and 90 edges. This icosahedron closely resembles a soccer ball. How many vertices does it have? Explain your reasoning.

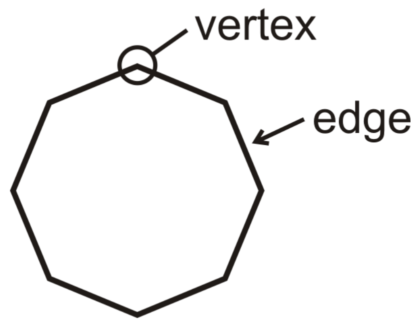
26. Use construction tools to construct a large equilateral triangle. Construct the three midsegments of the triangle. Cut out the equilateral triangle and fold along the midsegments. What net have you constructed?
27. Describe a method to construct a net for a regular octahedron.

For problems 28-30, we are going to connect the Platonic Solids to probability. A six sided die is the shape of a cube. The probability of any one side landing face up is $\frac{1}{6}$ because each of the six faces is congruent to each other.

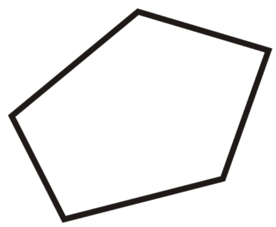
28. What shape would we make a die with 12 faces? If we number these faces 1 to 12, and each face has the same likelihood of landing face up, what is the probability of rolling a multiple of three?
29. I have a die that is a regular octahedron. Each face is labeled with a consecutive prime number starting with 2. What is the largest prime number on my die?
30. ***Challenge*** Rebecca wants to design a new die. She wants it to have one red face. The other faces will be yellow, blue or green. How many faces should her die have and how many of each color does it need so that: the probability of rolling yellow is eight times the probability of rolling red, the probability of rolling green is half the probability of rolling yellow and the probability of rolling blue is seven times the probability of rolling red?

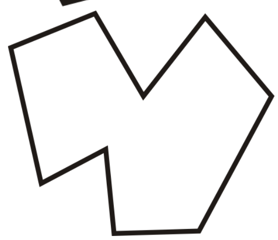
Review Queue Answers

a. There are 8 vertices and 8 edges in an octagon.



b. $5^2 = 25\ cm^2$
c. $\frac{1}{2} \cdot 10 \cdot 5\sqrt{3} = 25\sqrt{3}\ in^2$

a. 

b. 

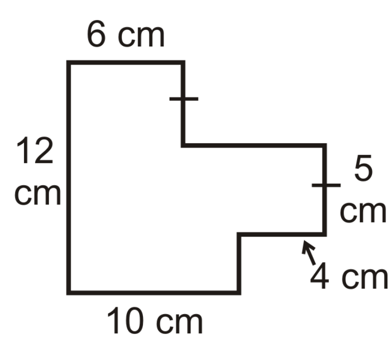
11.2 Surface Area of Prisms and Cylinders

Learning Objectives

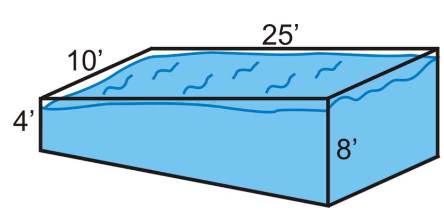
- Find the surface area of a prism.
- Find the surface area of a cylinder.

Review Queue

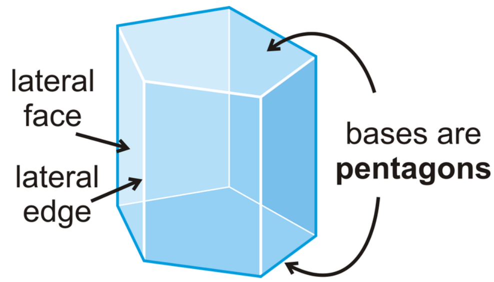
a. Find the area of a rectangle with sides:
   a. 6 and 9
   b. 11 and 4
   c. $5\sqrt{2}$ and $8\sqrt{6}$
b. If the area of a square is 36 $units^2$, what are the lengths of the sides?
c. If the area of a square is 45 $units^2$, what are the lengths of the sides?
d. Find the area of the shape. All sides are perpendicular. (Split the shape up into rectangles.)



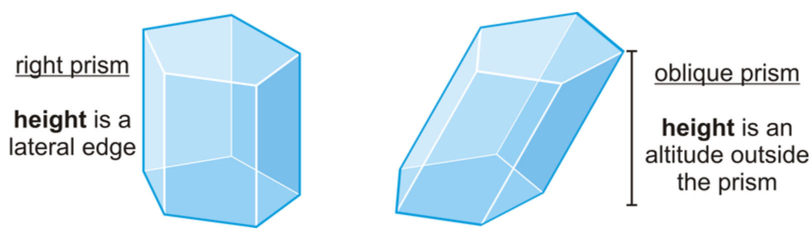
Know What? Your parents decide they want to put a pool in the backyard. They agree on a pool where the shallow end will be 4 ft. and the deep end will be 8 ft. The pool will be 10 ft. by 25 ft. How much siding do they need to buy to cover the sides and bottom of the pool? If the siding is $25.00 a square yard, how much will it cost to enclose the pool?



Parts of a Prism



In the last section, we defined a prism as a 3-dimensional figure with 2 congruent bases, in parallel planes with rectangular lateral faces. The edges between the ***lateral faces*** are called ***lateral edges***. All prisms are named by their bases, so the prism to the right is a pentagonal prism. This particular prism is called a ***right prism*** because the lateral faces are perpendicular to the bases. ***Oblique prisms*** lean to one side or the other and the height is outside the prism.



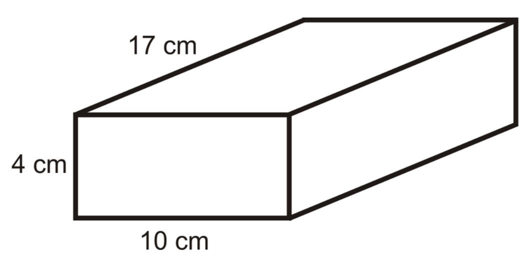
Surface Area of a Prism

Surface Area: The sum of the areas of the faces.

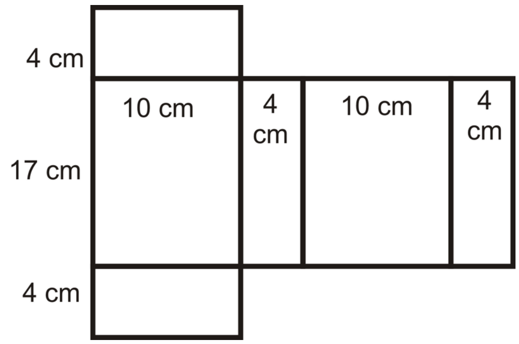
Lateral Area: The sum of the areas of the ***lateral*** faces.

You can use a net and the Area Addition Postulate to find the surface area of a right prism.

Example 1: Find the surface area of the prism below.



Solution: Open up the prism and draw the net. Determine the measurements for each rectangle in the net.



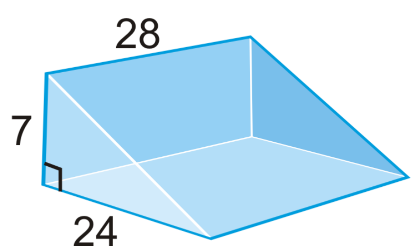
Using the net, we have:

$$\begin{aligned} SA_{prism} &= 2(4)(10) + 2(10)(17) + 2(17)(4) \\ &= 80 + 340 + 136 \\ &= 556\ cm^2 \end{aligned}$$

Because this is still area, the units are squared.

Surface Area of a Right Prism: The surface area of a right prism is the sum of the area of the bases and the area of each rectangular lateral face.

Example 2: Find the surface area of the prism below.

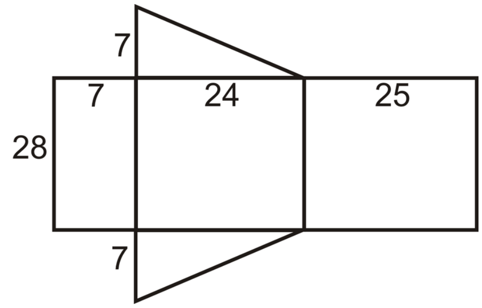


Solution: This is a right triangular prism. To find the surface area, we need to find the length of the hypotenuse of the base because it is the width of one of the lateral faces. Using the Pythagorean Theorem, the hypotenuse is

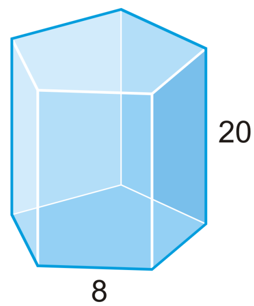
$$\begin{aligned} 7^2 + 24^2 &= c^2 \\ 49 + 576 &= c^2 \\ 625 &= c^2 \\ c &= 25 \end{aligned}$$

Looking at the net, the surface area is:

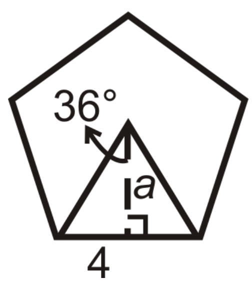
$$SA = 28(7) + 28(24) + 28(25) + 2\left(\frac{1}{2} \cdot 7 \cdot 24\right)$$
$$SA = 196 + 672 + 700 + 168 = 1736$$



Example 3: Find the surface area of the regular pentagonal prism.



Solution: For this prism, each lateral face has an area of 160 $units^2$. Then, we need to find the area of the regular pentagonal bases. Recall that the area of a regular polygon is $\frac{1}{2}asn$. $s = 8$ and $n = 5$, so we need to find a, the apothem.

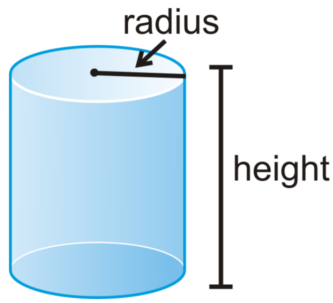


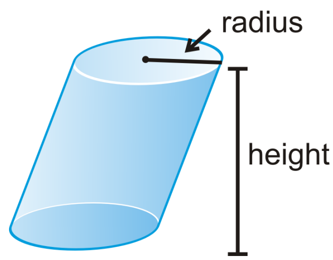
$$\tan 36^\circ = \frac{4}{a}$$
$$a = \frac{4}{\tan 36^\circ} \approx 5.51$$
$$SA = 5(160) + 2\left(\frac{1}{2} \cdot 5.51 \cdot 8 \cdot 5\right) = 1020.4$$

Cylinders

Cylinder: A solid with congruent circular bases that are in parallel planes. The space between the circles is enclosed.

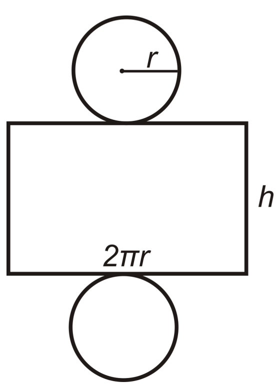
Just like a circle, the cylinder has a radius for each of the circular bases. Also, like a prism, a cylinder can be oblique, like the one to the right.





Surface Area of a Right Cylinder

Let's find the net of a right cylinder. One way for you to do this is to take a label off of a soup can or can of vegetables. When you take this label off, we see that it is a rectangle where the height is the height of the cylinder and the base is the circumference of the base. This rectangle and the two circular bases make up the net of a cylinder.



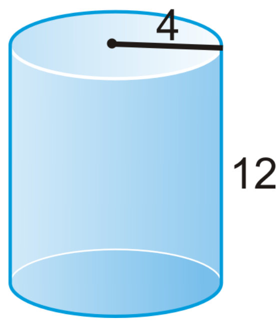
From the net, we can see that the surface area of a right cylinder is

$$\underbrace{2\pi r^2}_{\text{area of both circles}} + \underbrace{2\pi r}_{\text{length of rectangle}} h$$

Surface Area of a Right Cylinder: If r is the radius of the base and h is the height of the cylinder, then the surface area is $SA = 2\pi r^2 + 2\pi rh$.

To see an animation of the surface area, click http://www.rkm.com.au/ANIMATIONS/animation-Cylinder-Surface-Area-Derivation.html , by Russell Knightley.

Example 4: Find the surface area of the cylinder.



Solution: $r = 4$ and $h = 12$. Plug these into the formula.

$$\begin{aligned} SA &= 2\pi(4)^2 + 2\pi(4)(12) \\ &= 32\pi + 96\pi \\ &= 128\pi \end{aligned}$$

Example 5: The circumference of the base of a cylinder is 16π and the height is 21. Find the surface area of the cylinder.

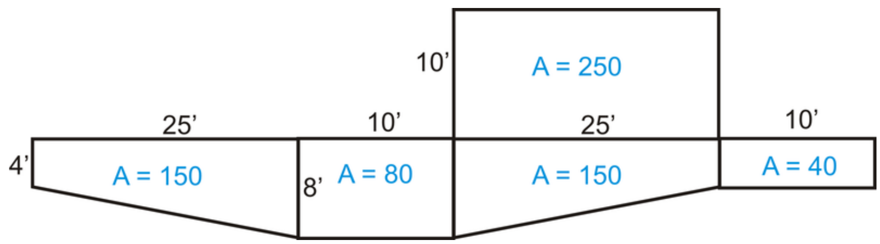
Solution: If the circumference of the base is 16π, then we can solve for the radius.

$$\begin{aligned} 2\pi r &= 16\pi \\ r &= 8 \end{aligned}$$

Now, we can find the surface area.

$$\begin{aligned} SA &= 2\pi(8)^2 + (16\pi)(21) \\ &= 128\pi + 336\pi \\ &= 464\pi \end{aligned}$$

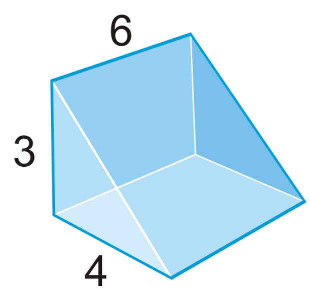
Know What? Revisited To the right is the net of the pool (minus the top). From this, we can see that your parents would need 670 square feet of siding. This means that the total cost would be $5583.33 for the siding.



Review Questions

1. How many square feet are in a square yard?
2. How many square centimeters are in a square meter?

Use the right triangular prism to answer questions 3-6.



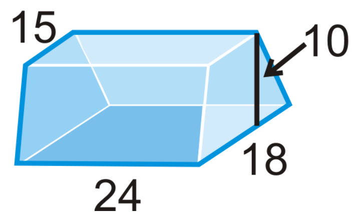
3. What shape are the bases of this prism? What are their areas?
4. What are the dimensions of each of the lateral faces? What are their areas?
5. Find the lateral surface area of the prism.
6. Find the total surface area of the prism.
7. ***Writing*** Describe the difference between lateral surface area and total surface area.
8. The lateral surface area of a cylinder is what shape? What is the area of this shape?
9. Fuzzy dice are cubes with 4 inch sides.

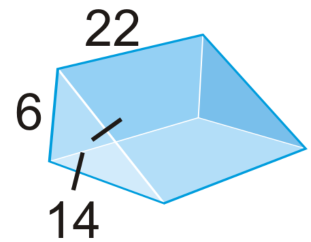
   a. What is the surface area of one die?
   b. Typically, the dice are sold in pairs. What is the surface area of two dice?

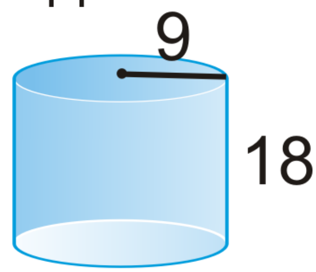
10. A right cylinder has a 7 cm radius and a height of 18 cm. Find the surface area.

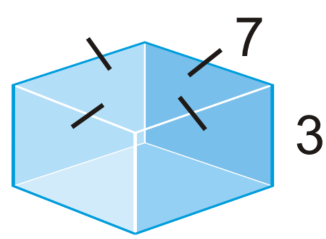
Find the surface area of the following solids. Leave answers in terms of π.

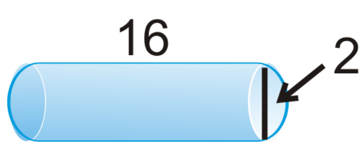
11. bases are isosceles trapezoids

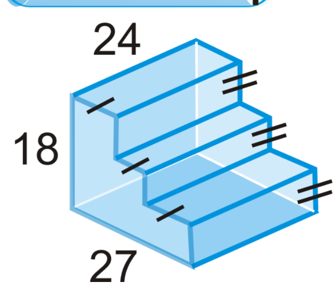


12. 

13. 

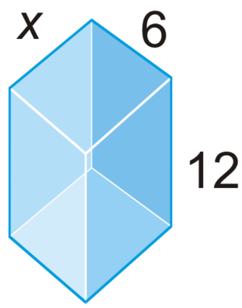
14. 

15. 

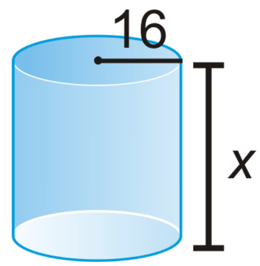
16. 

Algebra Connection Find the value of x, given the surface area.

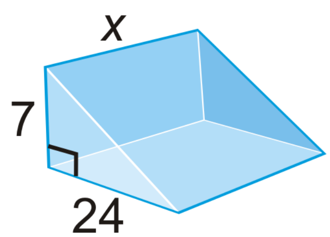
17. $SA = 432\ units^2$



18. $SA = 1536\pi\ units^2$

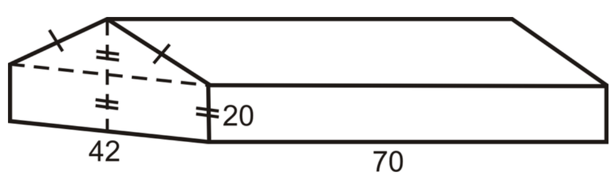


19. $SA = 1568\ units^2$

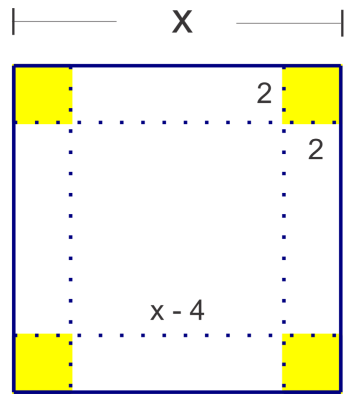


20. The area of the base of a cylinder is $25\pi\ in^2$ and the height is 6 in. Find the *lateral* surface area.
21. The circumference of the base of a cylinder is $80\pi\ cm$ and the height is 36 cm. Find the total surface area.
22. The lateral surface area of a cylinder is $30\pi\ m^2$. What is one possibility for height of the cylinder?

Use the diagram below for questions 23-27. The barn is shaped like a pentagonal prism with dimensions shown in feet.



23. What is the area of the roof? (Both sides)
24. What is the floor area of the barn?
25. What is the area of the sides of the barn?
26. The farmer wants to paint the sides of the roof (excluding the roof). If a gallon of paint covers 250 square feet, how many gallons will he need?
27. A gallon of paint costs $15.50. How much will it cost for him to paint the sides of the barn?
28. Charlie started a business canning artichokes. His cans are 5 in tall and have diameter 4 in. If the label must cover the entire lateral surface of the can and the ends must overlap by at least one inch, what are the dimensions and area of the label?
29. An open top box is made by cutting out 2 in by 2 in squares from the corners of a large square piece of cardboard. Using the picture as a guide, find an expression for the surface area of the box. If the surface area is $609\ in^2$, find the length of x. Remember, there is no top.



30. Find an expression for the surface area of a cylinder in which the ratio of the height to the diameter is 2:1. If x is the diameter, use your expression to find x if the surface area is 160π.

Review Queue Answers

a.
   a. 54
   b. 44
   c. $80\sqrt{3}$
b. $s = 6$
c. $s = 3\sqrt{5}$
d. $A = 60 + 30 + 20 = 110\ cm^2$

11.3 Surface Area of Pyramids and Cones

Learning Objectives

- Find the surface area of a pyramid.
- Find the surface area of a cone.

Review Queue

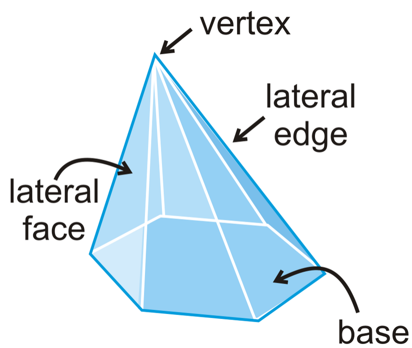
a. A rectangular prism has sides of 5 cm, 6 cm, and 7 cm. What is the surface area?
b. Triple the dimensions of the rectangular prism from #1. What is its surface area?
c. A cylinder has a diameter of 10 in and a height of 25 in. What is the surface area?
d. A cylinder has a circumference of $72\pi\ ft.$ and a height of 24 ft. What is the surface area?
e. Draw the net of a square pyramid.

Know What? A typical waffle cone is 6 inches tall and has a diameter of 2 inches. This happens to be your friend Jeff's favorite part of his ice cream dessert. You decide to use your mathematical prowess to figure out exactly how much waffle cone Jeff is eating. What is the surface area of the waffle cone? (You may assume that the cone is straight across at the top)

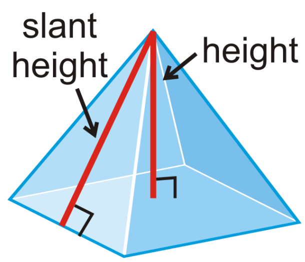
Jeff decides he wants a "king size" cone, which is 8 inches tall and has a diameter of 4 inches. What is the surface area of this cone?

Parts of a Pyramid

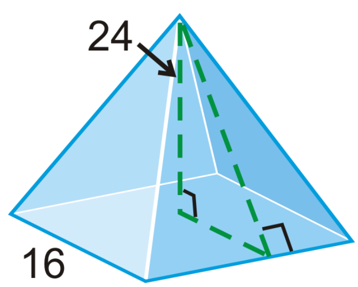
A pyramid has one ***base*** and all the ***lateral faces*** meet at a common ***vertex***. The edges between the lateral faces are ***lateral edges***. The edges between the base and the lateral faces are called ***base edges***. If we were to draw the height of the pyramid to the right, it would be off to the left side.



When a pyramid has a height that is directly in the center of the base, the pyramid is said to be regular. These pyramids have a regular polygon as the base. All regular pyramids also have a ***slant height*** that is the height of a lateral face. Because of the nature of regular pyramids, ***all slant heights are congruent***. A non-regular pyramid does not have a slant height.



Example 1: Find the slant height of the square pyramid.



Solution: Notice that the slant height is the hypotenuse of a right triangle formed by the height and half the base length. Use the Pythagorean Theorem.

$$\begin{aligned} 8^2 + 24^2 &= l^2 \\ 64 + 576 &= l^2 \\ 640 &= l^2 \\ l &= \sqrt{640} = 8\sqrt{10} \end{aligned}$$

Surface Area of a Regular Pyramid

Using the slant height, which is usually labeled l, the area of each triangular face is $A = \frac{1}{2}bl$.

Example 2: Find the surface area of the pyramid from Example 1.

Solution: The surface area of the four triangular faces are $4\left(\frac{1}{2}bl\right) = 2(16)\left(8\sqrt{10}\right) = 256\sqrt{10}$. To find the total surface area, we also need the area of the base, which is $16^2 = 256$. The total surface area is $256\sqrt{10} + 256 \approx 1065.54$.

From this example, we see that the formula for a square pyramid is:

$$SA = (area\ of\ the\ base) + 4(area\ of\ triangular\ faces)$$

$$SA = B + n\left(\frac{1}{2}bl\right) \qquad B \text{ is the area of the base and } n \text{ is the number of triangles.}$$

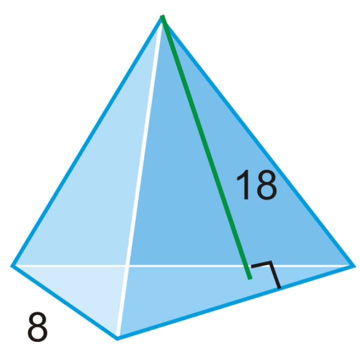
$$SA = B + \frac{1}{2}l(nb) \qquad \text{Rearranging the variables, } nb = P, \text{ the perimeter of the base.}$$

$$SA = B + \frac{1}{2}Pl$$

Surface Area of a Regular Pyramid: If B is the area of the base and P is the perimeter of the base and l is the slant height, then $SA = B + \frac{1}{2}Pl$.

If you ever forget this formula, use the net. Each triangular face is congruent, plus the area of the base. This way, you do not have to remember a formula, just a process, which is the same as finding the area of a prism.

Example 3: Find the area of the regular triangular pyramid.



Solution: The area of the base is $A = \frac{1}{4}s^2\sqrt{3}$ because it is an equilateral triangle.

$$B = \frac{1}{4}8^2\sqrt{3} = 16\sqrt{3}$$

$$SA = 16\sqrt{3} + \frac{1}{2}(24)(18) = 16\sqrt{3} + 216 \approx 243.71$$

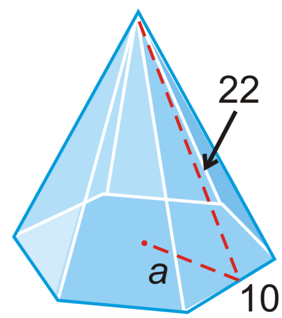
Example 4: If the lateral surface area of a square pyramid is $72\ ft^2$ and the base edge is equal to the slant height, what is the length of the base edge?

Solution: In the formula for surface area, the lateral surface area is $\frac{1}{2}Pl$ or $\frac{1}{2}nbl$. We know that $n = 4$ and $b = l$. Let's solve for b.

$$\begin{aligned}\frac{1}{2}nbl &= 72\ ft^2\\ \frac{1}{2}(4)b^2 &= 72\\ 2b^2 &= 72\\ b^2 &= 36\\ b &= 6\end{aligned}$$

Therefore, the base edges are all 6 units and the slant height is also 6 units.

Example 4: Find the area of the regular hexagonal pyramid below.

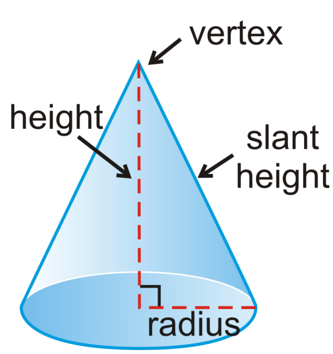


Solution: To find the area of the base, we need to find the apothem. If the base edges are 10 units, then the apothem is $5\sqrt{3}$ for a regular hexagon. The area of the base is $\frac{1}{2}asn = \frac{1}{2}\left(5\sqrt{3}\right)(10)(6) = 150\sqrt{3}$. The total surface area is:

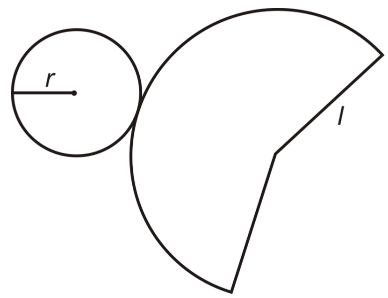
$$\begin{aligned}SA &= 150\sqrt{3} + \frac{1}{2}(6)(10)(22)\\ &= 150\sqrt{3} + 660 \approx 919.81\ units^2\end{aligned}$$

Surface Area of a Cone

Cone: A solid with a circular base and sides taper up towards a common vertex.



It is said that a cone is generated from rotating a right triangle around one leg in a circle. Notice that a cone has a slant height, just like a pyramid. The surface area of a cone is a little trickier, however. We know that the base is a circle, but we need to find the formula for the curved side that tapers up from the base. Unfolding a cone, we have the net:



From this, we can see that the lateral face's edge is $2\pi r$ and the sector of a circle with radius l. We can find the area of the sector by setting up a proportion.

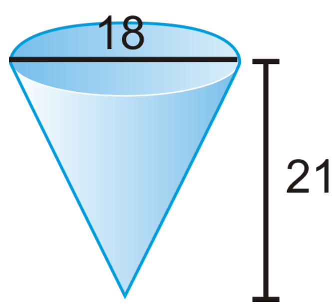
$$\frac{Area\ of\ circle}{Area\ of\ sector} = \frac{Circumference}{Arc\ length}$$
$$\frac{\pi l^2}{Area\ of\ sector} = \frac{2\pi l}{2\pi r} = \frac{l}{r}$$

Cross multiply:

$$l(Area\ of\ sector) = \pi r l^2$$
$$Area\ of\ sector = \pi r l$$

Surface Area of a Right Cone: The surface area of a right cone with slant height l and base radius r is $SA = \pi r^2 + \pi r l$.

Example 5: What is the surface area of the cone?



Solution: In order to find the surface area, we need to find the slant height. Recall from a pyramid, that the slant height forms a right triangle with the height and the radius. Use the Pythagorean Theorem.

$$l^2 = 9^2 + 21^2$$
$$= 81 + 441$$
$$l = \sqrt{522} \approx 22.85$$

The surface area would be $SA = \pi 9^2 + \pi(9)(22.85) \approx 900.54\ units^2$.

Example 6: The surface area of a cone is 36π and the slant height is 5 units. What is the radius?

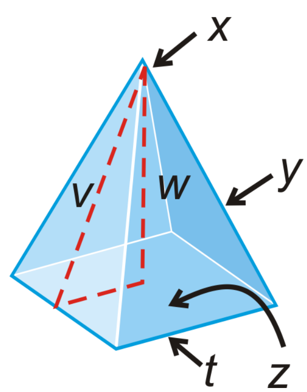
Solution: Plug in what you know into the formula for the surface area of a cone and solve for r.

$$
\begin{aligned}
36\pi &= \pi r^2 + \pi r(5) && \text{Because every term has } \pi, \text{ we can cancel it out.}\\
36 &= r^2 + 5r && \text{Set one side equal to zero, and this becomes a factoring problem.}\\
r^2 + 5r - 36 &= 0 &&\\
(r-4)(r+9) &= 0 && \text{The possible answers for } r \text{ are 4 and } -9. \text{ The radius must be positive,}\\
& && \text{so our answer is 4.}
\end{aligned}
$$

Know What? Revisited The standard cone has a surface area of $\pi + 6\pi = 7\pi \approx 21.99\ in^2$. The "king size" cone has a surface area of $4\pi + 16\pi = 20\pi \approx 62.83$, almost three times as large as the standard cone.

Review Questions

Fill in the blanks about the diagram to the left.

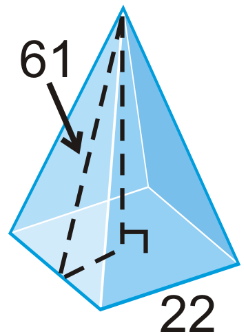


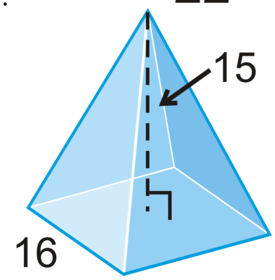
1. x is the ___________.
2. The slant height is ________.
3. y is the ___________.
4. The height is ________.
5. The base is _______.
6. The base edge is ________.
7. Sketch a right cone. Label the *height, slant height*, and *radius*.

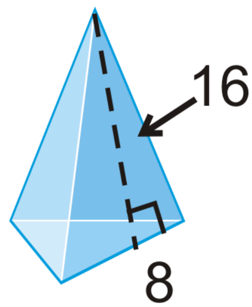
For questions 8-10, sketch each of the following solids and answer the question. Your drawings should be to scale, but not one-to-one. Leave your answer in simplest radical form.

8. Draw a right cone with a radius of 5 cm and a height of 15 cm. What is the slant height?
9. Draw a square pyramid with an edge length of 9 in and a 12 in height. Find the slant height.
10. Draw an equilateral triangle pyramid with an edge length of 6 cm and a height of 6 cm. ***Describe*** how you would find the slant height and then find it.

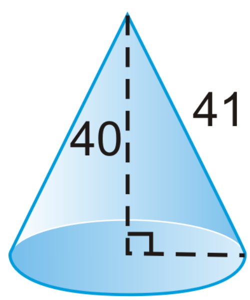
Find the area of a lateral face of the regular pyramid. Leave your answer in simplest radical form.

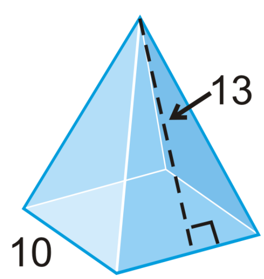
11. 

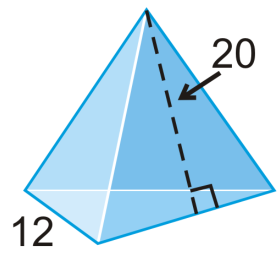
12. 

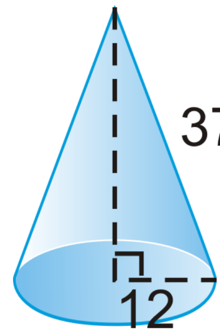
13. 

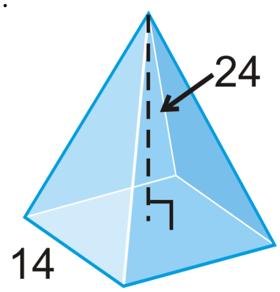
Find the surface area of the regular pyramids and right cones. Round your answers to 2 decimal places.

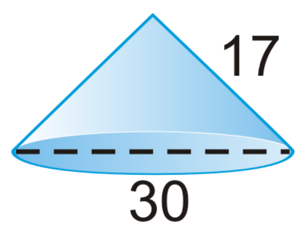
14. 

15. 

16. 

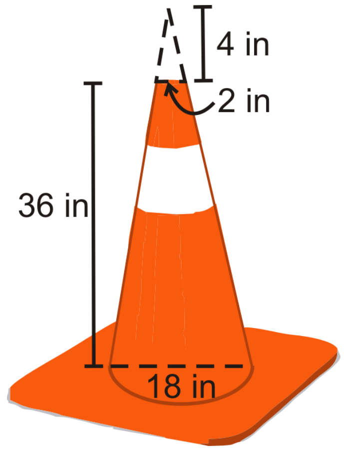
17. 

18. 

19. 

20. From these pictures, we see that a regular triangle pyramid does not have to have four congruent faces. How many faces must be congruent?
21. A ***regular tetrahedron*** has four equilateral triangles as its faces. Find the surface area of a regular tetrahedron with edge length of 6 units.
22. Using the formula for the area of an equilateral triangle, what is the surface area of a regular tetrahedron, with edge length s?

Challenge Find the surface area of the traffic cone with the given information. The gone is cut off at the top (4 inch cone) and the base is a square with sides of length 24 inches. Round answers to the nearest hundredth.



23. Find the area of the entire square. Then, subtract the area of the base of the cone.
24. Find the lateral area of the cone portion (include the 4 inch cut off top of the cone).
25. Now, subtract the cut-off top of the cone, to only have the lateral area of the cone portion of the traffic cone.
26. Combine your answers from #23 and #25 to find the entire surface area of the traffic cone.

For questions 27-30, consider the sector of a circle with radius 25 cm and arc length 14π.

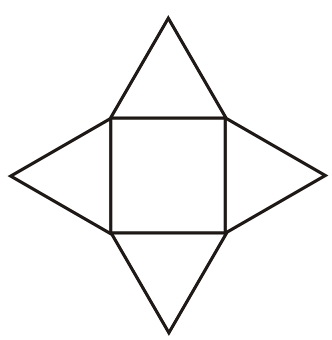
27. What is the central angle of this sector?
28. If this sector is rolled into a cone, what are the radius and area of the base of the cone?
29. What is the height of this cone?
30. What is the total surface are of the cone?

For questions 31-33, consider a square with diagonal length $10\sqrt{2}\ in$.

31. What is the length of a side of the square?
32. If this square is the base of a right pyramid with height 12, what is the slant height of the pyramid?
33. What is the surface area of the pyramid?

Review Queue Answers

a. $2(5\cdot 6)+2(5\cdot 7)+2(6\cdot 7)=214\ cm^2$
b. $2(15\cdot 18)+2(15\cdot 21)+2(18\cdot 21)=1926\ cm^2$
c. $2\cdot 25\pi+250\pi=300\pi\ in^2$
d. $36^2(2\pi)+72\pi(24)=4320\pi\ ft^2$

e. 

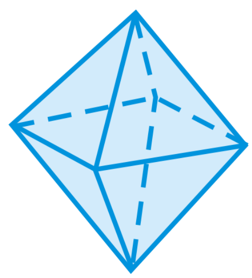
11.4 Volume of Prisms and Cylinders

Learning Objectives

- Find the volume of a prism.
- Find the volume of a cylinder.

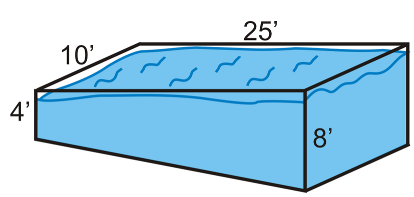
Review Queue

a. Define volume in your own words.
b. What is the surface area of a cube with 3 inch sides?
c. What is the surface area of a cube with $4\sqrt{2}$ inch sides?
d. A regular octahedron has 8 congruent equilateral triangles as the faces.



  a. If each edge is 4 cm, what is the surface area of the figure?
  b. If each edge is s, what is the surface area of the figure?

Know What? The pool is done and your family is ready to fill it with water. Recall that the shallow end is 4 ft. and the deep end is 8 ft. The pool is 10 ft. wide by 25 ft. long. How many gallons of water will it take to fill the pool? There are approximately 7.48 gallons in a cubic foot.



Volume of a Rectangular Prism

Volume: The measure of how much space a three-dimensional figure occupies.

Another way to define volume would be how much a three-dimensional figure can hold, water or sand, for example. The basic unit of volume is the cubic unit: cubic centimeter (cm^3), cubic inch (in^3), cubic meter (m^3), cubic foot (ft^3), etc. Each basic cubic unit has a measure of one for each: length, width, and height.

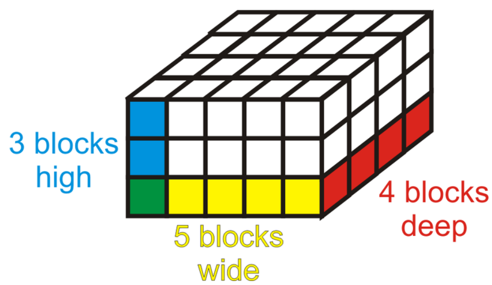
Volume of a Cube Postulate: The volume of a cube is the cube of the length of its side, or s^3.

What this postulate tells us is that every solid can be broken down into cubes, going along with our basic unit of measurement, the cubic unit. For example, if we wanted to find the volume of a cube with one inch sides, it would be $1^3 = 1\ in^3$. If we wanted to find the volume of a cube with 9 inch sides, it would be $9^3 = 729\ in^3$.

Volume Congruence Postulate: If two solids are congruent, then their volumes are congruent.

Volume Addition Postulate: The volume of a solid is the sum of the volumes of all of its non-overlapping parts.

Example 1: Find the volume of the right rectangular prism below.



Solution: A rectangular prism can be made from any square cubes. To find the volume, we would simply count the cubes. The bottom layer has 20 cubes, or 4 times 5, and there are 3 layers, or the same as the height. Therefore there are 60 cubes in this prism and the volume would be 60 $units^3$.

But, what if we didn't have cubes? Let's generalize this formula for any rectangular prism. Notice that each layer is the same as the area of the base. Then, we multiplied by the height. Here is our formula.

Volume of a Rectangular Prism: If a rectangular prism is h units high, w units wide, and l units long, then its volume is $V = l \cdot w \cdot h$.

Example 2: A typical shoe box is 8 in by 14 in by 6 in. What is the volume of the box?

Solution: We can assume that a shoe box is a rectangular prism. Therefore, we can use the formula above.

$$V = (8)(14)(6) = 672\ in^2$$

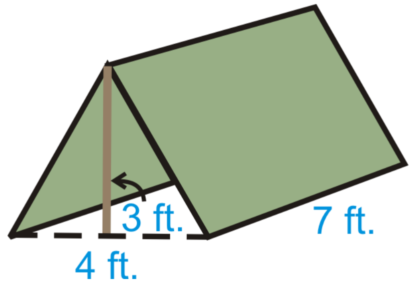
Volume of any Prism

If we further analyze the formula for the volume of a rectangular prism, we would see that $l \cdot w$ is equal to the area of the base of the prism, a rectangle. If the bases are not rectangles, this would still be true, however we would have to rewrite the equation a little.

Volume of a Prism: If the area of the base of a prism is B and the height is h, then the volume is $V = B \cdot h$.

Notice that "B" is not always going to be the same. So, to find the volume of a prism, you would first find the area of the base and then multiply it by the height.

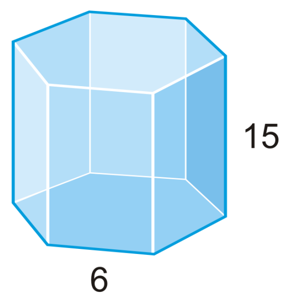
Example 3: You have a small, triangular prism shaped tent. How much volume does it have, once it is set up?



Solution: First, we need to find the area of the base. That is going to be $B = \frac{1}{2}(3)(4) = 6\ ft^2$. Multiplying this by 7 we would get the entire volume. The volume is $42\ ft^3$.

Even though the height in this problem does not look like a "height," it is, when referencing the formula. Usually, the height of a prism is going to be the last length you need to use.

Example 4: Find the volume of the regular hexagonal prism below.



Solution: Recall that a regular hexagon is divided up into six equilateral triangles. The height of one of those triangles would be the apothem. If each side is 6, then half of that is 3 and half of an equilateral triangle is a 30-60-90 triangle. Therefore, the apothem is going to be $3\sqrt{3}$. The area of the base is:

$$B = \frac{1}{2}\left(3\sqrt{3}\right)(6)(6) = 54\sqrt{3}\ units^2$$

And the volume will be:

$$V = Bh = \left(54\sqrt{3}\right)(15) = 810\sqrt{3}\ units^3$$

Cavalieri's Principle

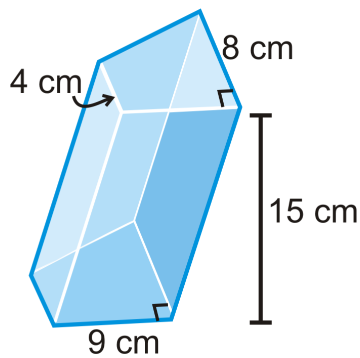
Recall that earlier in this section we talked about oblique prisms. These are prisms that lean to one side and the height is outside the prism. What would be the area of an oblique prism? To answer this question, we need to introduce Cavalieri's Principle. Consider to piles of books below.

Both piles have 15 books, therefore they will have the same volume. However, one pile is leaning. Cavalieri's Principle says that this does not matter, as long as the heights are the same and every horizontal cross section has the same area as the base, the volumes are the same.

Cavalieri's Principle: If two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

Basically, if an oblique prism and a right prism have the same base area and height, then they will have the same volume.

Example 5: Find the area of the oblique prism below.



Solution: This is an oblique right trapezoidal prism. First, find the area of the trapezoid.

$$B = \frac{1}{2}(9)(8+4) = 9(6) = 54\ cm^2$$

Then, multiply this by the height.

$$V = 54(15) = 810\ cm^3$$

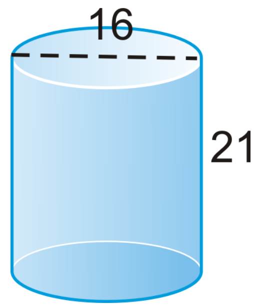
Volume of a Cylinder

If we use the formula for the volume of a prism, $V = Bh$, we can find the volume of a cylinder. In the case of a cylinder, the base, or B, would be the area of a circle. Therefore, the volume of a cylinder would be $V = (\pi r^2)h$, where πr^2 is the area of the base.

Volume of a Cylinder: If the height of a cylinder is h and the radius is r, then the volume would be $V = \pi r^2 h$.

Also, like a prism, Cavalieri's Principle holds. So, the volumes of an oblique cylinder and a right cylinder have the same formula.

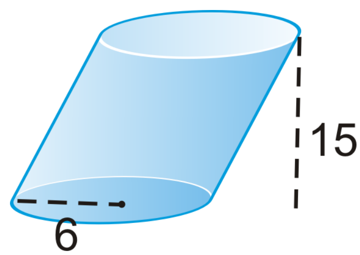
Example 6: Find the volume of the cylinder.



Solution: If the diameter is 16, then the radius is 8.

$$V = \pi 8^2(21) = 1344\pi \ units^3$$

Example 7: Find the volume of the cylinder.



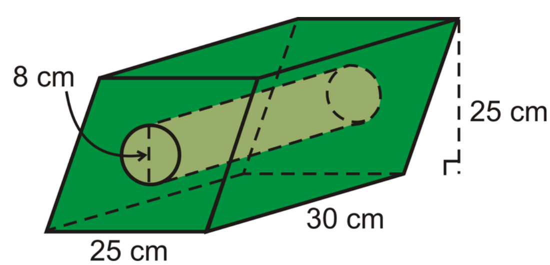
Solution: $V = \pi 6^2(15) = 540\pi \ units^3$

Example 8: If the volume of a cylinder is $484\pi \ in^3$ and the height is 4 in, what is the radius?

Solution: Plug in what you know to the volume formula and solve for r.

$$\begin{aligned}484\pi &= \pi r^2(4)\\ 121 &= r^2\\ 11 &= r\end{aligned}$$

Example 9: Find the volume of the solid below.



Solution: This solid is a parallelogram-based prism with a cylinder cut out of the middle. To find the volume, we need to find the volume of the prism and then subtract the volume of the cylinder.

$$\begin{aligned}V_{prism} &= (25 \cdot 25)30 = 18750 \ cm^3\\ V_{cylinder} &= \pi(4)^2(30) = 480\pi \ cm^3\end{aligned}$$

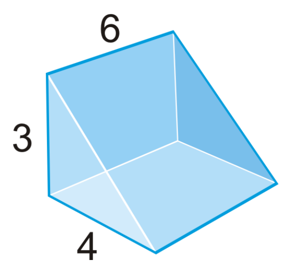
The total volume is $18750-480\pi \approx 17242.04\ cm^3$.

Know What? Revisited Even though it doesn't look like it, the trapezoid is considered the base of this prism. The area of the trapezoids are $\frac{1}{2}(4+8)25 = 150\ ft^2$. Multiply this by the height, 10 ft, and we have that the volume is $1500\ ft^3$. To determine the number of gallons that are needed, divide 1500 by 7.48. $\frac{1500}{7.48} \approx 200.53$ gallons are needed to fill the pool.

Review Questions

1. Two cylinders have the same surface area. Do they have the same volume? How do you know?
2. How many one-inch cubes can fit into a box that is 8 inches wide, 10 inches long, and 12 inches tall? Is this the same as the volume of the box?
3. A cereal box in 2 inches wide, 10 inches long and 14 inches tall. How much cereal does the box hold?
4. A can of soda is 4 inches tall and has a diameter of 2 inches. How much soda does the can hold? Round your answer to the nearest hundredth.
5. A cube holds 216 in^3. What is the length of each edge?
6. A cylinder has a volume of $486\pi\ ft.^3$. If the height is 6 ft., what is the diameter?

Use the right triangular prism to answer questions 7 and 8.

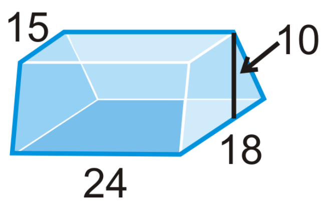


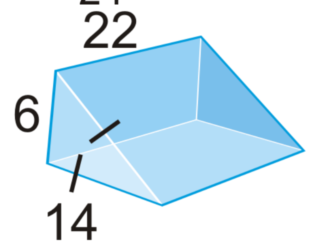
7. What is the length of the third base edge?
8. Find the volume of the prism.
9. Fuzzy dice are cubes with 4 inch sides.

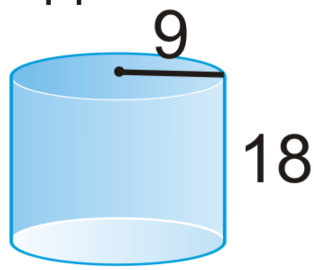
   a. What is the volume of one die?
   b. What is the volume of both dice?

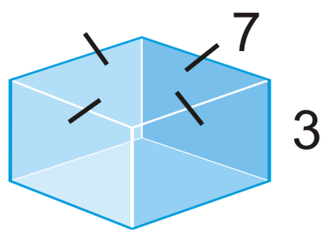
10. A right cylinder has a 7 cm radius and a height of 18 cm. Find the volume.

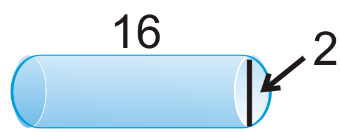
Find the volume of the following solids. Round your answers to the nearest hundredth.

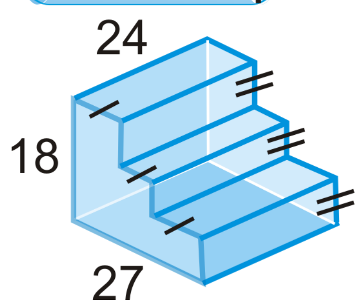
11. 

12. 

13. 

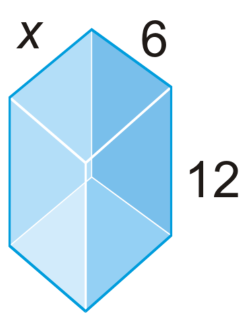
14. 

15. 

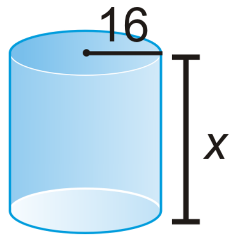
16. 

Algebra Connection Find the value of x, given the surface area.

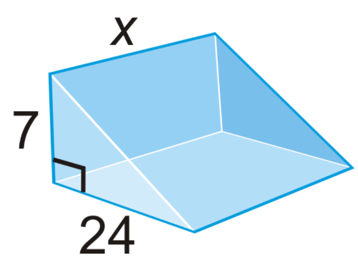
17. $V = 504$ $units^3$



18. $V = 6144\pi\ units^3$

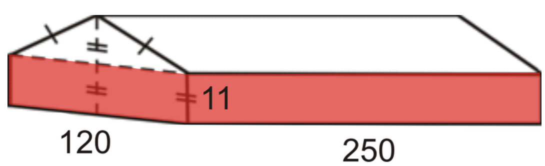


19. $V = 2688\ units^3$



20. The area of the base of a cylinder is $49\pi\ in^2$ and the height is 6 in. Find the volume.
21. The circumference of the base of a cylinder is $80\pi\ cm$ and the height is 15 cm. Find the volume.
22. The lateral surface area of a cylinder is $30\pi\ m^2$ and the circumference is $10\pi\ m$. What is the volume of the cylinder?

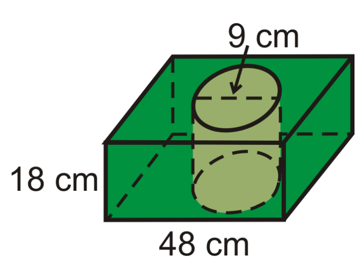
Use the diagram below for questions 23-25. The barn is shaped like a pentagonal prism with dimensions shown in feet.

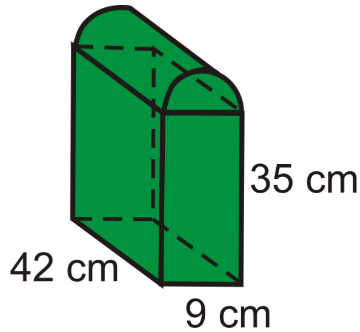


23. Find the volume of the red rectangular prism.
24. Find the volume of the triangular prism on top of the rectangular prism.
25. Find the total volume of the barn.

Find the volume of the composite solids below. Round your answers to the nearest hundredth.

26. The bases are squares.



27. 

28. The volume of a cylinder with height to radius ratio of 4:1 is $108\pi\ cm^3$. Find the radius and height of the cylinder.
29. The length of a side of the base of a hexagonal prism is 8 cm and its volume is $1056\sqrt{3}\ cm^3$. Find the height of the prism.
30. A cylinder fits tightly inside a rectangular prism with dimensions in the ratio 5:5:7 and volume 1400 in^3. Find the volume of the space inside the prism that is not contained in the cylinder.

Review Queue Answers

a. The amount a three-dimensional figure can hold.
b. 54 in^2
c. 192 in^2
   a. $8\left(\frac{1}{4}\cdot 4^2\sqrt{3}\right) = 32\sqrt{3}\ cm^2$
   b. $8\left(\frac{1}{4}\cdot s^2\sqrt{3}\right) = 2s^2\sqrt{3}$

11.5 Volume of Pyramids and Cones

Learning Objectives

- Find the volume of a pyramid.
- Find the volume of a cone.

Review Queue

a. Find the volume of a square prism with 8 inch base edges and a 12 inch height.
b. Find the volume of a cylinder with a *diameter* of 8 inches and a height of 12 inches.
c. In your answers from #1 and #2, which volume is bigger? Why do you think that is?
d. Find the surface area of a square pyramid with 10 inch base edges and a height of 12 inches.

Know What? The Khafre Pyramid is the second largest pyramid of the Ancient Egyptian Pyramids in Giza. It is a square pyramid with a base edge of 706 feet and an original height of 407.5 feet. What was the original volume of the Khafre Pyramid?

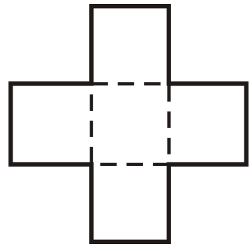
Volume of a Pyramid

Recall that the volume of a prism is Bh, where B is the area of the base. The volume of a pyramid is closely related to the volume of a prism with the same sized base.

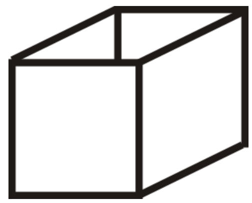
Investigation 11-1: Finding the Volume of a Pyramid

Tools needed: pencil, paper, scissors, tape, ruler, dry rice or sand.

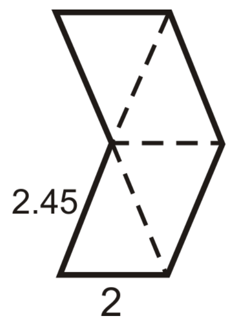
a. Make an open net (omit one base) of a cube, with 2 inch sides.



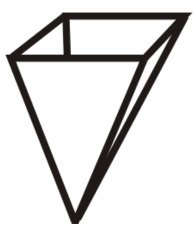
b. Cut out the net and tape up the sides to form an open cube.



c. Make an open net (no base) of a square pyramid, with lateral edges of 2.45 inches and base edges of 2 inches. This will make the overall height 2 inches.



d. Cut out the net and tape up the sides to form an open pyramid.

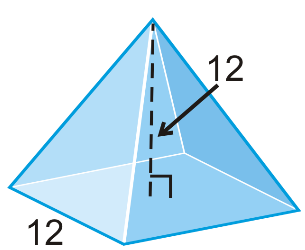


e. Fill the pyramid with dry rice. Then, dump the rice into the open cube. How many times do you have to repeat this to fill the cube?

Volume of a Pyramid: If B is the area of the base and h is the height, then the volume of a pyramid is $V = \frac{1}{3}Bh$.

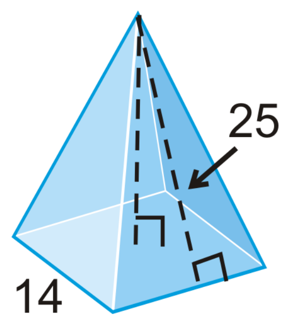
The investigation showed us that you would need to repeat this process three times to fill the cube. This means that the pyramid is one-third the volume of a prism with the same base.

Example 1: Find the volume of the pyramid.



Solution: $V = \frac{1}{3}(12^2)12 = 576\ units^3$

Example 2: Find the volume of the pyramid.

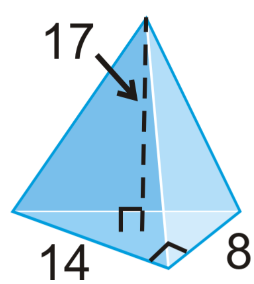


Solution: In this example, we are given the slant height. For volume, we need the height, so we need to use the Pythagorean Theorem to find it.

$$\begin{aligned} 7^2 + h^2 &= 25^2 \\ h^2 &= 576 \\ h &= 24 \end{aligned}$$

Using the height, the volume is $\frac{1}{3}(14^2)(24) = 1568\ units^3$.

Example 3: Find the volume of the pyramid.



Solution: The base of this pyramid is a right triangle. So, the area of the base is $\frac{1}{2}(14)(8) = 56\ units^2$.

$$V = \frac{1}{3}(56)(17) \approx 317.33\ units^3$$

Example 4: A rectangular pyramid has a base area of 56 cm^2 and a volume of 224 cm^3. What is the height of the pyramid?

Solution: The formula for the volume of a pyramid works for any pyramid, as long as you can find the area of the base.

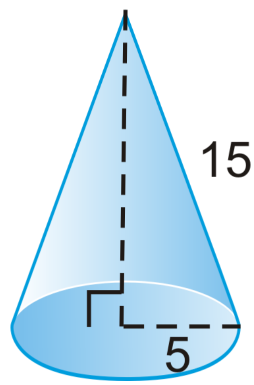
$$\begin{aligned} 224 &= 56h \\ 4 &= h \end{aligned}$$

Volume of a Cone

The volume of cone has the same relationship with a cylinder as pyramid does with a prism. If the bases of a cone and a cylinder are the same, then the volume of a cone will be one-third the volume of the cylinder.

Volume of a Cone: If r is the radius of a cone and h is the height, then the volume is $V = \frac{1}{3}\pi r^2 h$.

Example 5: Find the volume of the cone.



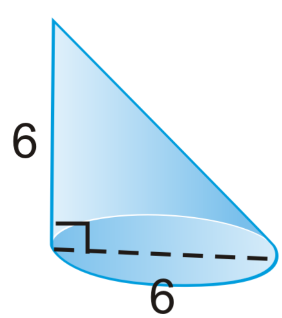
Solution: To find the volume, we need the height, so we have to use the Pythagorean Theorem.

$$\begin{aligned} 5^2 + h^2 &= 15^2 \\ h^2 &= 200 \\ h &= 10\sqrt{2} \end{aligned}$$

Now, we can find the volume.

$$V = \frac{1}{3}(5^2)\left(10\sqrt{2}\right)\pi \approx 370.24$$

Example 6: Find the volume of the cone.



Solution: Even though this doesn't look like the cone in Example 5, we can still find the volume in the same way. Use the *radius* in the formula.

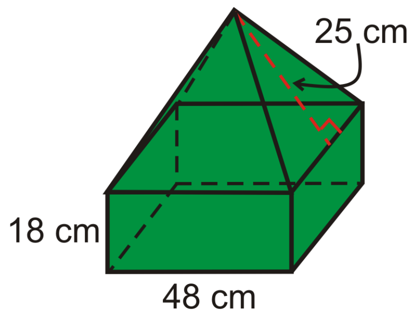
$$V = \frac{1}{3}\pi(3^2)(6) = 18\pi \approx 56.55$$

Example 7: The volume of a cone is $484\pi\ cm^3$ and the height is 12 cm. What is the radius?

Solution: Plug in what you know to the volume formula.

$$\begin{aligned} 484\pi &= \frac{1}{3}\pi r^2(12) \\ 121 &= r^2 \\ 11 &= r \end{aligned}$$

Example 8: Find the volume of the composite solid. All bases are squares.



Solution: This is a square prism with a square pyramid on top. Find the volume of each separeatly and then add them together to find the total volume. First, we need to find the height of the pyramid portion. The slant height is 25 and the edge is 48. Using have of the edge, we have a right triangle and we can use the Pythagorean Theorem. $h = \sqrt{25^2 - 24^2} = 7$

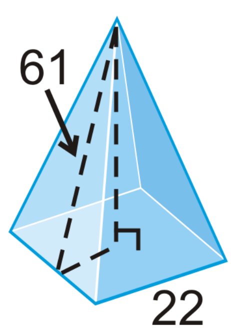
$$\begin{aligned} V_{prism} &= (48)(48)(18) = 41472\ cm^3 \\ V_{pyramid} &= \frac{1}{3}(48^2)(7) = 5376\ cm^3 \end{aligned}$$

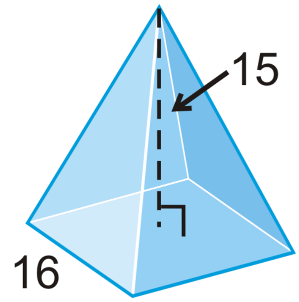
The total volume is $41472 + 5376 = 46,848\ cm^3$.

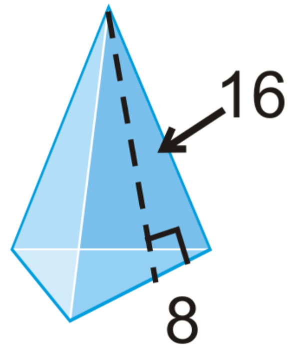
Know What? Revisited The original volume of the pyramid is $\frac{1}{3}(706^2)(407.5) \approx 67,704,223.33\ ft^3$.

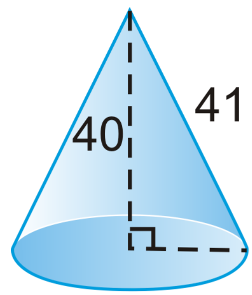
Review Questions

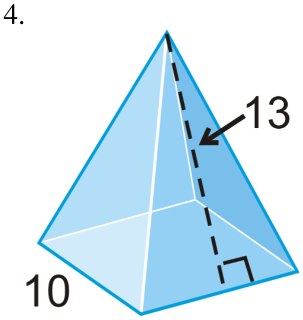
Find the volume of each regular pyramid and right cone. Round any decimal answers to the nearest hundredth. The bases of these pyramids are either squares or equilateral triangles.

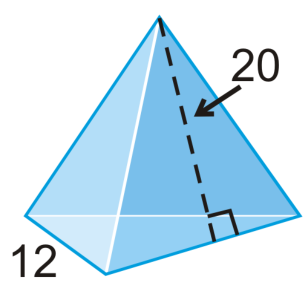
1. 

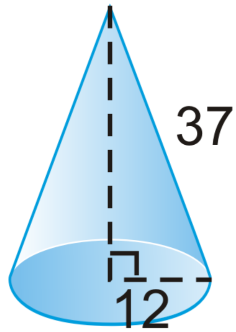
2. 

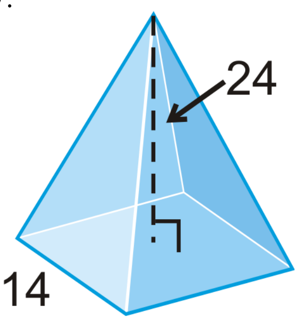
3. 

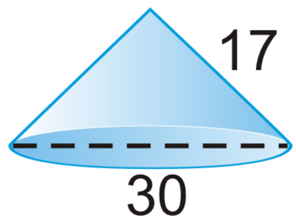
4. 

5. 

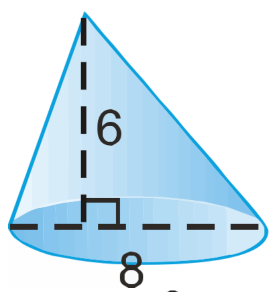
6. 

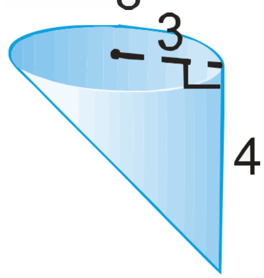
7. 

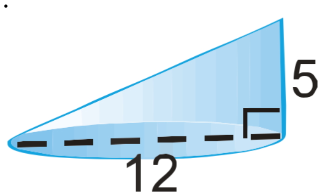
8. 

9. 

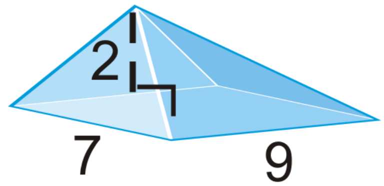
Find the volume of the following non-regular pyramids and cones. Round any decimal answers to the nearest hundredth.

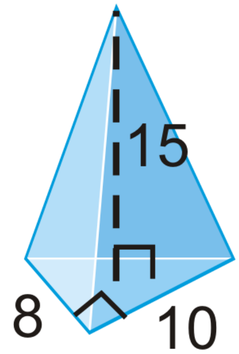
10. 

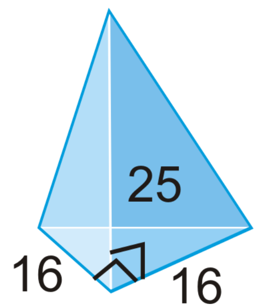
11. 

12. 

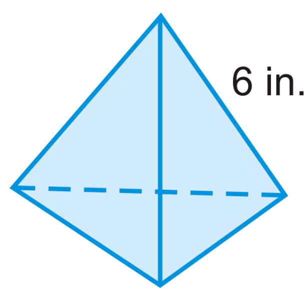
13. base is a rectangle



14. 

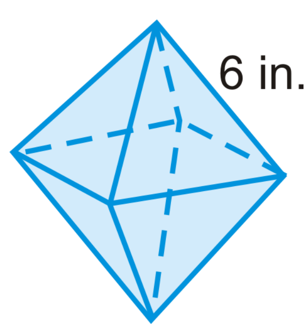
15. 

A ***regular tetrahedron*** has four equilateral triangles as its faces. Use the diagram to answer questions 16-19.



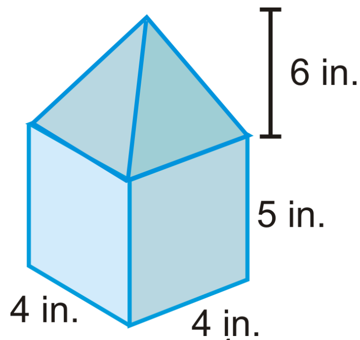
16. What is the area of the base of this regular etrahedron?
17. What is the height of this figure? Be careful!
18. Find the volume. Leave your answer in simplest radical form.
19. ***Challenge*** If the sides are length *s*, what is the volume?

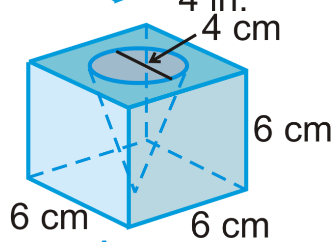
A ***regular octahedron*** has eight equilateral triangles as its faces. Use the diagram to answer questions 19-21.

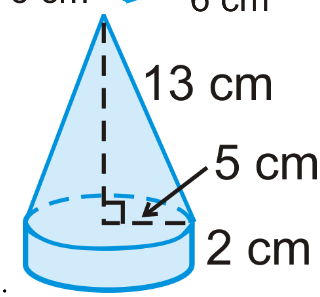


20. *Describe* how you would find the volume of this figure.
21. Find the volume. Leave your answer in simplest radical form.
22. ***Challenge*** If the sides are length s, what is the volume?
23. The volume of a square pyramid is 72 square inches and the base edge is 4 inches. What is the height?
24. If the volume of a cone is $30\pi\ cm^2$ and the radius is 5 cm, what is the height?
25. If the volume of a cone is $105\pi\ cm^2$ and the height is 35 cm, what is the radius?

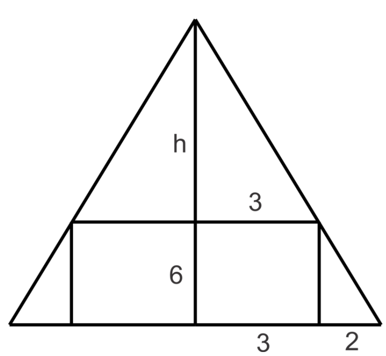
Find the volume of the composite solids. Round your answer to the nearest hundredth.

26. 

27. 

28. 

29. The ratio of the height to radius in a cone is 3:2. If the volume is $108\pi\ m^3$, find the height and radius of the cone.
30. A teepee is to be built such that there is a minimal cylindrical shaped central living space contained within the cone shape of diameter 6 ft and height 6 ft. If the radius of the entire teepee is 5 ft, find the total height of the teepee.



Review Queue Answers

a. $(8^2)(12) = 768\ in^3$
b. $(4^2)(12)\pi = 192\pi \approx 603.19$
c. The volume of the square prism is greater because the square base is larger than a circle with the same diameter as the square's edge.
d. Find slant height, $l = 13$. $SA = 100 + \frac{1}{2}(40)(13) = 360\ in^2$

11.6 Surface Area and Volume of Spheres

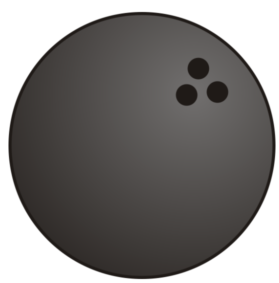
Learning Objectives

- Find the surface area of a sphere.
- Find the volume of a sphere.

Review Queue

a. List three spheres you would see in real life.
b. Find the area of a circle with a 6 cm radius.
c. Find the volume of a cylinder with the circle from #2 as the base and a height of 5 cm.
d. Find the volume of a cone with the circle from #2 as the base and a height of 5 cm.

Know What? A regulation bowling ball is a sphere that weighs between 12 and 16 pounds. The maximum circumference of a bowling ball is 27 inches. Using this number, find the radius of a bowling ball, its surface area and volume. You may assume the bowling ball does not have any finger holes. Round your answers to the nearest hundredth.

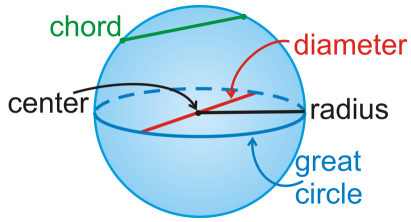


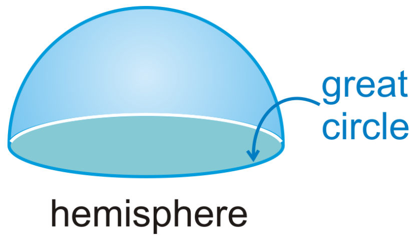
Defining a Sphere

A sphere is the last of the three-dimensional shapes that we will find the surface area and volume of. Think of a sphere as a three-dimensional circle. You have seen spheres in real-life countless times; tennis balls, basketballs, volleyballs, golf balls, and baseballs. Now we will analyze the parts of a sphere.

Sphere: The set of all points, in three-dimensional space, which are equidistant from a point.

A sphere has a ***center***, radius and diameter, just like a circle. The ***radius*** has an endpoint on the sphere and the other is on the center. The ***diameter*** must contain the center. If it does not, it is a ***chord***. The ***great circle*** is a plane that contains the diameter. It would be the largest circle cross section in a sphere. There are infinitely many great circles. ***The circumference of a sphere is the circumference of a great circle***. Every great circle divides a sphere into two congruent hemispheres, or two half spheres. Also like a circle, spheres can have tangent lines and secants. These are defined just like they are in a circle.





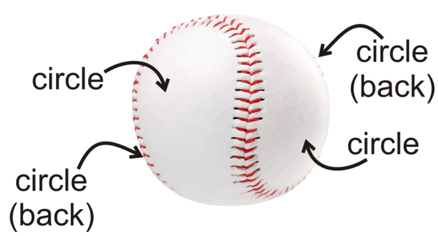
Example 1: The circumference of a sphere is 26π *feet*. What is the radius of the sphere?

Solution: The circumference is referring to the circumference of a great circle. Use $C = 2\pi r$.

$$\begin{aligned} 2\pi r &= 26\pi \\ r &= 13\ ft. \end{aligned}$$

Surface Area of a Sphere

One way to find the formula for the surface area of a sphere is to look at a baseball. We can best *approximate* the cover of the baseball by 4 circles. The area of a circle is πr^2, so the surface area of a sphere is $4\pi r^2$. While the covers of a baseball are not four perfect circles, they are stretched and skewed.



Another way to show the surface area of a sphere is to watch the link by Russell Knightley, http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Surface-Area-Derivation.html . It is a great visual interpretation of the formula.

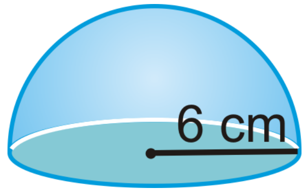
Surface Area of a Sphere: If r is the radius, then the surface area of a sphere is $SA = 4\pi r^2$.

Example 2: Find the surface area of a sphere with a radius of 14 feet.

Solution: Use the formula, $r = 14\ ft$.

$$\begin{aligned} SA &= 4\pi(14)^2 \\ &= 784\pi\ ft^2 \end{aligned}$$

Example 3: Find the surface area of the figure below.



Solution: This is a hemisphere. Be careful when finding the surface area of a hemisphere because you need to include the area of the base. If the question asked for the *lateral surface area*, then your answer would *not* include the bottom.

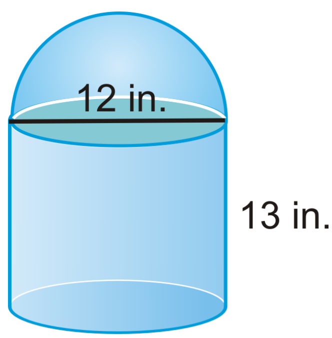
$$\begin{aligned} SA &= \pi r^2 + \frac{1}{2}4\pi r^2 \\ &= \pi(6^2) + 2\pi(6^2) \\ &= 36\pi + 72\pi = 108\pi \ cm^2 \end{aligned}$$

Example 4: The surface area of a sphere is $100\pi \ in^2$. What is the radius?

Solution: Plug in what you know to the formula and solve for r.

$$\begin{aligned} 100\pi &= 4\pi r^2 \\ 25 &= r^2 \\ 5 &= r \end{aligned}$$

Example 5: Find the surface area of the following solid.

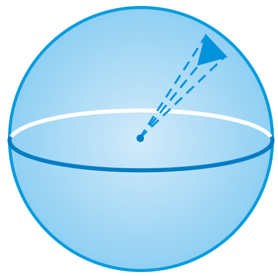


Solution: This solid is a cylinder with a hemisphere on top. Because it is one fluid solid, we would not include the bottom of the hemisphere or the top of the cylinder because they are no longer on the surface of the solid. Below, "*LA*" stands for *lateral area*.

$$\begin{aligned} SA &= LA_{cylinder} + LA_{hemisphere} + A_{base\ circle} \\ &= \pi rh + \frac{1}{2}4\pi r^2 + \pi r^2 \\ &= \pi(6)(13) + 2\pi 6^2 + \pi 6^2 \\ &= 78\pi + 72\pi + 36\pi \\ &= 186\pi \ in^2 \end{aligned}$$

Volume of a Sphere

A sphere can be thought of as a regular polyhedron with an infinite number of congruent regular polygon faces. As n, the number of faces increases to an infinite number, the figure approaches becoming a sphere. So a sphere can be thought of as a polyhedron with an infinite number faces. Each of those faces is the base of a pyramid whose vertex is located at the center of the sphere. Each of the pyramids that make up the sphere would be congruent to the pyramid shown. The volume of this pyramid is given by $V = \frac{1}{3}Bh$.



To find the volume of the sphere, you need to add up the volumes of an infinite number of infinitely small pyramids.

$$\begin{aligned} V(all\ pyramids) &= V_1 + V_2 + V_3 + \ldots + V_n \\ &= \frac{1}{3}(B_1h + B_2h + B_3h + \ldots + B_nh) \\ &= \frac{1}{3}h(B_1 + B_2 + B_3 + \ldots + B_n) \end{aligned}$$

The sum of all of the bases of the pyramids is the surface area of the sphere. Since you know that the surface area of the sphere is $4\pi r^2$, you can substitute this quantity into the equation above.

$$= \frac{1}{3}h\left(4\pi r^2\right)$$

In the picture above, we can see that the height of each pyramid is the radius, so $h = r$.

$$\begin{aligned} &= \frac{4}{3}r(\pi r^2) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

To see an animation of the volume of a sphere, see http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Volume-Derivation.html by Russell Knightley. It is a slightly different interpretation than our derivation.

Volume of a Sphere: If a sphere has a radius r, then the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Example 6: Find the volume of a sphere with a radius of 9 m.

Solution: Use the formula above.

$$\begin{aligned} V &= \frac{4}{3}\pi 6^3 \\ &= \frac{4}{3}\pi(216) \\ &= 288\pi \end{aligned}$$

Example 7: A sphere has a volume of 14137.167 ft^3, what is the radius?

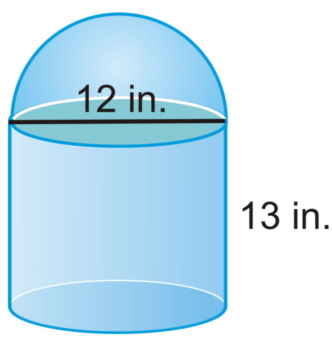
Solution: Because we have a decimal, our radius might be an approximation. Plug in what you know to the formula and solve for r.

$$\begin{aligned} 14137.167 &= \frac{4}{3}\pi r^3 \\ \frac{3}{4\pi} \cdot 14137.167 &= r^3 \\ 3375 &= r^3 \end{aligned}$$

At this point, you will need to take the ***cubed root*** of 3375. Depending on your calculator, you can use the $\sqrt[3]{x}$ function or $\wedge\left(\frac{1}{3}\right)$. The cubed root is the inverse of cubing a number, just like the square root is the inverse, or how you undo, the square of a number.

$$\sqrt[3]{3375} = 15 = r \qquad \text{The radius is } 15\ ft.$$

Example 8: Find the volume of the following solid.



Solution: To find the volume of this solid, we need the volume of a cylinder and the volume of the hemisphere.

$$\begin{aligned} V_{cylinder} &= \pi 6^2(13) = 78\pi \\ V_{hemisphere} &= \frac{1}{2}\left(\frac{4}{3}\pi 6^3\right) = 36\pi \\ V_{total} &= 78\pi + 36\pi = 114\pi\ in^3 \end{aligned}$$

Know What? Revisited If the maximum circumference of a bowling ball is 27 inches, then the maximum radius would be $27 = 2\pi r$, or $r = 4.30$ inches. Therefore, the surface area would be $4\pi 4.3^2 \approx 232.35\ in^2$, and the volume would be $\frac{4}{3}\pi\ 4.3^3 \approx 333.04\ in^3$. The weight of the bowling ball refers to its density, how heavy something is. The volume of the ball tells us how much it can hold.

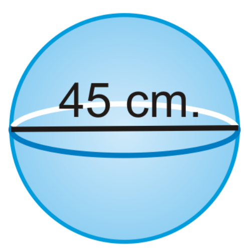
Review Questions

1. Are there any cross-sections of a sphere that are not a circle? Explain your answer.

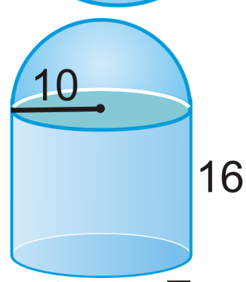
Find the surface area and volume of a sphere with: (Leave your answer in terms of π)

2. a radius of 8 in.
3. a diameter of 18 cm.
4. a radius of 20 ft.
5. a diameter of 4 m.
6. a radius of 15 ft.
7. a diameter of 32 in.
8. a circumference of 26π *cm*.
9. a circumference of 50π *yds*.
10. The surface area of a sphere is $121\pi\ in^2$. What is the radius?
11. The volume of a sphere is $47916\pi\ m^3$. What is the radius?
12. The surface area of a sphere is $4\pi\ ft^2$. What is the volume?
13. The volume of a sphere is $36\pi\ mi^3$. What is the surface area?
14. Find the radius of the sphere that has a volume of $335\ cm^3$. Round your answer to the nearest hundredth.
15. Find the radius of the sphere that has a surface area $225\pi\ ft^2$.

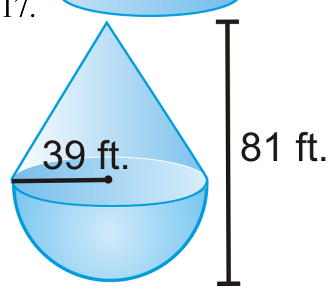
Find the surface area of the following shapes. Leave your answers in terms of π.



16.

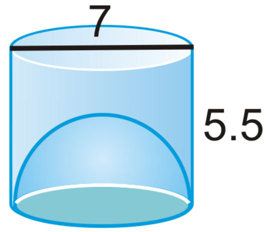


17.



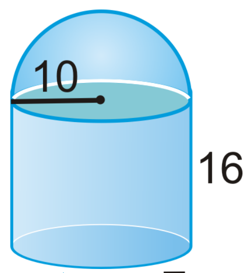
18.

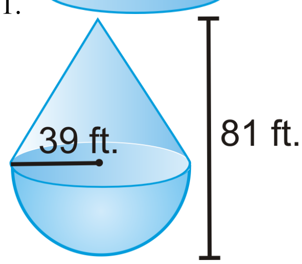
19. You may assume the bottom is *open*.

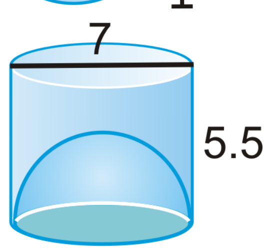


Find the volume of the following shapes. Round your answers to the nearest hundredth.

20. 

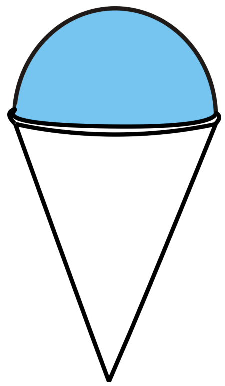
21. 

22. 

23. 

24. A sphere has a radius of 5 cm. A right cylinder has the same radius and volume. Find the height and total surface area of the cylinder.
25. Tennis balls with a 3 inch diameter are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space *not* occupied by the tennis balls? Round your answer to the nearest hundredth.

26. One hot day at a fair you buy yourself a snow cone. The height of the cone shaped container is 5 in and its radius is 2 in. The shaved ice is perfectly rounded on top forming a hemisphere. What is the volume of the ice in your frozen treat? If the ice melts at a rate of 2 cm^3 per minute, how long do you have to eat your treat before it all melts? Give your answer to the nearest minute.



Multi-Step Problems

27.
    a. What is the surface area of a cylinder?
    b. Adjust your answer from part a for the case where $r = h$.
    c. What is the surface area of a sphere?
    d. What is the relationship between your answers to parts b and c? Can you explain this?
28. At the age of 81, Mr. Luke Roberts began collecting string. He had a ball of string 3 feet in diameter.
    a. Find the volume of Mr. Roberts' ball of string in cubic inches.
    b. Assuming that each cubic inch weighs 0.03 pounds, find the weight of his ball of string.
    c. To the nearest inch, how big (diameter) would a 1 ton ball of string be? $(1\ ton = 2000\ lbs)$

For problems 29-31, use the fact that the earth's radius is approximately 4,000 miles.

29. Find the length of the equator.
30. Find the surface area of earth, rounding your answer to the nearest million square miles.
31. Find the volume of the earth, rounding your answer to the nearest billion cubic miles.

Review Queue Answers

a. *Answers will vary.* Possibilities are any type of ball, certain lights, or the 76/Unical orb.
b. 36π
c. 180π
d. 60π

11.7 Exploring Similar Solids

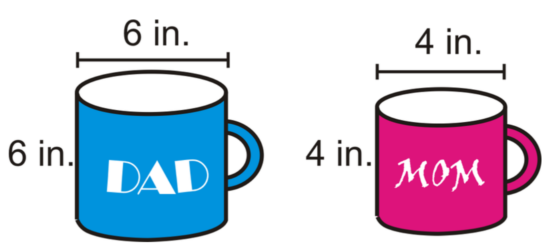
Learning Objectives

- Find the relationship between similar solids and their surface areas and volumes.

Review Queue

a. We know that every circle is similar, is every sphere similar?
b. Find the volume of a sphere with a 12 in radius. Leave your answer in terms of π.
c. Find the volume of a sphere with a 3 in radius. Leave your answer in terms of π.
d. Find the scale factor of the spheres from #2 and #3. Then find the ratio of the volumes and reduce it. What do you notice?
e. Two squares have a scale factor of 2:3. What is the ratio of their areas?
f. The smaller square from #5 has an area of 16 cm^2. What is the area of the larger square?
g. The ratio of the areas of two similar triangles is 1:25. The height of the larger triangle is 20 cm, what is the height of the smaller triangle?

Know What? Your mom and dad have cylindrical coffee mugs with the dimensions to the right. Are the mugs similar? (You may ignore the handles.) If the mugs are similar, find the volume of each, the scale factor and the ratio of the volumes.

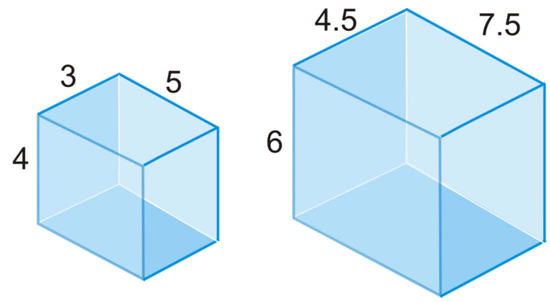


Similar Solids

Recall that two shapes are similar if all the corresponding angles are congruent and the corresponding sides are proportional.

Similar Solids: Two solids are similar if and only if they are the same type of solid and their corresponding linear measures (radii, heights, base lengths, etc.) are proportional.

Example 1: Are the two rectangular prisms similar? How do you know?

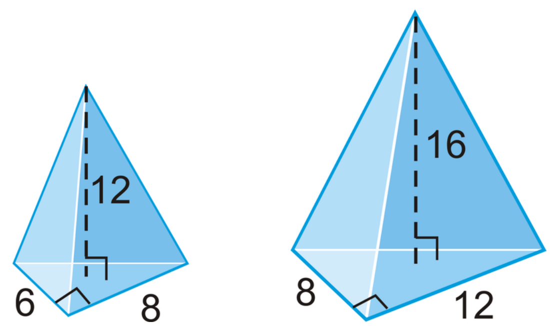


Solution: Match up the corresponding heights, widths, and lengths to see if the rectangular prisms are proportional.

$$\frac{small\ prism}{large\ prism} = \frac{3}{4.5} = \frac{4}{6} = \frac{5}{7.5}$$

The congruent ratios tell us the two prisms are similar.

Example 2: Determine if the two triangular pyramids similar.



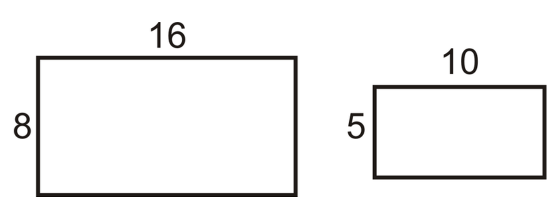
Solution: Just like Example 1, let's match up the corresponding parts.

$\frac{6}{8} = \frac{12}{16} = \frac{3}{4}$ however, $\frac{8}{12} = \frac{2}{3}$.

Because one of the base lengths is not in the same proportion as the other two lengths, these right triangle pyramids are not similar.

Surface Areas of Similar Solids

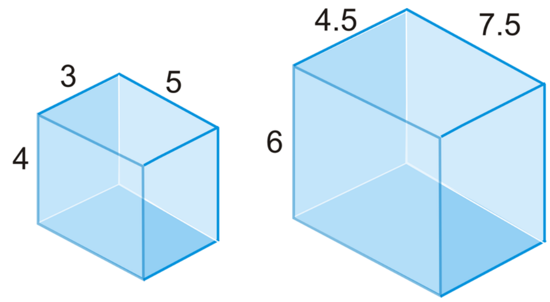
Recall that ***when two shapes are similar, the ratio of the area is a square of the scale factor.***



For example, the two rectangles to the left are similar because their sides are in a ratio of 5:8. The area of the larger rectangle is $8(16) = 128\ units^2$ and the area of the smaller rectangle is $5(10) = 50\ units^2$. If we compare the areas in a ratio, it is $50 : 128 = 25 : 64 = 5^2 = 8^2$.

So, what happens with the surface areas of two similar solids? Let's look at Example 1 again.

Example 3: Find the surface area of the two similar rectangular prisms.



Solution:

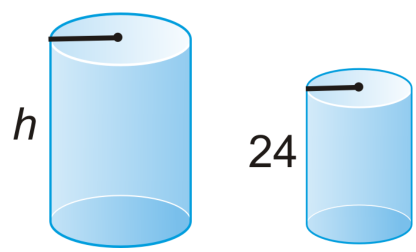
$$\begin{aligned} SA_{smaller} &= 2(4 \cdot 3) + 2(4 \cdot 5) + 2(3 \cdot 5) \\ &= 24 + 40 + 30 = 94 \ units^2 \end{aligned}$$

$$\begin{aligned} SA_{larger} &= 2(6 \cdot 4.5) + 2(4.5 \cdot 7.5) + 2(6 \cdot 7.5) \\ &= 54 + 67.5 + 90 = 211.5 \ units^2 \end{aligned}$$

Now, find the ratio of the areas. $\frac{94}{211.5} = \frac{4}{9} = \frac{2^2}{3^2}$. The sides are in a ratio of $\frac{4}{6} = \frac{2}{3}$, so the surface areas have the same relationship as the areas of two similar shapes.

Surface Area Ratio: If two solids are similar with a scale factor of $\frac{a}{b}$, then the surface areas are in a ratio of $\left(\frac{a}{b}\right)^2$.

Example 4: Two similar cylinders are below. If the ratio of the areas is 16:25, what is the height of the taller cylinder?



Solution: First, we need to take the square root of the area ratio to find the scale factor, $\sqrt{\frac{16}{25}} = \frac{4}{5}$. Now we can set up a proportion to find h.

$$\begin{aligned} \frac{4}{5} &= \frac{24}{h} \\ 4h &= 120 \\ h &= 30 \end{aligned}$$

Example 5: Using the cylinders from Example 4, if the surface area of the smaller cylinder is $1536\pi \ cm^2$, what is the surface area of the larger cylinder?

Solution: Set up a proportion using the ratio of the areas, 16:25.

$$\frac{16}{25} = \frac{1536\pi}{A}$$
$$16A = 38400\pi$$
$$A = 2400\pi\ cm^2$$

Volumes of Similar Solids

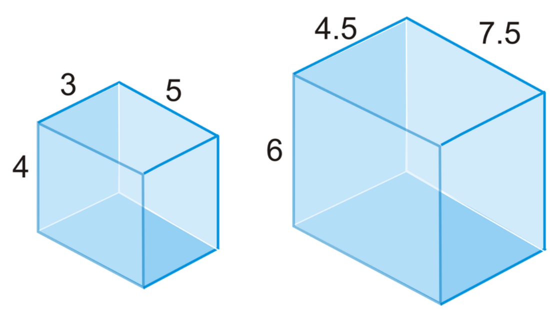
Let's look at what we know about similar solids so far.

TABLE 11.3:

	Ratios	***Units***
Scale Factor	$\frac{a}{b}$	in, ft, cm, m, etc.
Ratio of the Surface Areas	$\left(\frac{a}{b}\right)^2$	in^2, ft^2, cm^2, m^2, etc.
Ratio of the Volumes	??	in^3, ft^3, cm^3, m^3, etc.

It looks as though there is a pattern. If the ratio of the volumes follows the pattern from above, it should be the cube of the scale factor. We will do an example and test our theory.

Example 6: Find the volume of the following rectangular prisms. Then, find the ratio of the volumes.



Solution:

$$V_{smaller} = 3(4)(5) = 60$$
$$V_{larger} = 4.5(6)(7.5) = 202.5$$

The ratio is $\frac{60}{202.5}$, which reduces to $\frac{8}{27} = \frac{2^3}{3^3}$.

It seems as though our prediction based on the patterns is correct.

Volume Ratio: If two solids are similar with a scale factor of $\frac{a}{b}$, then the volumes are in a ratio of $\left(\frac{a}{b}\right)^3$.

Example 7: Two spheres have radii in a ratio of 3:4. What is the ratio of their volumes?

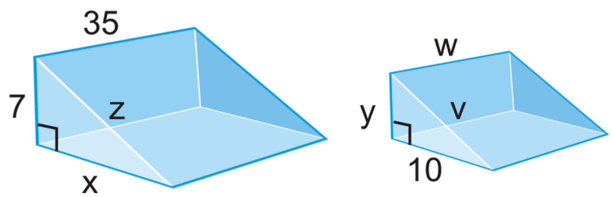
Solution: If we cube 3 and 4, we will have the ratio of the volumes. Therefore, $3^3 : 4^3$ or 27:64 is the ratio of the volumes.

Example 8: If the ratio of the volumes of two similar prisms is 125:8, what is their scale factor?

Solution: This example is the opposite of the previous example. We need to take the cubed root of 125 and 8 to find the scale factor.

$$\sqrt[3]{125} : \sqrt[3]{8} = 5 : 2$$

Example 9: Two similar right triangle prisms are below. If the ratio of the volumes is 343:125, find the missing sides in both figures.



Solution: If the ratio of the volumes is 343:125, then the scale factor is 7:5, the cubed root of each. With the scale factor, we can now set up several proportions.

$$\begin{array}{lllll}
\frac{7}{5}=\frac{7}{y} & \frac{7}{5}=\frac{x}{10} & \frac{7}{5}=\frac{35}{w} & 7^2+x^2=z^2 & \frac{7}{5}=\frac{z}{v} \\
y=5 & x=14 & w=25 & 7^2+14^2=z^2 & \\
 & & & z=\sqrt{245}=7\sqrt{5} & \frac{7}{5}=\frac{7\sqrt{5}}{v} \rightarrow v=5\sqrt{5}
\end{array}$$

Example 10: The ratio of the surface areas of two similar cylinders is 16:81. If the volume of the smaller cylinder is $96\pi\ in^3$, what is the volume of the larger cylinder?

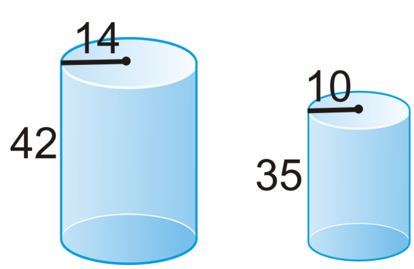
Solution: First we need to find the scale factor from the ratio of the surface areas. If we take the square root of both numbers, we have that the ratio is 4:9. Now, we need cube this to find the ratio of the volumes, $4^3 : 9^3 = 64 : 729$. At this point we can set up a proportion to solve for the volume of the larger cylinder.

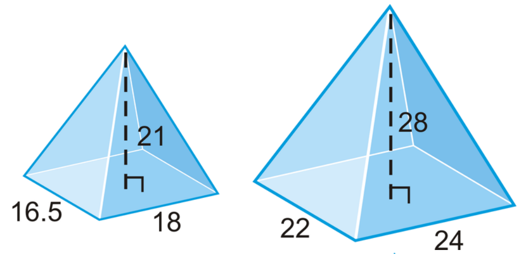
$$\begin{aligned}
\frac{64}{729} &= \frac{96\pi}{V} \\
64V &= 69984\pi \\
V &= 1093.5\pi\ in^3
\end{aligned}$$

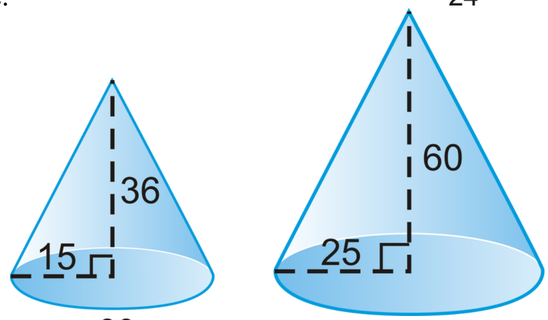
Know What? Revisited The coffee mugs are similar because the heights and radii are in a ratio of 2:3, which is also their scale factor. The volume of Dad's mug is $54\pi\ in^3$ and Mom's mug is $16\pi\ in^3$. The ratio of the volumes is $54\pi : 16\pi$, which reduces to 8:27.

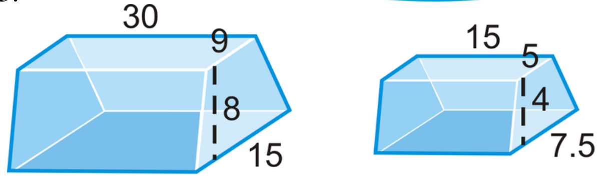
Review Questions

Determine if each pair of right solids are similar. *Explain* your reasoning.

1. 

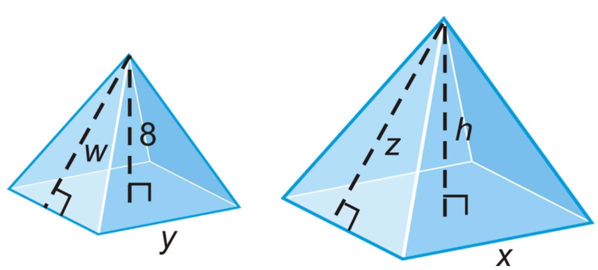
2. 

3. 

4. 

5. Are all cubes similar? Why or why not?
6. Two prisms have a scale factor of 1:4. What is the ratio of their surface areas?
7. Two pyramids have a scale factor of 2:7. What is the ratio of their volumes?
8. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
9. The surface area of two similar cones is in a ratio of 64:121. What is the scale factor?
10. The volume of two hemispheres is in a ratio of 125:1728. What is the scale factor?
11. A cone has a volume of 15π and is similar to another larger cone. If the scale factor is 5:9, what is the volume of the larger cone?
12. A cube has sides of length x and is enlarged so that the sides are $4x$. How does the volume change?
13. The ratio of the volumes of two similar pyramids is 8:27. What is the ratio of their total surface areas?
14. The ratio of the volumes of two tetrahedrons is 1000:1. The smaller tetrahedron has a side of length 6 cm. What is the side length of the larger tetrahedron?
15. The ratio of the surface areas of two cubes is 64:225. If the volume of the smaller cube is 13824 m^3, what is the volume of the larger cube?

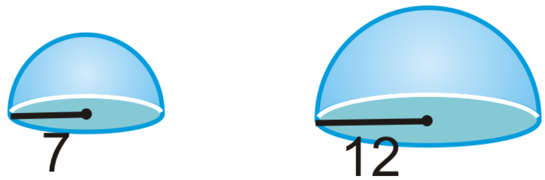
Below are two similar square pyramids with a volume ratio of 8:27. The base lengths are equal to the heights. Use this to answer questions 16-21.



16. What is the scale factor?
17. What is the ratio of the surface areas?
18. Find h, x and y.
19. Find w and z.

20. Find the volume of both pyramids.
21. Find the *lateral* surface area of both pyramids.

Use the hemispheres below to answer questions 22-25.



22. Are the two hemispheres similar? How do you know?
23. Find the ratio of the surface areas and volumes.
24. Find the *lateral* surface areas of both hemispheres.
25. Determine the ratio of the lateral surface areas for the hemispheres. Is it the same as the ratio of the total surface area? Why or why not?

Animal A and animal B are similar (meaning the size and shape of their bones and bodies are similar) and *the strength of their respective bones are proportional to the cross sectional area of their bones*. Answer the following questions given that the ratio of the height of animal A to the height of animal B is 3:5. You may assume the lengths of their bones are in the same ratio.

26. Find the ratio of the strengths of the bones. How much stronger are the bones in animal B?
27. If their weights are proportional to their volumes, find the ratio of their weights.
28. Which animal has a skeleton more capable of supporting its own weight? Explain.

Two sizes of cans of beans are similar. The thickness of the walls and bases are the same in both cans. The ratio of their surface areas is 4:9.

29. If the surface area of the smaller can is 36 sq in, what is the surface area of the larger can?
30. If the sheet metal used to make the cans costs $0.006 per square inch, how much does it cost to make each can?
31. What is the ratio of their volumes?
32. If the smaller can is sold for $0.85 and the larger can is sold for $2.50, which is a better deal?

Review Queue Answers

a. Yes, every sphere is similar because the similarity only depends on one length, the radius.
b. $\frac{4}{3}12^3\pi = 2304\pi\ in^3$
c. $\frac{4}{3}3^3\pi = 27\pi\ in^3$
d. The scale factor is 4:1, the volume ratio is 2304:36 or 64:1
e. $\frac{4}{9}$
f. $\frac{4}{9} = \frac{16}{A} \rightarrow A = 36\ cm^2$
g. $\frac{1}{5} = \frac{x}{20} \rightarrow x = 4\ cm$

11.8 Chapter 11 Review

Keywords, Theorems, Formulas

Polyhedron
A 3-dimensional figure that is formed by polygons that enclose a region in space.

Face
Each polygon in a polyhedron is called a ***face***.

Edge
The line segment where two faces intersect is called an ***edge***

Vertex
the point of intersection of two edges is a ***vertex***.

Prism
A polyhedron with two congruent bases, in parallel planes, and the lateral sides are rectangles.

Pyramid
A polyhedron with one base and all the lateral sides meet at a common vertex. The lateral sides are triangles.

Euler's Theorem
The number of faces (F), vertices (V), and edges (E) of a polyhedron can be related such that $F+V=E+2$.

Regular Polyhedron
A polyhedron where all the faces are congruent regular polygons.

Regular Tetrahedron
A 4-faced polyhedron where all the faces are equilateral triangles.

Cube
A 6-faced polyhedron where all the faces are squares.

Regular Octahedron
An 8-faced polyhedron where all the faces are equilateral triangles.

Regular Dodecahedron
A 12-faced polyhedron where all the faces are regular pentagons.

Regular Icosahedron
A 20-faced polyhedron where all the faces are equilateral triangles.

Cross-Section
The intersection of a plane with a solid.

Net

An unfolded, flat representation of the sides of a three-dimensional shape.

Lateral Face

A face that is not the base.

Lateral Edge

The edges between the ***lateral faces*** are called ***lateral edges***.

Base Edge

The edges between the base and the lateral faces are called ***base edges***.

Right Prism

All prisms are named by their bases, so the prism to the right is a pentagonal prism. This particular prism is called a ***right prism***

Oblique Prism

Oblique prisms lean to one side or the other and the height is outside the prism.

Surface Area

The sum of the areas of the faces.

Lateral Area

The sum of the areas of the ***lateral*** faces.

Surface Area of a Right Prism

The surface area of a right prism is the sum of the area of the bases and the area of each rectangular lateral face.

Cylinder

A solid with congruent circular bases that are in parallel planes. The space between the circles is enclosed.

Surface Area of a Right Cylinder

If r is the radius of the base and h is the height of the cylinder, then the surface area is $SA = 2\pi r^2 + 2\pi rh$.

Surface Area of a Regular Pyramid

If B is the area of the base and P is the perimeter of the base and l is the slant height, then $SA = B + \frac{1}{2}Pl$.

Cone

A solid with a circular base and sides taper up towards a common vertex.

Slant Height

All regular pyramids also have a ***slant height*** that is the height of a lateral face. Because of the nature of regular pyramids, ***all slant heights are congruent***. A non-regular pyramid does not have a slant height.

Surface Area of a Right Cone

The surface area of a right cone with slant height l and base radius r is $SA = \pi r^2 + \pi rl$.

Volume
The measure of how much space a three-dimensional figure occupies.

Volume of a Cube Postulate
The volume of a cube is the cube of the length of its side, or s^3.

Volume Congruence Postulate
If two solids are congruent, then their volumes are congruent.

Volume Addition Postulate
The volume of a solid is the sum of the volumes of all of its non-overlapping parts.

Volume of a Rectangular Prism
If a rectangular prism is h units high, w units wide, and l units long, then its volume is $V = l \cdot w \cdot h$.

Volume of a Prism
If the area of the base of a prism is B and the height is h, then the volume is $V = B \cdot h$.

Cavalieri's Principle
If two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

Volume of a Cylinder
If the height of a cylinder is h and the radius is r, then the volume would be $V = \pi r^2 h$.

Volume of a Pyramid
If B is the area of the base and h is the height, then the volume of a pyramid is $V = \frac{1}{3}Bh$.

Volume of a Cone
If r is the radius of a cone and h is the height, then the volume is $V = \frac{1}{3}\pi r^2 h$.

Sphere
The set of all points, in three-dimensional space, which are equidistant from a point.

Great Circle
The ***great circle*** is a plane that contains the diameter.

Surface Area of a Sphere
If r is the radius, then the surface area of a sphere is $SA = 4\pi r^2$.

Volume of a Sphere
If a sphere has a radius r, then the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

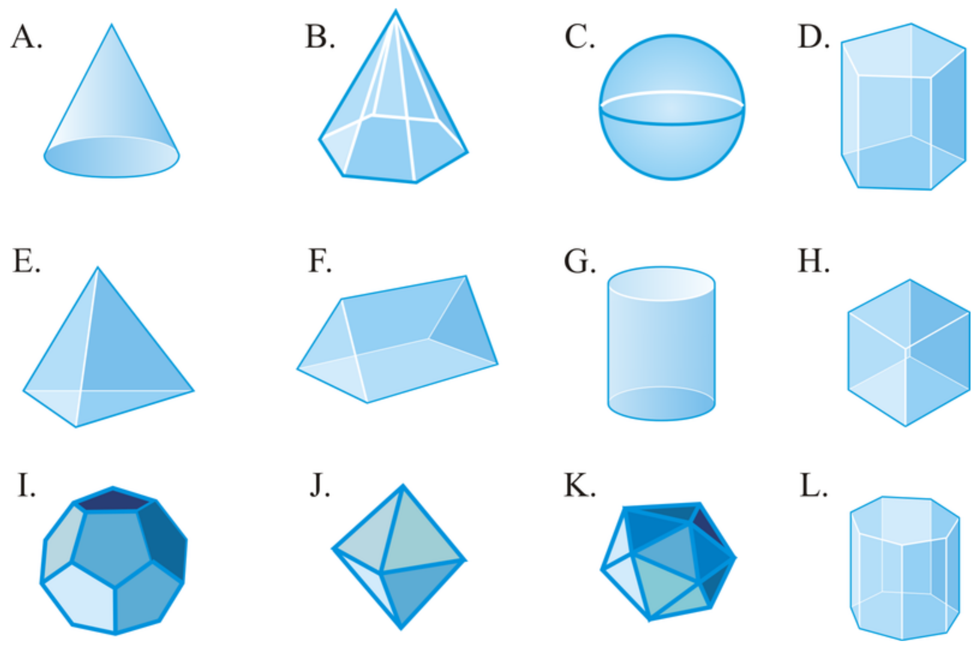
Similar Solids
Two solids are similar if and only if they are the same type of solid and their corresponding linear measures (radii, heights, base lengths, etc.) are proportional.

Surface Area Ratio
If two solids are similar with a scale factor of $\frac{a}{b}$, then the surface areas are in a ratio of $\left(\frac{a}{b}\right)^2$.

Volume Ratio
If two solids are similar with a scale factor of $\frac{a}{b}$, then the volumes are in a ratio of $\left(\frac{a}{b}\right)^3$.

Review Questions

Match the shape with the correct name.



1. Triangular Prism
2. Icosahedron
3. Cylinder
4. Cone
5. Tetrahedron
6. Pentagonal Prism
7. Octahedron
8. Hexagonal Pyramid
9. Octagonal Prism
10. Sphere
11. Cube
12. Dodecahedron

Match the formula with its description.

13. Volume of a Prism - A. $\frac{1}{3}\pi r^2 h$
14. Volume of a Pyramid - B. $\pi r^2 h$
15. Volume of a Cone - C. $4\pi r^2$
16. Volume of a Cylinder - D. $\frac{4}{3}\pi r^3$
17. Volume of a Sphere - E. $\pi r^2 + \pi rl$
18. Surface Area of a Prism - F. $2\pi r^2 + 2\pi rh$
19. Surface Area of a Pyramid - G. $\frac{1}{3}Bh$
20. Surface Area of a Cone - H. Bh
21. Surface Area of a Cylinder - I. $B + \frac{1}{2}Pl$
22. Surface Area of a Sphere - J. The sum of the area of the bases and the area of each rectangular lateral face.

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9696 .

CHAPTER 12 Circles

Chapter Outline

Finally, we dive into a different shape, circles. First, we will define all the parts of circles and then explore the properties of tangent lines, arcs, inscribed angles, and chords. Next, we will learn about the properties of angles within circles that are formed by chords, tangents and secants. Lastly, we will place circles in the coordinate plane, find the equations of, and graph circles.

12.1 Parts of Circles & Tangent Lines

Learning Objectives

- Define circle, center, radius, diameter, chord, tangent, and secant of a circle.
- Explore the properties of tangent lines and circles.

Review Queue

a. Find the equation of the line with $m = -2$ and passes through (4, -5).
b. Find the equation of the line that passes though (6, 2) and (-3, -1).
c. Find the equation of the line *perpendicular* to the line in #2 and passes through (-8, 11).

Know What? The clock to the right is an ancient astronomical clock in Prague. It has a large background circle that tells the local time and the "ancient time" and then the smaller circle rotates around on the orange line to show the current astrological sign. The yellow point is the center of the larger clock. How does the orange line relate to the small and larger circle? How does the hand with the moon on it (black hand with the circle) relate to both circles? Are the circles concentric or tangent?

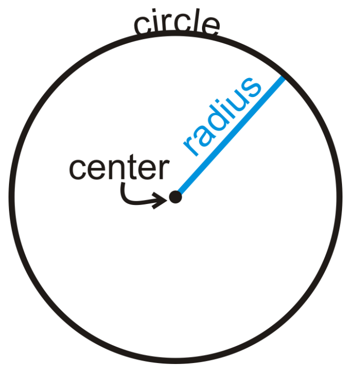
For more information on this clock, see: http://en.wikipedia.org/wiki/Prague_Astronomical_Clock

Defining Terms

Circle: The set of all points that are the same distance away from a specific point, called the **center.**

Radius: The distance from the center to the circle.

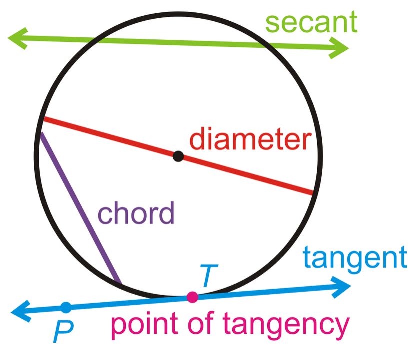
The center is typically labeled with a capital letter because it is a point. If the center is A, we would call this circle, "circle A," and labeled $\odot A$. Radii (the plural of radius) are line segments. There are infinitely many radii in any circle.



Chord: A line segment whose endpoints are on a circle.

Diameter: A chord that passes through the center of the circle.

Secant: A line that intersects a circle in two points.

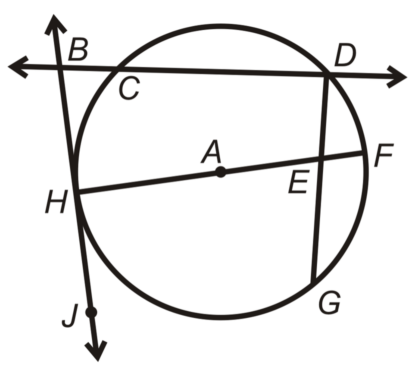


Tangent: A line that intersects a circle in exactly one point.

Point of Tangency: The point where the tangent line touches the circle.

Notice that the tangent ray $\overrightarrow{TP}$ and tangent segment $\overline{TP}$ are also called tangents. The length of a diameter is two times the length of a radius.

Example 1: Identify the parts of $\odot A$ that best fit each description.



a) A radius

b) A chord

c) A tangent line

d) The point of tangency

e) A diameter

f) A secant

Solution:

a) $\overline{HA}$ or $\overline{AF}$

b) $\overline{CD}, \overline{HF}$, or $\overline{DG}$

c) $\overleftrightarrow{BJ}$

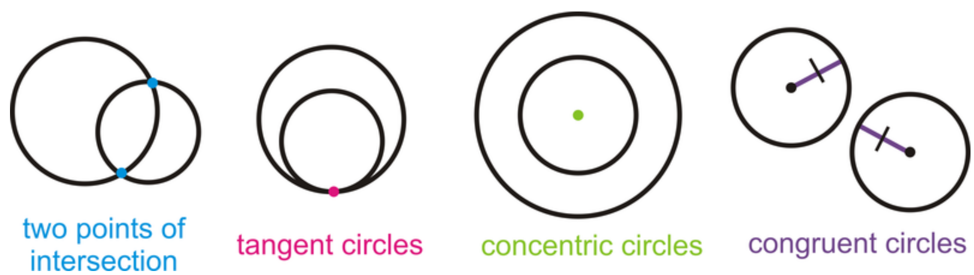
d) *Point H*

e) $\overline{HF}$

f) $\overleftrightarrow{BD}$

Coplanar Circles

Two circles can intersect in two points, one point, or no points. If two circles intersect in one point, they are called *tangent circles.*

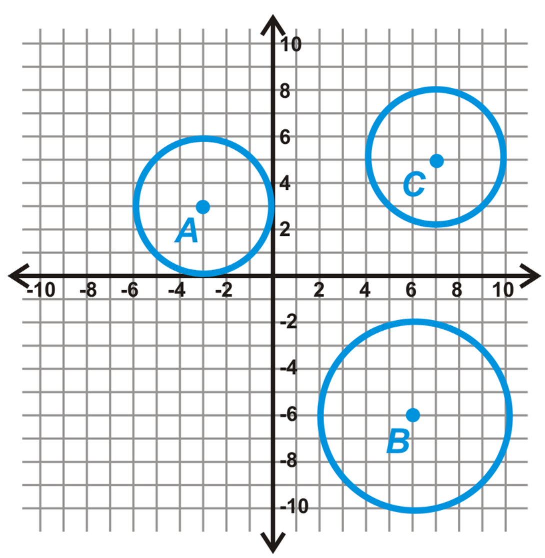


Congruent Circles: Two circles with the same radius, but different centers.

Concentric Circles: When two circles have the same center, but different radii.

If two circles have different radii, they are similar. *All circles are similar.*

Example 2: Determine if any of the following circles are congruent.



Solution: From each center, count the units to the circle. It is easiest to count vertically or horizontally. Doing this, we have:

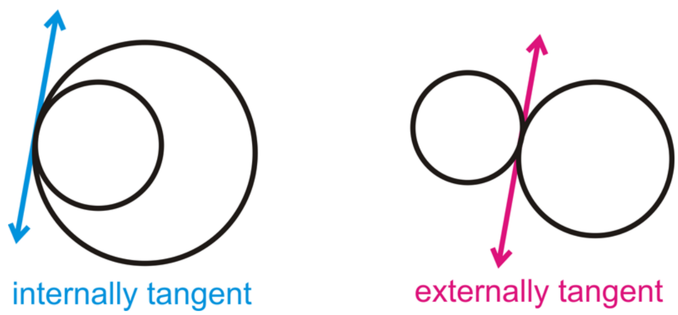
Radius of $\odot A = 3\ units$
Radius of $\odot B = 4\ units$
Radius of $\odot C = 3\ units$

From these measurements, we see that $\odot A \cong \odot C$.

Notice that two circles are congruent, just like two triangles or quadrilaterals. Only the *lengths* of the radii are equal.

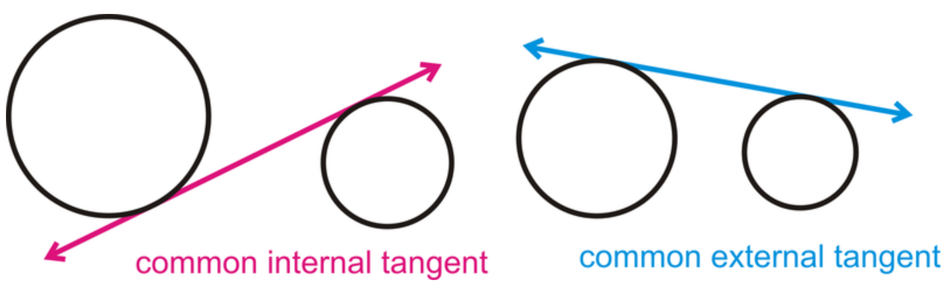
Tangent Lines

We just learned that two circles can be tangent to each other. Two triangles can be tangent in two different ways, either *internally* tangent or *externally* tangent.



If the circles are not tangent, they can share a tangent line, called a *common* tangent. Common tangents can be internally tangent and externally tangent too.

Notice that the common internal tangent passes through the space between the two circles. Common external tangents stay on the top or bottom of both circles.



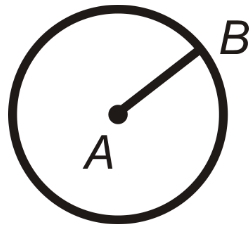
Tangents and Radii

The tangent line and the radius drawn to the point of tangency have a unique relationship. Let's investigate it here.

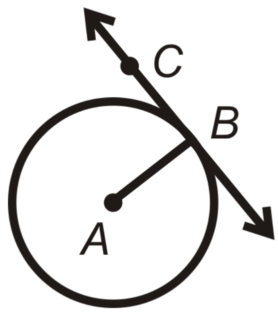
Investigation 9-1: Tangent Line and Radius Property

Tools needed: compass, ruler, pencil, paper, protractor

a. Using your compass, draw a circle. Locate the center and draw a radius. Label the radius $\overline{AB}$, with A as the center.



b. Draw a tangent line, $\overleftrightarrow{BC}$, where B is the point of tangency. To draw a tangent line, take your ruler and line it up with point B. Make sure that B is the only point on the circle that the line passes through.

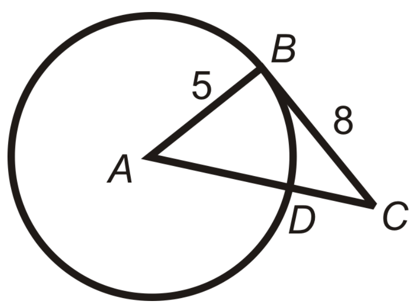


c. Using your protractor, find $m\angle ABC$.

Tangent to a Circle Theorem: A line is tangent to a circle if and only if the line is perpendicular to the radius drawn to the point of tangency.

To prove this theorem, the easiest way to do so is indirectly (proof by contradiction). Also, notice that this theorem uses the words "if and only if," making it a biconditional statement. Therefore, the converse of this theorem is also true.

Example 3: In $\odot A, \overline{CB}$ is tangent at point B. Find AC. Reduce any radicals.



Solution: Because $\overline{CB}$ is tangent, $\overline{AB} \perp \overline{CB}$, making $\triangle ABC$ a right triangle. We can use the Pythagorean Theorem to find AC.

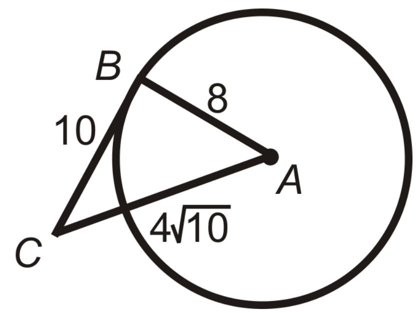
$$\begin{aligned} 5^2+8^2 &= AC^2 \\ 25+64 &= AC^2 \\ 89 &= AC^2 \\ AC &= \sqrt{89} \end{aligned}$$

Example 4: Find DC, in $\odot A$. Round your answer to the nearest hundredth.

Solution:

$$DC = AC - AD$$
$$DC = \sqrt{89} - 5 \approx 4.43$$

Example 5: Determine if the triangle below is a right triangle. Explain why or why not.

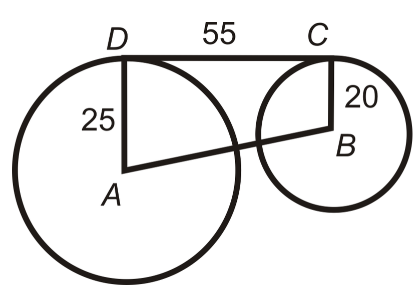


Solution: To determine if the triangle is a right triangle, use the Pythagorean Theorem. $4\sqrt{10}$ is the longest length, so we will set it equal to c in the formula.

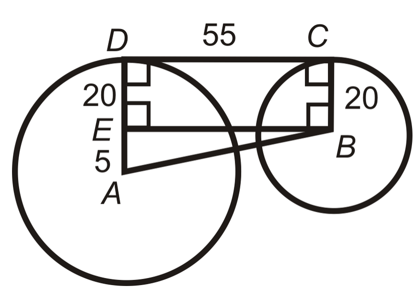
$$8^2 + 10^2 \ ? \ \left(4\sqrt{10}\right)^2$$
$$64 + 100 \neq 160$$

$\triangle ABC$ is not a right triangle. And, from the converse of the Tangent to a Circle Theorem, $\overline{CB}$ is not tangent to $\odot A$.

Example 6: Find the distance between the centers of the two circles. Reduce all radicals.



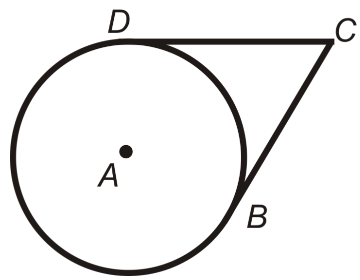
Solution: The distance between the two circles is AB. They are not tangent, however, $\overline{AD} \perp \overline{DC}$ and $\overline{DC} \perp \overline{CB}$. Let's add $\overline{BE}$, such that $EDCB$ is a rectangle. Then, use the Pythagorean Theorem to find AB.



$$\begin{aligned} 5^2 + 55^2 &= AC^2 \\ 25 + 3025 &= AC^2 \\ 3050 &= AC^2 \\ AC &= \sqrt{3050} = 5\sqrt{122} \end{aligned}$$

Tangent Segments

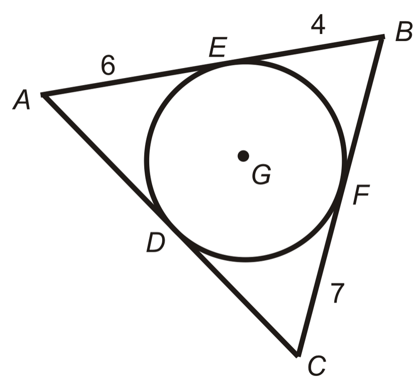
Let's look at two tangent segments, drawn from the same external point. If we were to measure these two segments, we would find that they are equal.



Theorem 10-2: If two tangent segments are drawn from the same external point, then the segments are equal.

The proof of Theorem 10-2 is in the review exercises.

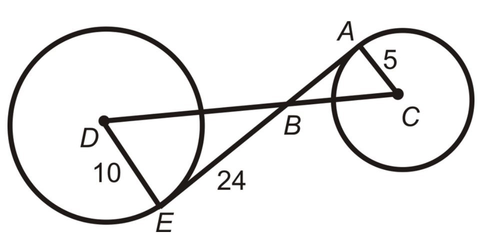
Example 7: Find the perimeter of $\triangle ABC$.



Solution: $AE = AD, EB = BF$, and $CF = CD$. Therefore, the perimeter of $\triangle ABC = 6+6+4+4+7+7 = 34$.

We say that $\odot G$ is ***inscribed*** in $\triangle ABC$. A circle is inscribed in a polygon, if every side of the polygon is tangent to the circle.

Example 8: If D and A are the centers and AE is tangent to both circles, find DC.



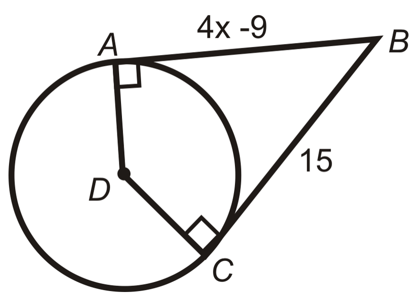
Solution: Because AE is tangent to both circles, it is perpendicular to both radii and $\triangle ABC$ and $\triangle DBE$ are similar. To find DB, use the Pythagorean Theorem.

$$\begin{aligned}10^2+24^2 &= DB^2\\100+576 &= 676\\DB &= \sqrt{676}=26\end{aligned}$$

To find BC, use similar triangles.

$$\frac{5}{10}=\frac{BC}{26}\longrightarrow BC=13$$
$$DC=AB+BC=26+13=39$$

Example 9: ***Algebra Connection*** Find the value of x.



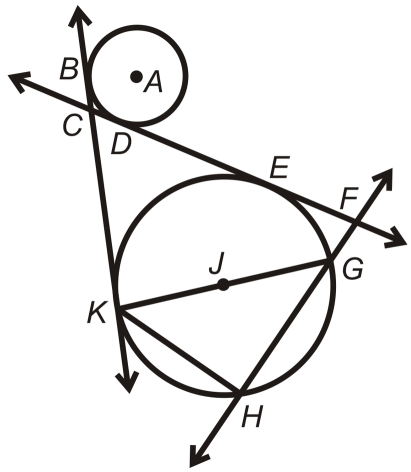
Solution: Because $\overline{AB}\perp\overline{AD}$ and $\overline{DC}\perp\overline{CB}$, $\overline{AB}$ and $\overline{CB}$ are tangent to the circle and also congruent. Set $AB=CB$ and solve for x.

$$\begin{aligned}4x-9 &= 15\\4x &= 24\\x &= 6\end{aligned}$$

Know What? Revisited Refer to the photograph in the "Know What?" section at the beginning of this chapter. The orange line (which is normally black, but outlined for the purpose of this exercise) is a diameter of the smaller circle. Since this line passes through the center of the larger circle (yellow point, also outlined), it is part of one of its diameters. The "moon" hand is a diameter of the larger circle, but a secant of the smaller circle. The circles are not concentric because they do not have the same center and are not tangent because the sides of the circles do not touch.

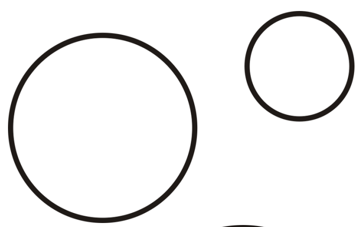
Review Questions

Determine which term best describes each of the following parts of $\odot P$.



1. $\overline{KG}$
2. $\overleftrightarrow{FH}$
3. $\overline{KH}$
4. E
5. $\overleftrightarrow{BK}$
6. $\overleftrightarrow{CF}$
7. A
8. $\overline{JG}$
9. What is the longest chord in any circle?

Copy each pair of circles. Draw in all common tangents.



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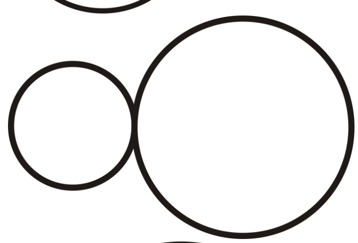

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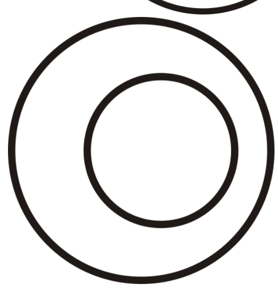
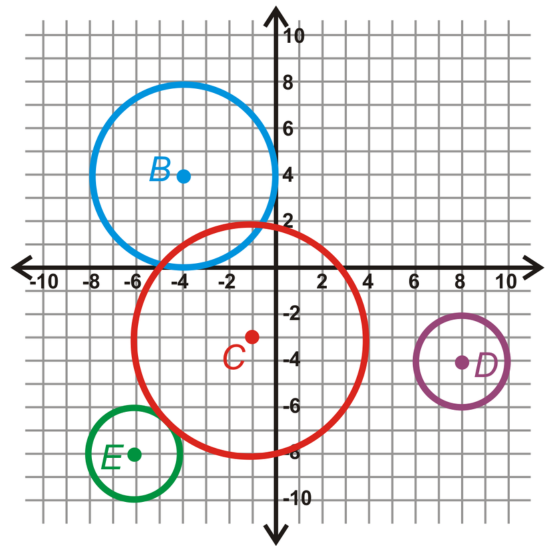


12.

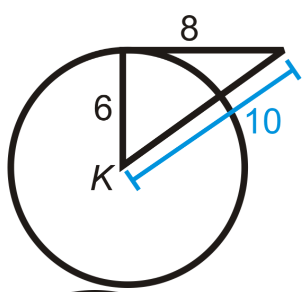
Coordinate Geometry Use the graph below to answer the following questions.

13. Find the radius of each circle.
14. Are any circles congruent? How do you know?

15. Find all the common tangents for $\odot B$ and $\odot C$.
16. $\odot C$ and $\odot E$ are externally tangent. What is CE?
17. Find the equation of $\overline{CE}$.



Determine whether the given segment is tangent to $\odot K$.



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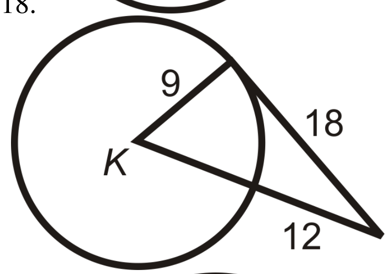

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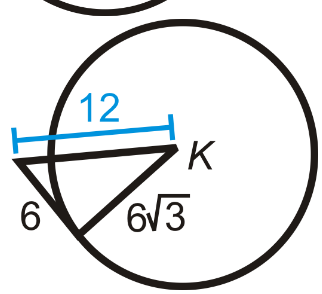
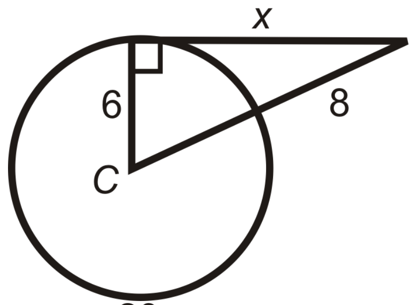


20.

Algebra Connection Find the value of the indicated length(s) in $\odot C$. A and B are points of tangency. Simplify all radicals.

21.



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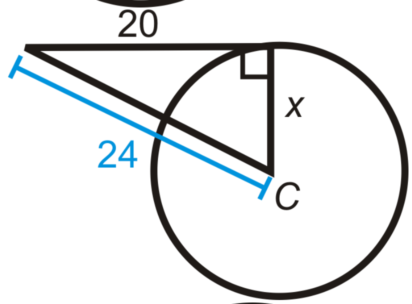

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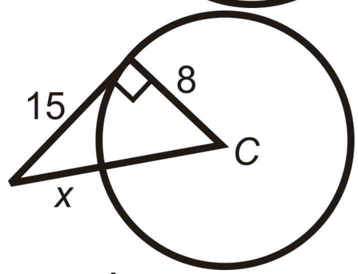

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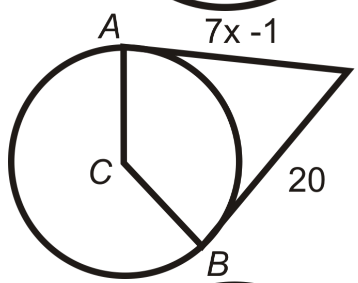

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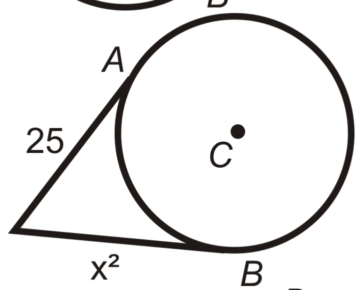
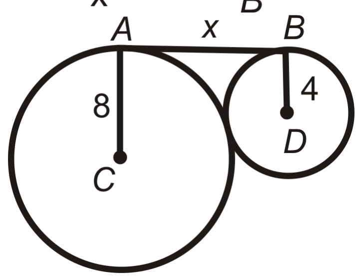
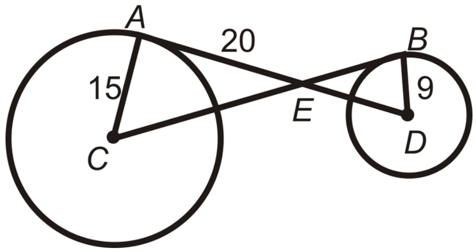


26.



27. A and B are points of tangency for $\odot C$ and $\odot D$, respectively.



a. Is $\triangle AEC \sim \triangle BED$? Why?
b. Find BC.
c. Find AD.
d. Using the trigonometric ratios, find $m\angle C$. Round to the nearest tenth of a degree.

28. Fill in the blanks in the proof of Theorem 10-2. Given: $\overline{AB}$ and $\overline{CB}$ with points of tangency at A and C. $\overline{AD}$ and $\overline{DC}$ are radii. Prove: $\overline{AB} \cong \overline{CB}$

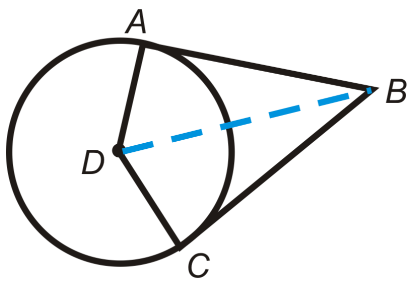


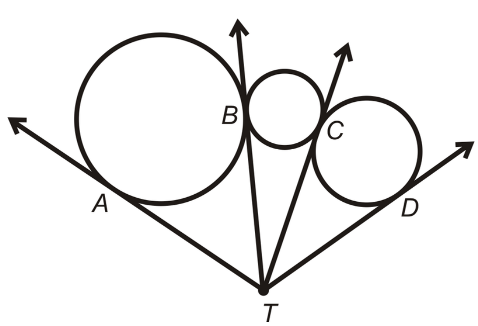
TABLE 12.1:

Statement	Reason
1.	
2. $\overline{AD} \cong \overline{DC}$	
3. $\overline{DA} \perp \overline{AB}$ and $\overline{DC} \perp \overline{CB}$	
4.	Definition of perpendicular lines
5.	Connecting two existing points
6. $\triangle ADB$ and $\triangle DCB$ are right triangles	
7. $\overline{DB} \cong \overline{DB}$	
8. $\triangle ABD \cong \triangle CBD$	
9. $\overline{AB} \cong \overline{CB}$	

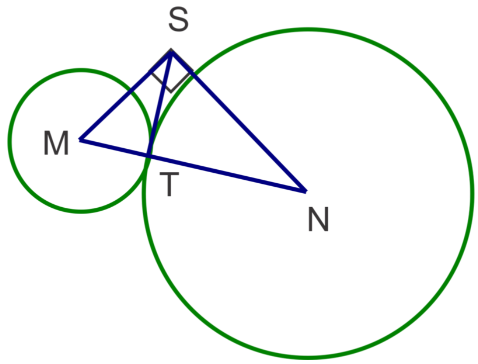
29. From the above proof, we can also conclude (fill in the blanks):

a. $ABCD$ is a ______________ (type of quadrilateral).
b. The line that connects the ____________ and the external point B __________ $\angle ADC$ and $\angle ABC$.

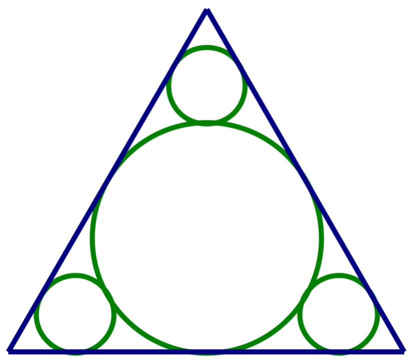
30. Points A, B, C, and D are all points of tangency for the three tangent circles. *Explain* why $\overline{AT} \cong \overline{BT} \cong \overline{CT} \cong \overline{DT}$.



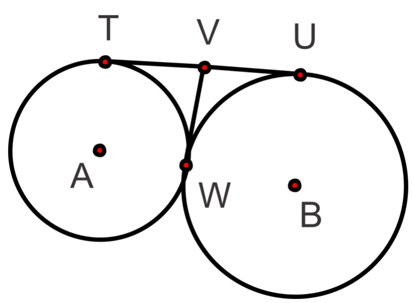
31. Circles tangent at T are centered at M and N. $\overline{ST}$ is tangent to both circles at T. Find the radius of the smaller circle if $\overline{SN} \perp \overline{SM}$, $SM = 22, TN = 25$ and $m\angle SNT = 40^\circ$.



32. Four circles are arranged inside an equilateral triangle as shown. If the triangle has sides equal to 16 cm, what is the radius of the bigger circle? What are the radii of the smaller circles?



33. Circles centered at A and B are tangent at W. Explain why A, B and W are collinear. $\overline{TU}$ is a common external tangent to the two circles. $\overline{VW}$ is tangent to both circles. Justify the fact that $\overline{TV} \cong \overline{VU} \cong \overline{VW}$.



Review Queue Answers

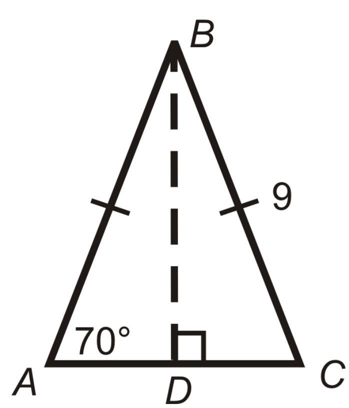
a. $y = -2x + 3$
b. $y = \frac{1}{3}x$
c. $y = -3x - 13$

12.2 Properties of Arcs

Learning Objectives

- Define and measure central angles in circles.
- Define minor arcs and major arcs.

Review Queue



1. What kind of triangle is $\triangle ABC$?

2. How does $\overline{BD}$ relate to $\triangle ABC$?

3. Find $m\angle ABC$ and $m\angle ABD$.

Round to the nearest tenth.

4. Find AD.

5. Find AC.

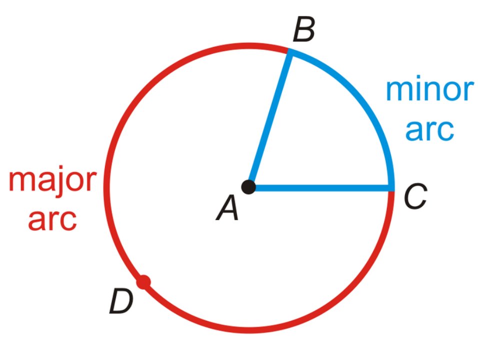
Know What? The Ferris wheel to the right has equally spaced seats, such that the central angle is 20°. How many seats are there? Why do you think it is important to have equally spaced seats on a Ferris wheel?

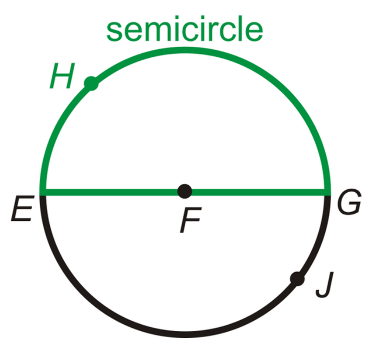
If the radius of this Ferris wheel is 25 ft., how far apart are two adjacent seats? Round your answer to the nearest tenth. *The shortest distance between two points is a straight line.*

Central Angles & Arcs

Central Angle: The angle formed by two radii of the circle with its vertex at the center of the circle.

In the picture to the right, the central angle would be $\angle BAC$. Every central angle divides a circle into two ***arcs.*** In this case the arcs are $\widehat{BC}$ and $\widehat{BDC}$. Notice the $\odot$ above the letters. To label an arc, always use this curve above the letters. Do not confuse $\overline{BC}$ and $\widehat{BC}$.





Arc: A section of the circle.

If D was not on the circle, we would not be able to tell the difference between $\widehat{BC}$ and $\widehat{BDC}$. There are 360° in a circle, where a semicircle is half of a circle, or 180°. $m\angle EFG = 180^\circ$, because it is a straight angle, so $m\widehat{EHG} = 180^\circ$ and $m\widehat{EJG} = 180^\circ$.

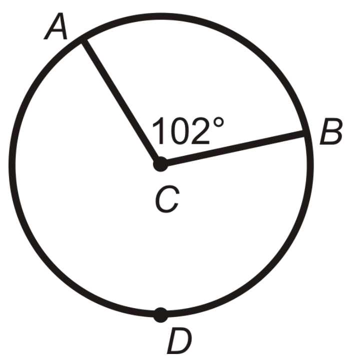
Semicircle: An arc that measures 180°.

Minor Arc: An arc that is less than 180°.

Major Arc: An arc that is greater than 180°. ***Always*** use 3 letters to label a major arc.

An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this chapter we will use degree measure. ***The measure of the minor arc is the same as the measure of the central angle*** that corresponds to it. The measure of the major arc equals to 360° minus the measure of the minor arc. In order to prevent confusion, major arcs are always named with three letters; the letters that denote the endpoints of the arc and any other point on the major arc. When referring to the measure of an arc, always place an "*m*" in from of the label.

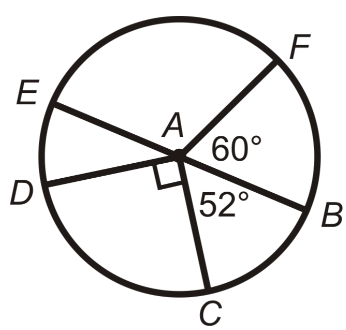
Example 1: Find $m\widehat{AB}$ and $m\widehat{ADB}$ in $\odot C$.



Solution: $m\widehat{AB}$ is the same as $m\angle ACB$. So, $m\widehat{AB} = 102^\circ$. The measure of $m\widehat{ADB}$, which is the major arc, is equal to 360° minus the minor arc.

$m\widehat{ADB} = 360^\circ - m\widehat{AB} = 360^\circ - 102^\circ = 258^\circ$

Example 2: Find the measures of the arcs in $\odot A$. $\overline{EB}$ is a diameter.

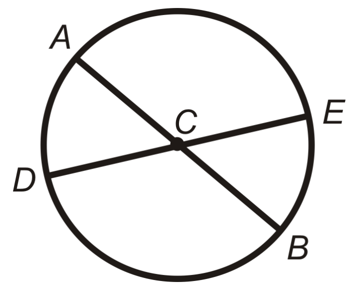


Solution: Because $\overline{EB}$ is a diameter, $m\angle EAB = 180^\circ$. Each arc is the same as its corresponding central angle.

$$
\begin{aligned}
m\widehat{BF} &= m\angle FAB = 60^\circ \\
m\widehat{EF} &= m\angle EAF = 120^\circ && \rightarrow m\angle EAB - m\angle FAB \\
m\widehat{ED} &= m\angle EAD = 38^\circ && \rightarrow m\angle EAB - m\angle BAC - m\angle CAD \\
m\widehat{DC} &= m\angle DAC = 90^\circ \\
m\widehat{BC} &= m\angle BAC = 52^\circ
\end{aligned}
$$

Congruent Arcs: Two arcs are congruent if their central angles are congruent.

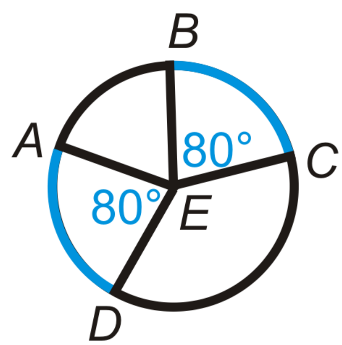
Example 3: List all the congruent arcs in $\odot C$ below. $\overline{AB}$ and $\overline{DE}$ are diameters.



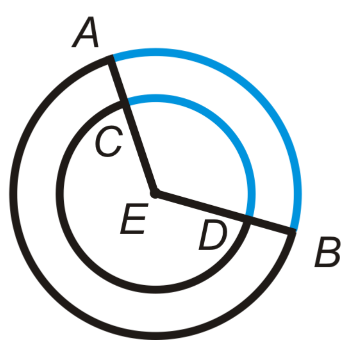
Solution: From the picture, we see that $\angle ACD$ and $\angle ECB$ are vertical angles. $\angle DCB$ and $\angle ACE$ are also vertical angles. Because all vertical angles are equal and these four angles are all central angles, we know that $\widehat{AD} \cong \widehat{EB}$ and $\widehat{AE} \cong \widehat{DB}$.

Example 4: Are the blue arcs congruent? Explain why or why not.

a)



b)



Solution: In part a, $\widehat{AD} \cong \widehat{BC}$ because they have the same central angle measure. In part b, the two arcs do have the same measure, but are not congruent because the circles are not congruent.

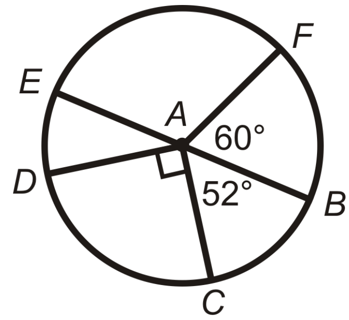
Arc Addition Postulate

Just like the Angle Addition Postulate and the Segment Addition Postulate, there is an Arc Addition Postulate. It is very similar.

Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.

Using the picture from Example 3, we would say $m\widehat{AE} + m\widehat{EB} = m\widehat{AEB}$.

Example 5: Reusing the figure from Example 2, find the measure of the following arcs in $\odot A$. $\overline{EB}$ is a diameter.



a) $m\widehat{FED}$

b) $m\widehat{CDF}$

c) $m\widehat{BD}$

d) $m\widehat{DFC}$

Solution: Use the Arc Addition Postulate.

a) $m\widehat{FED} = m\widehat{FE} + m\widehat{ED} = 120^\circ + 38^\circ = 158^\circ$

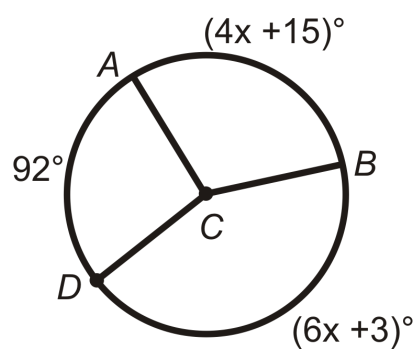
We could have labeled $\widehat{FED}$ as $\widehat{FD}$ because it is less than 180°.

b) $m\widehat{CDF} = m\widehat{CD} + m\widehat{DE} + m\widehat{EF} = 90^\circ + 38^\circ + 120^\circ = 248^\circ$

c) $m\widehat{BD} = m\widehat{BC} + m\widehat{CD} = 52^\circ + 90^\circ = 142^\circ$

d) $m\widehat{DFC} = 38^\circ + 120^\circ + 60^\circ + 52^\circ = 270^\circ$ or $m\widehat{DFC} = 360^\circ - m\widehat{CD} = 360^\circ - 90^\circ = 270^\circ$

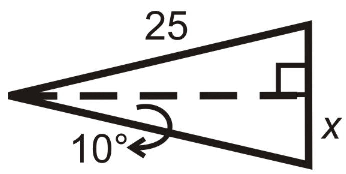
Example 6: *Algebra Connection* Find the value of x for $\odot C$ below.



Solution: There are 360° in a circle. Let's set up an equation.

$$\begin{aligned}
m\widehat{AB} + m\widehat{AD} + m\widehat{DB} &= 360^\circ \\
(4x+15)^\circ + 92^\circ + (6x+3)^\circ &= 360^\circ \\
10x + 110^\circ &= 360^\circ \\
10x &= 250^\circ \\
x &= 25^\circ
\end{aligned}$$

Know What? Revisited Because the seats are 20° apart, there will be $\frac{360^\circ}{20^\circ} = 18$ seats. It is important to have the seats evenly spaced for balance. To determine how far apart the adjacent seats are, use the triangle to the right. We will need to use sine to find x and then multiply it by 2.

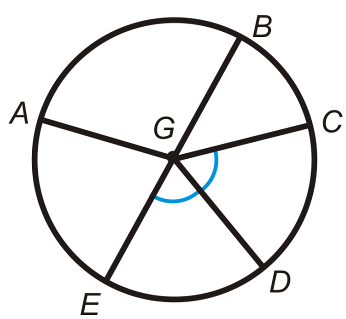


$$\sin 10^\circ = \frac{x}{25}$$
$$x = 25\sin 10^\circ = 4.3\ ft.$$

The total distance apart is 8.6 feet.

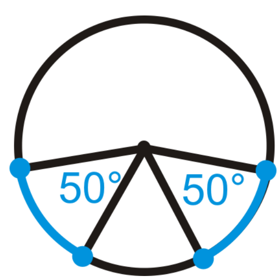
Review Questions

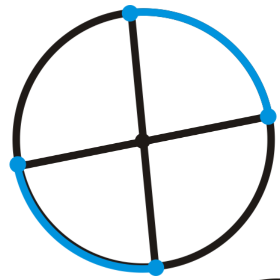
Determine if the arcs below are a minor arc, major arc, or semicircle of $\odot G$. $\overline{EB}$ is a diameter.



1. $\widehat{AB}$
2. $\widehat{ABD}$
3. $\widehat{BCE}$
4. $\widehat{CAE}$
5. $\widehat{ABC}$
6. $\widehat{EAB}$
7. Are there any congruent arcs? If so, list them.
8. If $m\widehat{BC} = 48^\circ$, find $m\widehat{CD}$.
9. Using #8, find $m\widehat{CAE}$.

Determine if the blue arcs are congruent. If so, state why.

10. 



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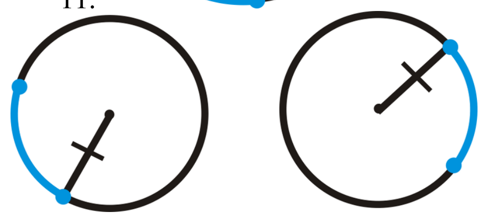
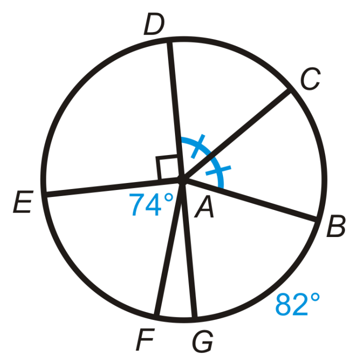


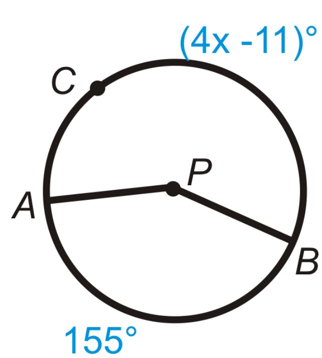
12.

Find the measure of the indicated arcs or central angles in $\odot A$. $\overline{DG}$ is a diameter.

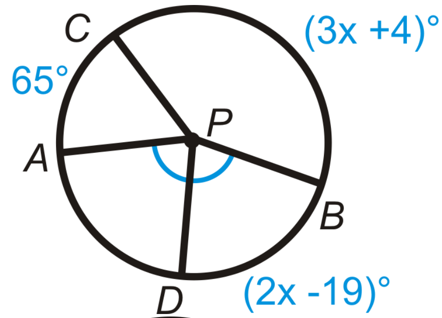


13. $\widehat{DE}$
14. $\widehat{DC}$
15. $\angle GAB$
16. $\widehat{FG}$
17. $\widehat{EDB}$
18. $\angle EAB$
19. $\widehat{DCF}$
20. $\widehat{DBE}$

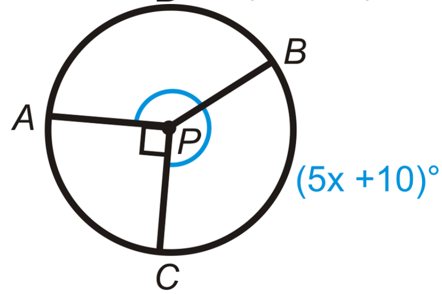
Algebra Connection Find the measure of x in $\odot P$.



21.

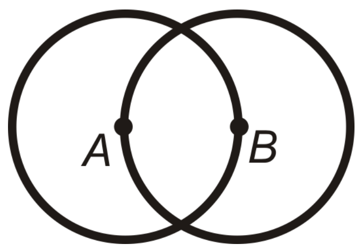


22.

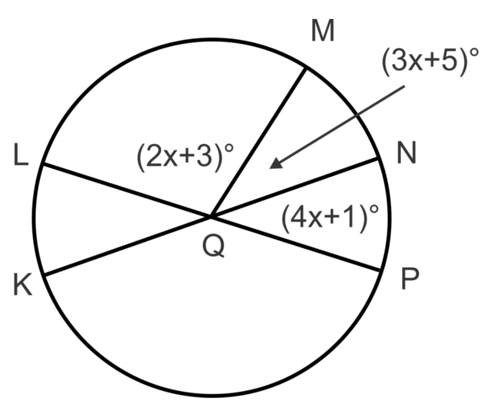


23.

24. What can you conclude about $\odot A$ and $\odot B$?

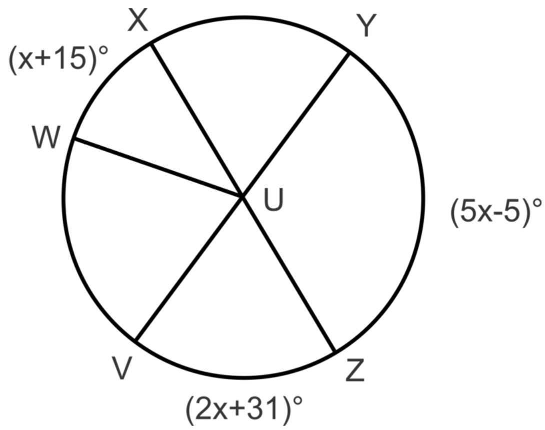


Use the diagram below to find the measures of the indicated arcs in problems 25-30.



25. $m\widehat{MN}$
26. $m\widehat{LK}$
27. $m\widehat{MP}$
28. $m\widehat{MK}$
29. $m\widehat{NPL}$
30. $m\widehat{LKM}$

Use the diagram below to find the measures indicated in problems 31-36.



31. $m\angle VUZ$
32. $m\angle YUZ$
33. $m\angle WUV$
34. $m\angle XUV$
35. $m\widehat{YWZ}$
36. $m\widehat{WYZ}$

Review Queue Answers

a. isosceles
b. $\overline{BD}$ is the angle bisector of $\angle ABC$ and the perpendicular bisector of $\overline{AC}$.
c. $m\angle ABC = 40^\circ, m\angle ABD = 25^\circ$
d. $\cos 70^\circ = \frac{AD}{9} \rightarrow AD = 9 \cdot \cos 70^\circ = 3.1$
e. $AC = 2 \cdot AD = 2 \cdot 3.1 = 6.2$

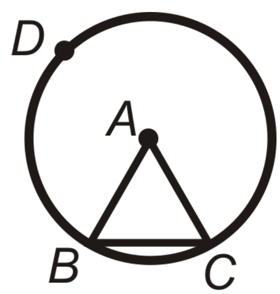
12.3 Properties of Chords

Learning Objectives

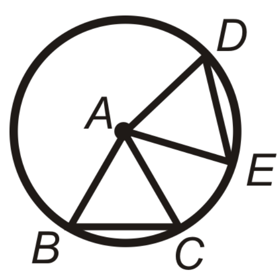
- Find the lengths of chords in a circle.
- Discover properties of chords and arcs.

Review Queue

a. Draw a chord in a circle.
b. Draw a diameter in the circle from #1. Is a diameter a chord?
c. $\triangle ABC$ is an equilateral triangle in $\odot A$. Find $m\widehat{BC}$ and $m\widehat{BDC}$.



d. $\triangle ABC$ and $\triangle ADE$ are equilateral triangles in $\odot A$. List a pair of congruent arcs and chords.



Know What? To the right is the Gran Teatro Falla, in Cadiz, Andalucía, Spain. This theater was built in 1905 and hosts several plays and concerts. It is an excellent example of circles in architecture. Notice the five windows, $A-E$. $\odot A \cong \odot E$ and $\odot B \cong \odot C \cong \odot D$. Each window is topped with a 240° arc. The gold chord in each circle connects the rectangular portion of the window to the circle. Which chords are congruent? How do you know?

Recall from the first section, that a chord is a line segment whose endpoints are on a circle. A diameter is the longest chord in a circle. There are several theorems that explore the properties of chords.

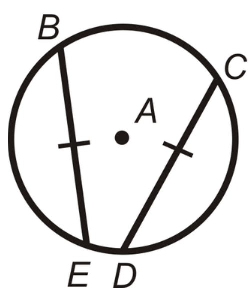
Congruent Chords & Congruent Arcs

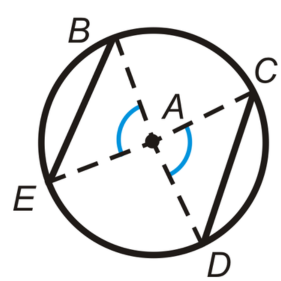
From #4 in the Review Queue above, we noticed that $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$. This leads to our first theorem.

Theorem 10-3: In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.

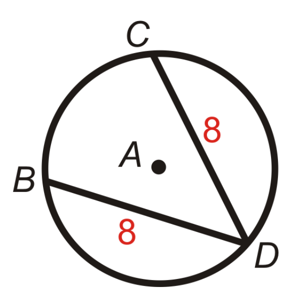
Notice the "if and only if" in the middle of the theorem. This means that Theorem 10-3 is a biconditional statement. Taking this theorem one step further, any time two central angles are congruent, the chords and arcs from the endpoints of the sides of the central angles are also congruent.

In both of these pictures, $\overline{BE} \cong \overline{CD}$ and $\widehat{BE} \cong \widehat{CD}$. In the second picture, we have $\triangle BAE \cong \triangle CAD$ because the central angles are congruent and $\overline{BA} \cong \overline{AC} \cong \overline{AD} \cong \overline{AE}$ because they are all radii (SAS). By CPCTC, $\overline{BE} \cong \overline{CD}$.





Example 1: Use $\odot A$ to answer the following.



a) If $m\widehat{BD} = 125^\circ$, find $m\widehat{CD}$.

b) If $m\widehat{BC} = 80^\circ$, find $m\widehat{CD}$.

Solution:

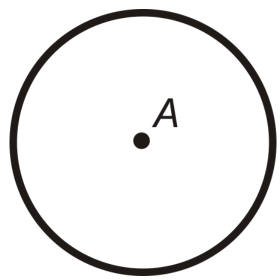
a) From the picture, we know $BD = CD$. Because the chords are equal, the arcs are too. $m\widehat{CD} = 125^\circ$.

b) To find $m\widehat{CD}$, subtract 80° from 360° and divide by 2. $m\widehat{CD} = \frac{360^\circ - 80^\circ}{2} = \frac{280^\circ}{2} = 140^\circ$

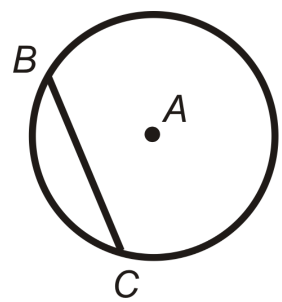
Investigation 9-2: Perpendicular Bisector of a Chord

Tools Needed: paper, pencil, compass, ruler

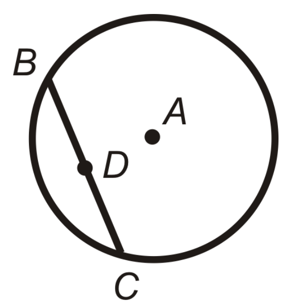
a. Draw a circle. Label the center A.



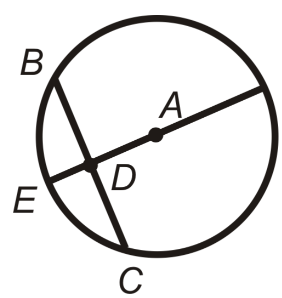
b. Draw a chord in $\odot A$. Label it $\overline{BC}$.



c. Find the midpoint of $\overline{BC}$ by using a ruler. Label it D.



d. Connect A and D to form a diameter. How does $\overline{AD}$ relate to the chord, $\overline{BC}$?

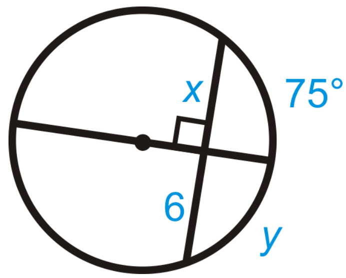


Theorem 10-4: The perpendicular bisector of a chord is also a diameter.

In the picture to the left, $\overline{AD} \perp \overline{BC}$ and $\overline{BD} \cong \overline{DC}$. From this theorem, we also notice that $\overline{AD}$ also bisects the corresponding arc at E, so $\widehat{BE} \cong \widehat{EC}$.

Theorem 10-5: If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.

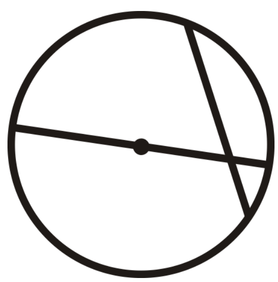
Example 2: Find the value of x and y.



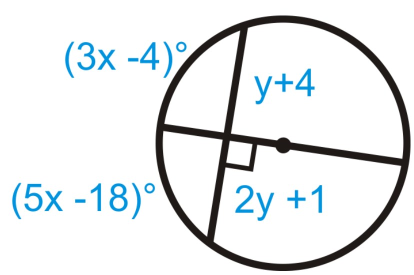
Solution: The diameter here is also perpendicular to the chord. From Theorem 10-5, $x = 6$ and $y = 75^\circ$.

Example 3: Is the converse of Theorem 10-4 true?

Solution: The converse of Theorem 10-4 would be: A diameter is also the perpendicular bisector of a chord. This is not a true statement, see the counterexample to the right.



Example 4: ***Algebra Connection*** Find the value of x and y.



Solution: Because the diameter is perpendicular to the chord, it also bisects the chord and the arc. Set up an equation for x and y.

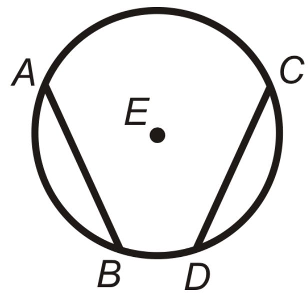
$$\begin{aligned} (3x-4)^\circ &= (5x-18)^\circ \\ 14^\circ &= 2x \\ 7^\circ &= x \end{aligned} \qquad\qquad \begin{aligned} y+4 &= 2y+1 \\ 3 &= y \end{aligned}$$

Equidistant Congruent Chords

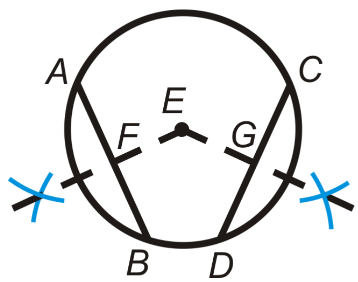
Investigation 9-3: Properties of Congruent Chords

Tools Needed: pencil, paper, compass, ruler

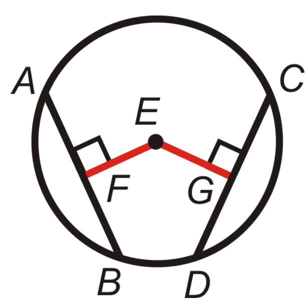
a. Draw a circle with a radius of 2 inches and two chords that are both 3 inches. Label as in the picture to the right. *This diagram is drawn to scale.*



b. From the center, draw the perpendicular segment to $\overline{AB}$ and $\overline{CD}$. You can either use your ruler, a protractor or Investigation 3-2 (Constructing a Perpendicular Line through a Point not on the line. We will show arc marks for Investigation 3-2.



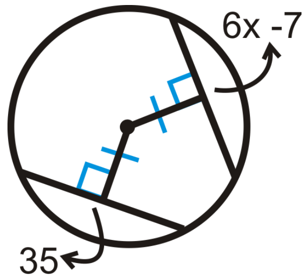
c. Erase the arc marks and lines beyond the points of intersection, leaving $\overline{FE}$ and $\overline{EG}$. Find the measure of these segments. What do you notice?



Theorem 10-6: In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.

Recall that two lines are equidistant from the same point if and only if the shortest distance from the point to the line is congruent. The shortest distance from any point to a line is the perpendicular line between them. In this theorem, the fact that $FE = EG$ means that $\overline{AB}$ and $\overline{CD}$ are equidistant to the center and $\overline{AB} \cong \overline{CD}$.

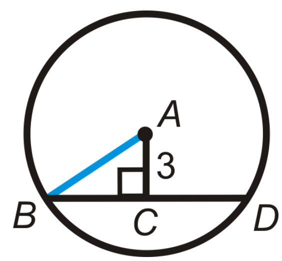
Example 5: ***Algebra Connection*** Find the value of x.



Solution: Because the distance from the center to the chords is congruent and perpendicular to the chords, then the chords are equal.

$$\begin{aligned} 6x - 7 &= 35 \\ 6x &= 42 \\ x &= 7 \end{aligned}$$

Example 6: $BD = 12$ and $AC = 3$ in $\odot A$. Find the radius and $m\widehat{BD}$.



Solution: First find the radius. In the picture, $\overline{AB}$ is a radius, so we can use the right triangle $\triangle ABC$, such that $\overline{AB}$ is the hypotenuse. From 10-5, $BC = 6$.

$$\begin{aligned} 3^2 + 6^2 &= AB^2 \\ 9 + 36 &= AB^2 \\ AB &= \sqrt{45} = 3\sqrt{5} \end{aligned}$$

In order to find $m\widehat{BD}$, we need the corresponding central angle, $\angle BAD$. We can find half of $\angle BAD$ because it is an acute angle in $\triangle ABC$. Then, multiply the measure by 2 for $m\widehat{BD}$.

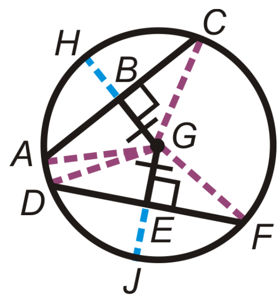
$$\begin{aligned} \tan^{-1}\left(\frac{6}{3}\right) &= m\angle BAC \\ m\angle BAC &\approx 63.43^\circ \end{aligned}$$

This means that $m\angle BAD \approx 126.9^\circ$ and $m\widehat{BD} \approx 126.9^\circ$ as well.

Know What? Revisited In the picture, the chords from $\odot A$ and $\odot E$ are congruent and the chords from $\odot B$, $\odot C$, and $\odot D$ are also congruent. We know this from Theorem 10-3. All five chords are not congruent because all five circles are not congruent, even though the central angle for the circles is the same.

Review Questions

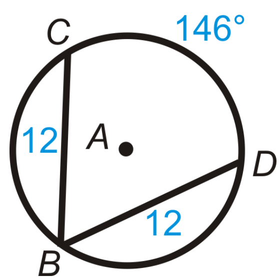
1. Two chords in a circle are perpendicular and congruent. Does one of them have to be a diameter? Why or why not? Fill in the blanks.



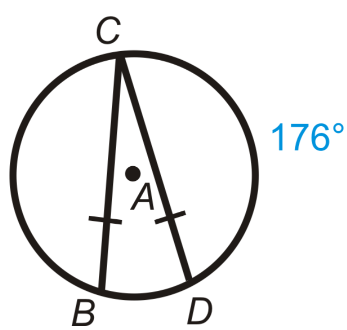
2. ___ $\cong \overline{DF}$
3. $\widehat{AC} \cong$ ___
4. $\widehat{DJ} \cong$ ___
5. ___ $\cong \overline{EJ}$
6. $\angle AGH \cong$ ___
7. $\angle DGF \cong$ ___
8. List all the congruent radii in $\odot G$.

Find the value of the indicated arc in $\odot A$.

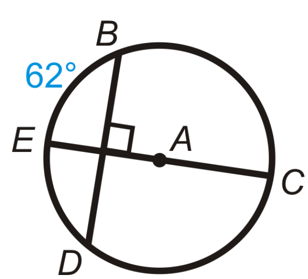
9. $m\widehat{BC}$



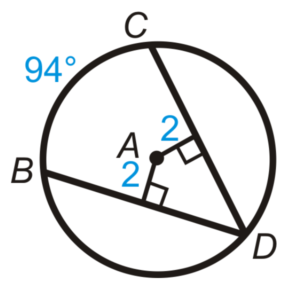
10. $m\widehat{BD}$



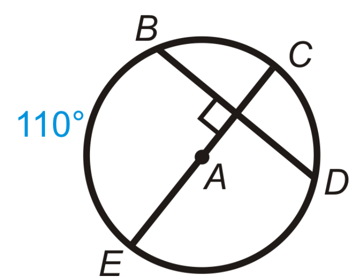
11. $m\widehat{BC}$



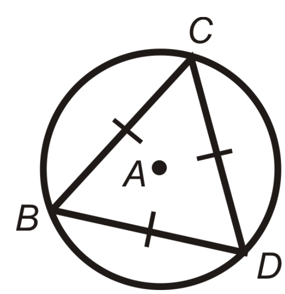
12. $m\widehat{BD}$



13. $m\widehat{BD}$

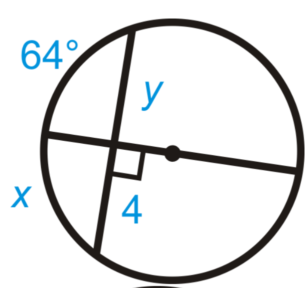


14. $m\widehat{BD}$



Algebra Connection Find the value of x and/or y.

15.



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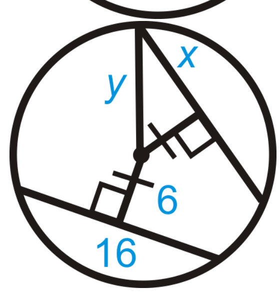

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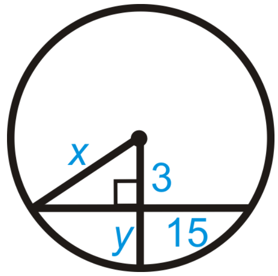
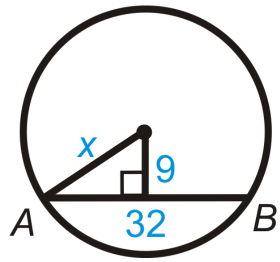


18. $AB = 32$



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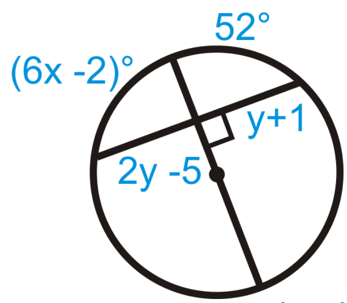

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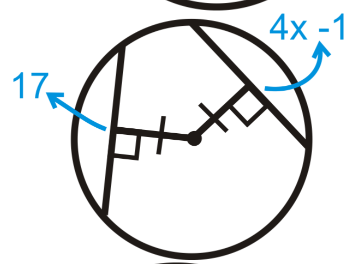

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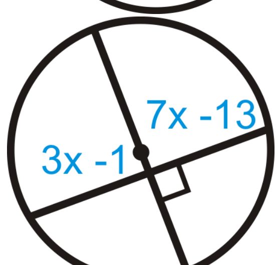

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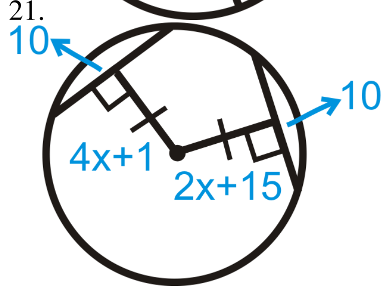
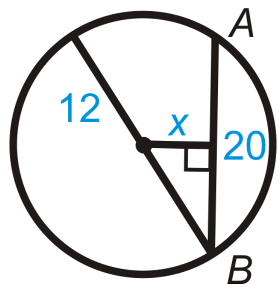
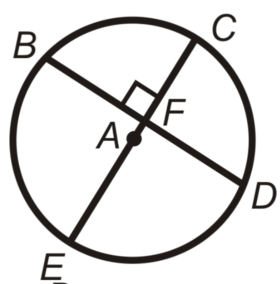




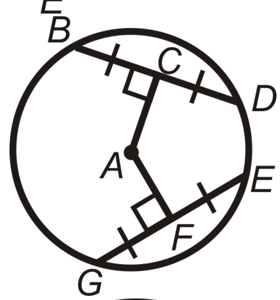
23.

24. Find $m\widehat{AB}$ in Question 18. Round your answer to the nearest tenth of a degree.
25. Find $m\widehat{AB}$ in Question 23. Round your answer to the nearest tenth of a degree.

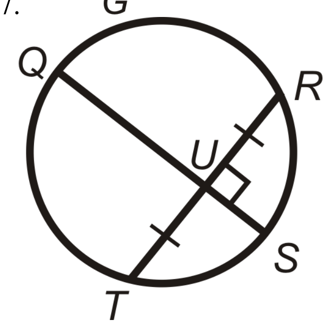
In problems 26-28, what can you conclude about the picture? State a theorem that justifies your answer. You may assume that A is the center of the circle.



26.

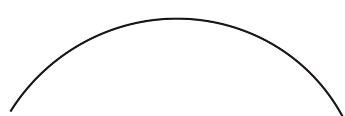


27.



28.

29. Trace the arc below onto your paper then follow the steps to locate the center using a compass and straightedge.



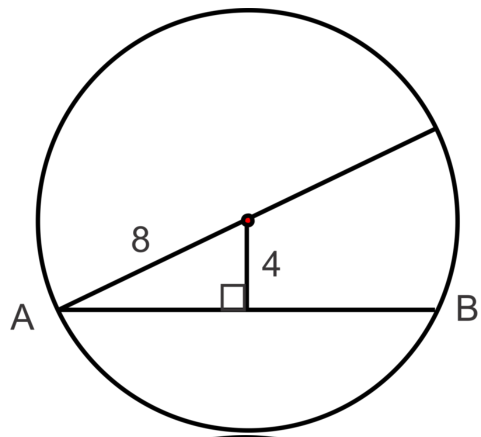
a. Use your straightedge to make a chord in the arc.
b. Use your compass and straightedge to construct the perpendicular bisector of this chord.
c. Repeat steps a and b so that you have two chords and their perpendicular bisectors.
d. What is the significance of the point where the perpendicular bisectors intersect?
e. Verify your answer to part d by using the point and your compass to draw the rest of the circle.

30. ***Algebra Connection*** Let's repeat what we did in problem 29 using coordinate geometry skills. Given the points $A(-3,5), B(5,5)$ and $C(4,-2)$ on the circle (an arc could be drawn through these points from A to C). The following steps will walk you through the process to find the equation of the perpendicular bisector of

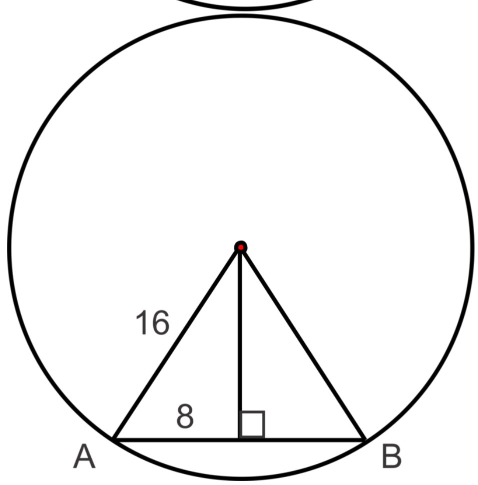
a chord, and use two of these perpendicular bisectors to locate the center of the circle. Let's first find the perpendicular bisector of chord $\overline{AB}$.

a. Since the perpendicular bisector passes through the midpoint of a segment we must first find the midpoint between A and B.
b. Now the perpendicular line must have a slope that is the opposite reciprocal of the slope of $\overleftrightarrow{AB}$. Find the slope of $\overleftrightarrow{AB}$ and then its opposite reciprocal.
c. Finally, you can write the equation of the perpendicular bisector of $\overline{AB}$ using the point you found in part a and the slope you found in part b.
d. Repeat steps a-c for chord $\overline{BC}$.
e. Now that we have the two perpendicular bisectors of the chord we can use algebra to find their intersection. Solve the system of linear equations to find the center of the circle.
f. Find the radius of the circle by finding the distance from the center (point found in part e) to any of the three given points on the circle.

31. Find the measure of $\widehat{AB}$ in each diagram below.



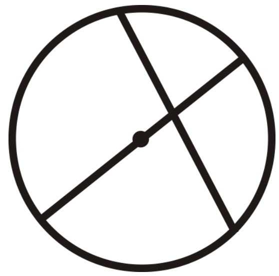
a.



b.

Review Queue Answers

1 & 2. Answers will vary



3. $m\widehat{BC} = 60^\circ, m\widehat{BDC} = 300^\circ$
4. $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$

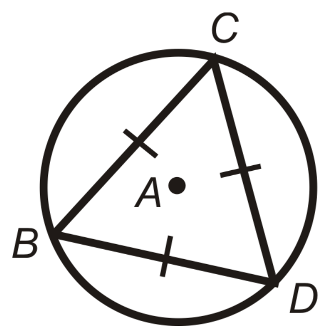
12.4 Inscribed Angles

Learning Objectives

- Find the measure of inscribed angles and the arcs they intercept.

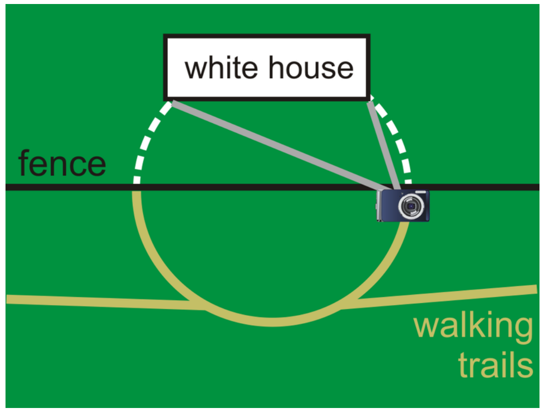
Review Queue

We are going to use #14 from the homework in the previous section.



a. What is the measure of each angle in the triangle? How do you know?
b. What do you know about the three arcs?
c. What is the measure of each arc?
d. What is the relationship between the angles in the triangles and the measure of each arc?

Know What? Your family went to Washington DC over the summer and saw the White House. The closest you can get to the White House are the walking trails on the far right. You got as close as you could (on the trail) to the fence to take a picture (you were not allowed to walk on the grass). Where else could you have taken your picture from to get the same frame of the White House? Where do you think the best place to stand would be? *Your line of sight in the camera is marked in the picture as the grey lines. The white dotted arcs do not actually exist, but were added to help with this problem.*



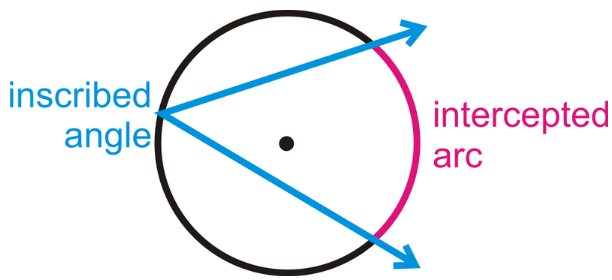
Inscribed Angles

We have discussed central angles so far in this chapter. We will now introduce another type of angle, the inscribed angle.

Inscribed Angle: An angle with its vertex is the circle and its sides contain chords.

Intercepted Arc: The arc that is on the interior of the inscribed angle and whose endpoints are on the angle.

The vertex of an inscribed angle can be anywhere on the circle as long as its sides intersect the circle to form an intercepted arc.

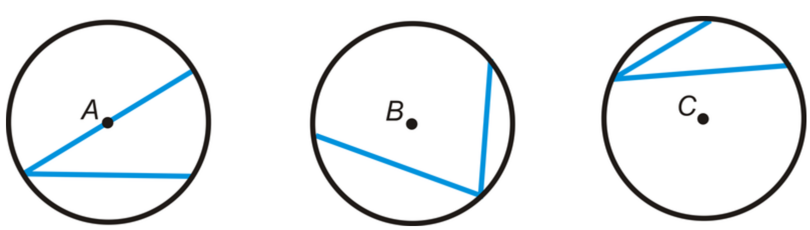


Now, we will investigation the relationship between the inscribed angle, the central angle and the arc they intercept.

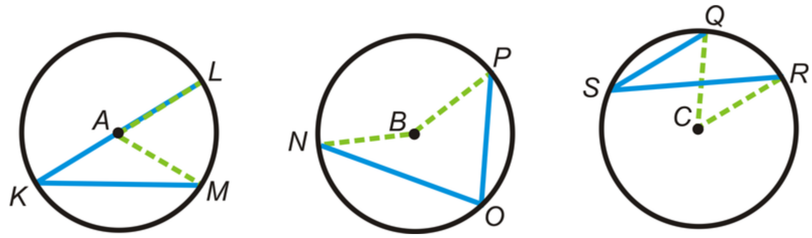
Investigation 9-4: Measuring an Inscribed Angle

Tools Needed: pencil, paper, compass, ruler, protractor

1. Draw three circles with three different inscribed angles. For $\odot A$, make one side of the inscribed angle a diameter, for $\odot B$, make B inside the angle and for $\odot C$ make C outside the angle. Try to make all the angles different sizes.



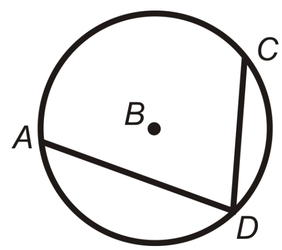
2. Using your ruler, draw in the corresponding central angle for each angle and label each set of endpoints.



3. Using your protractor measure the six angles and determine if there is a relationship between the central angle, the inscribed angle, and the intercepted arc.

$m\angle LAM =$ ____	$m\angle NBP =$ ____	$m\angle QCR =$ ____
$m\widehat{LM} =$ ____	$m\widehat{NP} =$ ____	$m\widehat{QR} =$ ____
$m\angle LKM =$ ____	$m\angle NOP =$ ____	$m\angle QSR =$ ____

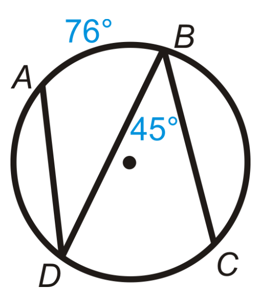
Inscribed Angle Theorem: The measure of an inscribed angle is half the measure of its intercepted arc.



In the picture, $m\angle ADC = \frac{1}{2}m\widehat{AC}$. If we had drawn in the central angle $\angle ABC$, we could also say that $m\angle ADC = \frac{1}{2}m\angle ABC$ because the measure of the central angle is equal to the measure of the intercepted arc.

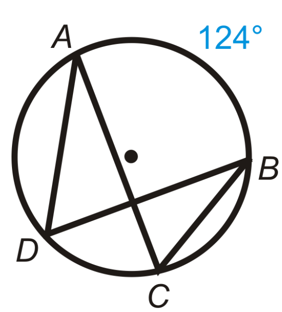
To prove the Inscribed Angle Theorem, you would need to split it up into three cases, like the three different angles drawn from Investigation 9-4. We will touch on the algebraic proofs in the review exercises.

Example 1: Find $m\widehat{DC}$ and $m\angle ADB$.



Solution: From the Inscribed Angle Theorem, $m\widehat{DC} = 2 \cdot 45^\circ = 90^\circ$. $m\angle ADB = \frac{1}{2} \cdot 76^\circ = 38^\circ$.

Example 2: Find $m\angle ADB$ and $m\angle ACB$.



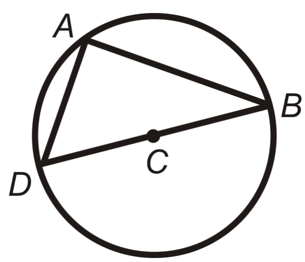
Solution: The intercepted arc for both angles is $\widehat{AB}$. Therefore, $m\angle ADB = m\angle ACB = \frac{1}{2} \cdot 124^\circ = 62^\circ$

This example leads us to our next theorem.

Theorem 9-8: Inscribed angles that intercept the same arc are congruent.

To prove Theorem 9-8, you would use the similar triangles that are formed by the chords.

Example 3: Find $m\angle DAB$ in $\odot C$.



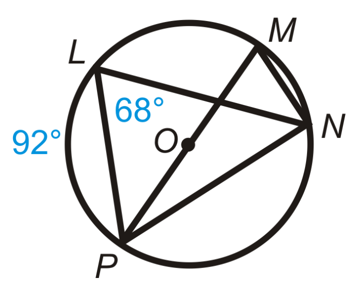
Solution: Because C is the center, $\overline{DB}$ is a diameter. Therefore, $\angle DAB$ inscribes semicircle, or 180°. $m\angle DAB = \frac{1}{2} \cdot 180^\circ = 90^\circ$.

Theorem 9-9: An angle that intercepts a semicircle is a right angle.

In Theorem 9-9 we could also say that the angle is inscribed in a semicircle. Anytime a right angle is inscribed in a circle, the endpoints of the angle are the endpoints of a diameter. Therefore, the converse of Theorem 9-9 is also true.

When the three vertices of a triangle are on the circle, like in Example 3, we say that the triangle is *inscribed* in the circle. We can also say that the circle is *circumscribed* around (or about) the triangle. Any polygon can be inscribed in a circle.

Example 4: Find $m\angle PMN, m\widehat{PN}, m\angle MNP, m\angle LNP$, and $m\widehat{LN}$.



Solution:

$m\angle PMN = m\angle PLN = 68^\circ$ by Theorem 9-8.

$m\widehat{PN} = 2 \cdot 68^\circ = 136^\circ$ from the Inscribed Angle Theorem.

$m\angle MNP = 90^\circ$ by Theorem 9-9.

$m\angle LNP = \frac{1}{2} \cdot 92^\circ = 46^\circ$ from the Inscribed Angle Theorem.

To find $m\widehat{LN}$, we need to find $m\angle LPN$. $\angle LPN$ is the third angle in $\triangle LPN$, so $68^\circ + 46^\circ + m\angle LPN = 180^\circ$. $m\angle LPN = 66^\circ$, which means that $m\widehat{LN} = 2 \cdot 66^\circ = 132^\circ$.

Inscribed Quadrilaterals

The last theorem for this section involves inscribing a quadrilateral in a circle.

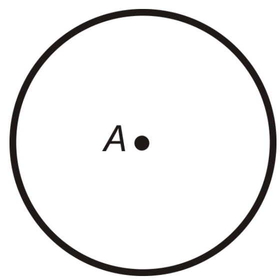
Inscribed Polygon: A polygon where every vertex is on a circle.

Note, that not every quadrilateral or polygon can be inscribed in a circle. Inscribed quadrilaterals are also called *cyclic quadrilaterals.* For these types of quadrilaterals, they must have one special property. We will investigate it here.

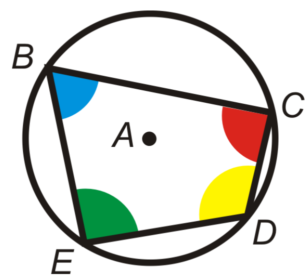
Investigation 9-5: Inscribing Quadrilaterals

Tools Needed: pencil, paper, compass, ruler, colored pencils, scissors

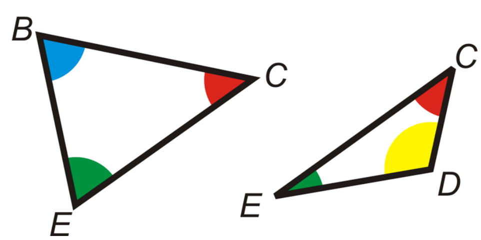
a. Draw a circle. Mark the center point A.



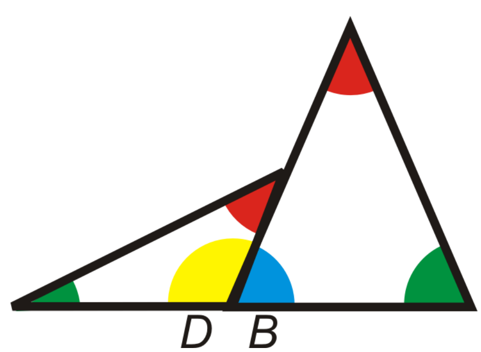
b. Place four points on the circle. Connect them to form a quadrilateral. Color the 4 angles of the quadrilateral 4 different colors.



c. Cut out the quadrilateral. Then cut the quadrilateral into two triangles, by cutting on a diagonal.



d. Line up $\angle B$ and $\angle D$ so that they are adjacent angles. What do you notice? What does this show?

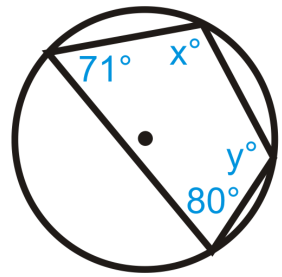


This investigation shows that the opposite angles in an inscribed quadrilateral are supplementary. By cutting the quadrilateral in half, through the diagonal, we were able to show that the other two angles (that we did not cut through) formed a linear pair when matched up.

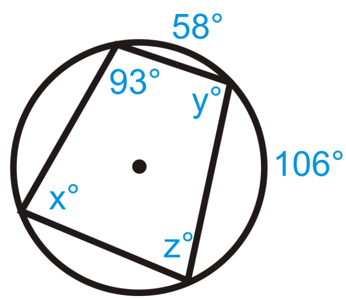
Theorem 9-10: A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

Example 5: Find the value of the missing variables.

a)



b)



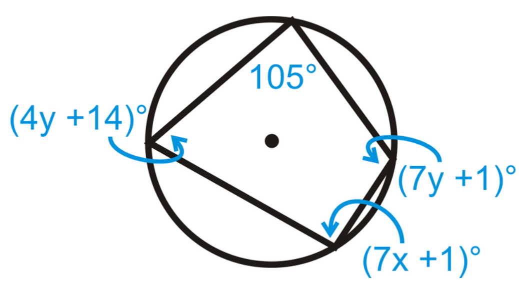
Solution:

a) $x+80^\circ = 180^\circ$ by Theorem 9-10. $x = 100^\circ$.

$y+71^\circ = 180^\circ$ by Theorem 9-10. $y = 109^\circ$.

b) It is easiest to figure out z first. It is supplementary with 93°, so $z = 87^\circ$. Second, we can find x. x is an inscribed angle that intercepts the arc $58^\circ + 106^\circ = 164^\circ$. Therefore, by the Inscribed Angle Theorem, $x = 82^\circ$. y is supplementary with x, so $y = 98^\circ$.

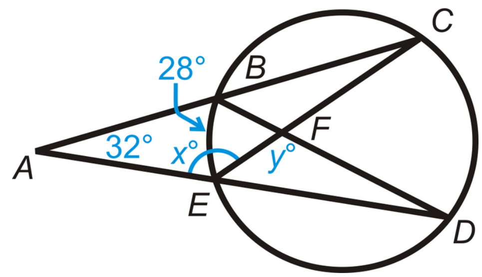
Example 6: ***Algebra Connection*** Find x and y in the picture below.



Solution: The opposite angles are supplementary. Set up an equation for x and y.

$$\begin{aligned}(7x+1)^\circ + 105^\circ &= 180^\circ \\ 7x+106^\circ &= 180^\circ \\ 7x &= 84^\circ \\ x &= 12^\circ\end{aligned} \qquad \begin{aligned}(4y+14)^\circ + (7y+1)^\circ &= 180^\circ \\ 11y+15^\circ &= 180^\circ \\ 11y &= 165^\circ \\ y &= 15^\circ\end{aligned}$$

Example 7: Find x and y in the picture below.



Solution: To find x, use $\triangle ACE$. $m\angle ACE = 14^\circ$ because it is half of $m\widehat{BE}$ by the Inscribed Angle Theorem.

$$\begin{aligned} 32^\circ + 14^\circ + x^\circ &= 180^\circ \\ x &= 134^\circ \end{aligned}$$

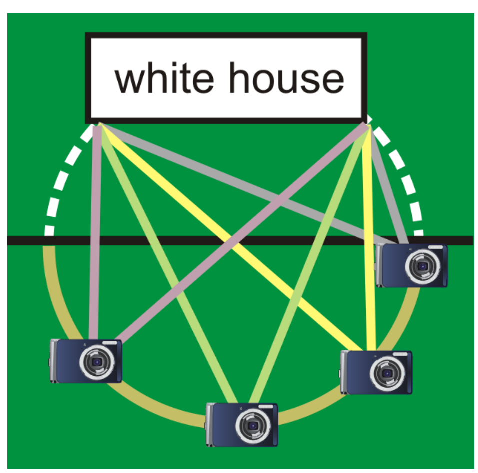
To find y, we will use $\triangle EFD$.

$$\begin{aligned} m\angle FED &= 180^\circ - x^\circ \\ m\angle FED &= 180^\circ - 134^\circ = 46^\circ \end{aligned}$$

$m\angle BDE = m\angle ACE = 14^\circ$ because they intercept the same arc, Theorem 9-8. Let's solve for y in $\triangle EFD$, using the Triangle Sum Theorem.

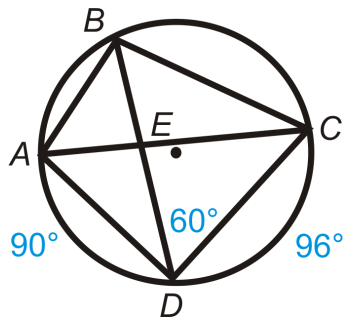
$$\begin{aligned} 46^\circ + 14^\circ + y^\circ &= 180^\circ \\ y &= 120^\circ \end{aligned}$$

Know What? Revisited You can take the picture from anywhere on the semicircular walking path. The best place to take the picture is subjective, but most would think the pale green frame, straight-on, would be the best view.



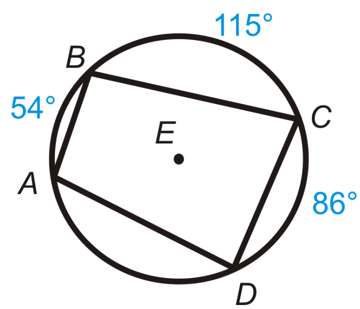
Review Questions

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



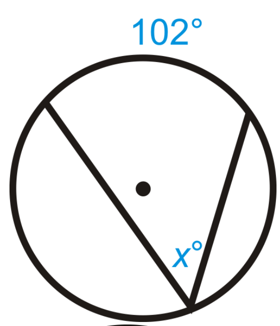
1. $m\angle DBC$
2. $m\widehat{BC}$
3. $m\widehat{AB}$
4. $m\angle ACD$
5. $m\angle ADC$
6. $m\angle ACB$

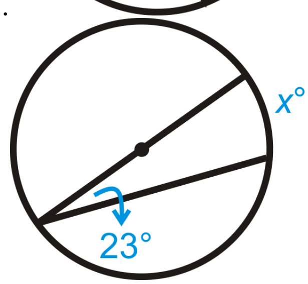
Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:

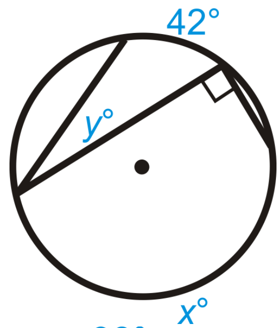


7. $m\angle A$
8. $m\angle B$
9. $m\angle C$
10. $m\angle D$

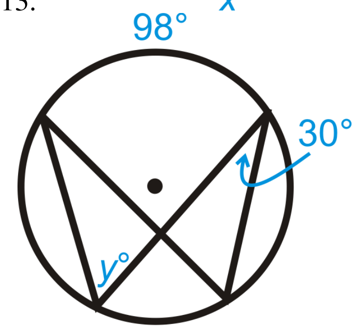
Find the value of x and/or y in $\odot A$.

11. 

12. 



13.



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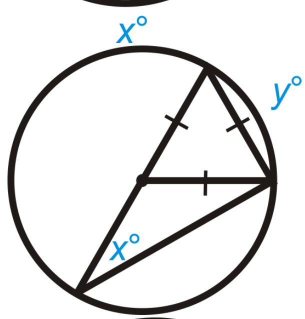

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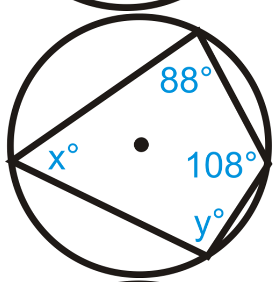

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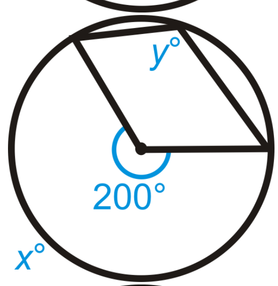
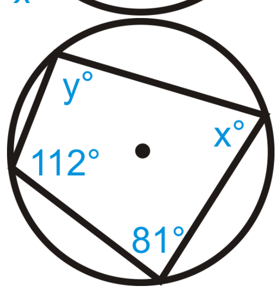
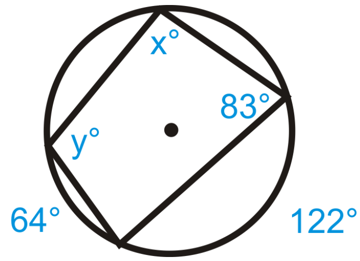


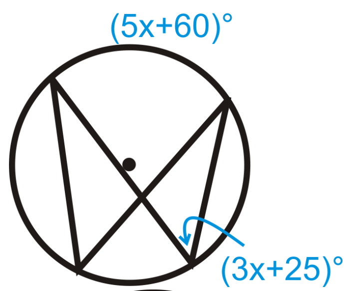
17.

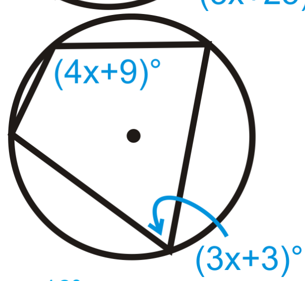


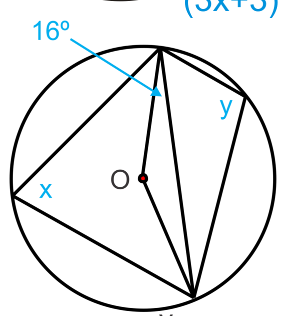
18.

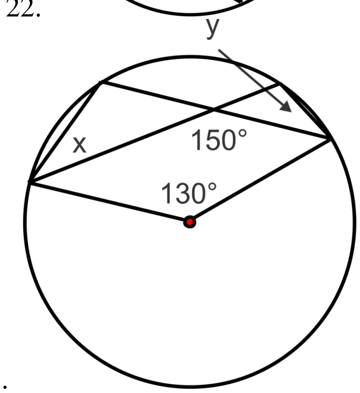
19. 

Algebra Connection Solve for the variables.

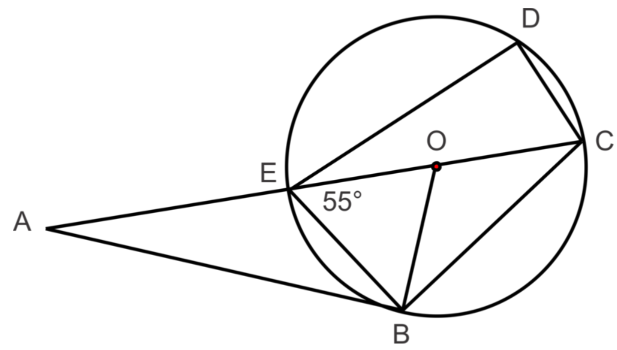
20. 

21. 

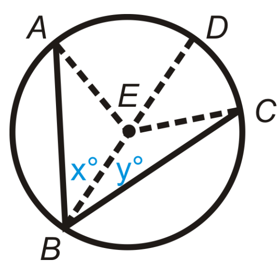
22. 

23. 

Use the diagram below to find the measures of the indicated angles and arcs in problems 24-28.



24. $m\angle EBO$
25. $m\angle EOB$
26. $m\widehat{BC}$
27. $m\angle ABO$
28. $m\angle A$
29. $m\angle EDC$
30. Fill in the blanks of one proof of the Inscribed Angle Theorem.

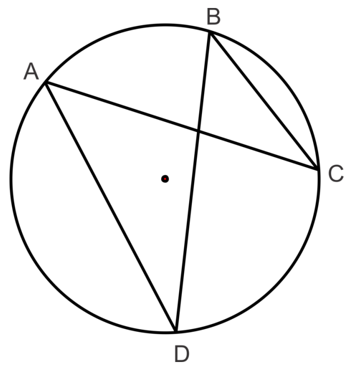


Given: Inscribed $\angle ABC$ and diameter $\overline{BD}$Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

TABLE 12.2:

Statement	***Reason***
1. Inscribed $\angle ABC$ and diameter $\overline{BD}$ $m\angle ABE = x^\circ$ and $m\angle CBE = y^\circ$	
2. $x^\circ + y^\circ = m\angle ABC$	
3.	All radii are congruent
4.	Definition of an isosceles triangle
5. $m\angle EAB = x^\circ$ and $m\angle ECB = y^\circ$	
6. $m\angle AED = 2x^\circ$ and $m\angle CED = 2y^\circ$	
7. $m\widehat{AD} = 2x^\circ$ and $m\widehat{DC} = 2y^\circ$	
8.	Arc Addition Postulate
9. $m\widehat{AC} = 2x^\circ + 2y^\circ$	
10.	Distributive PoE
11. $m\widehat{AC} = 2m\angle ABC$	
12. $m\angle ABC = \frac{1}{2}m\widehat{AC}$	

31. Use the diagram below to write a proof of Theorem 9-8.



32. Suppose that $\overline{AB}$ is a diameter of a circle centered at O, and C is any other point on the circle. Draw the line through O that is parallel to $\overline{AC}$, and let D be the point where it meets $\widehat{BC}$. Prove that D is the midpoint of $\widehat{BC}$.

Review Queue Answers

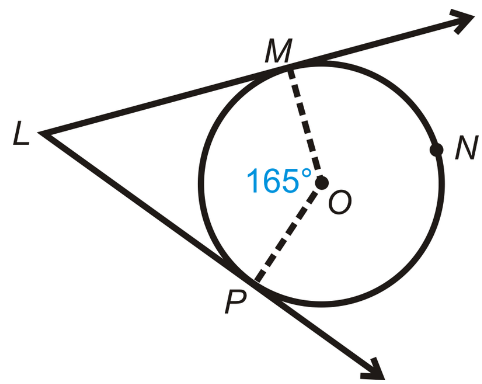
a. 60°, it is an equilateral triangle.
b. They are congruent because the chords are congruent.
c. $\frac{360^\circ}{3} = 120^\circ$
d. The arcs are double each angle.

12.5 Angles of Chords, Secants, and Tangents

Learning Objectives

- Find the measures of angles formed by chords, secants, and tangents.

Review Queue



a. What is $m\angle OML$ and $m\angle OPL$? How do you know?
b. Find $m\angle MLP$.
c. Find $m\widehat{MNP}$.
d. Find $\frac{m\widehat{MNP}-m\widehat{MP}}{2}$. What is it the same as?

Know What? The sun's rays hit the Earth such that the tangent rays determine when daytime and night time are. The time and Earth's rotation determine when certain locations have sun. If the arc that is exposed to sunlight is 178°, what is the angle at which the sun's rays hit the earth (x°)?

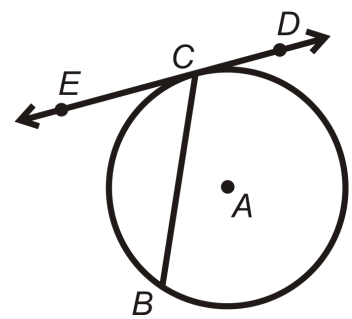
Angle

When an angle is on a circle, the vertex is on the circumference of the circle. One type of angle *on* a circle is the inscribed angle, from the previous section. Recall that ***an inscribed angle is formed by two chords and is half the measure of the intercepted arc.*** Another type of angle *on* a circle is one formed by a tangent and a chord.

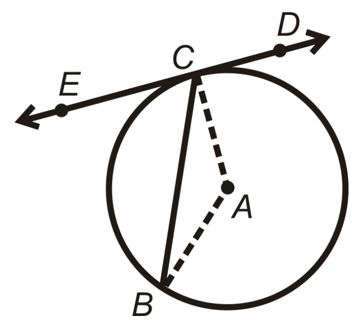
Investigation 9-6: The Measure of an Angle formed by a Tangent and a Chord

Tools Needed: pencil, paper, ruler, compass, protractor

a. Draw $\odot A$ with chord $\overline{BC}$ and tangent line $\overleftrightarrow{ED}$ with point of tangency C.



b. Draw in central angle $\angle CAB$. Then, using your protractor, find $m\angle CAB$ and $m\angle BCE$.



c. Find $m\widehat{BC}$ (the minor arc). How does the measure of this arc relate to $m\angle BCE$?

What other angle that you have learned about is this type of angle similar to?

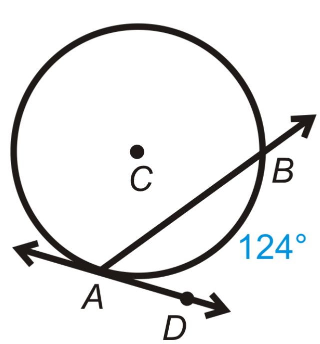
This investigation proves Theorem 9-11.

Theorem 9-11: The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

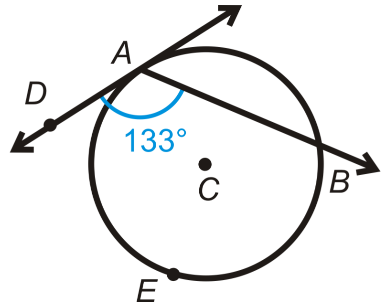
From Theorem 9-11, we now know that there are two types of angles that are half the measure of the intercepted arc; an inscribed angle and an angle formed by a chord and a tangent. Therefore, ***any angle with its vertex on a circle will be half the measure of the intercepted arc***.

Example 1: Find:

a) $m\angle BAD$



b) $m\widehat{AEB}$

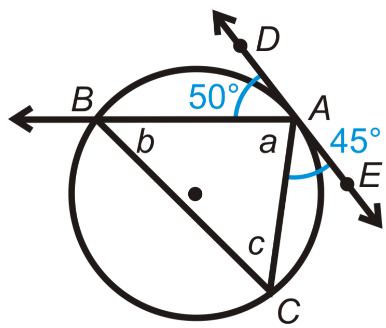


Solution: Use Theorem 9-11.

a) $m\angle BAD = \frac{1}{2} m\widehat{AB} = \frac{1}{2} \cdot 124^\circ = 62^\circ$

b) $m\widehat{AEB} = 2 \cdot m\angle DAB = 2 \cdot 133^\circ = 266^\circ$

Example 2: Find a, b, and c.



Solution: To find a, it is in line with 50° and 45°. The three angles add up to 180°. $50^\circ + 45^\circ + m\angle a = 180^\circ, m\angle a = 85^\circ$.

b is an inscribed angle, so its measure is half of $m\widehat{AC}$. From Theorem 9-11, $m\widehat{AC} = 2 \cdot m\angle EAC = 2 \cdot 45^\circ = 90^\circ$.

$m\angle b = \frac{1}{2} \cdot m\widehat{AC} = \frac{1}{2} \cdot 90^\circ = 45^\circ$.

To find c, you can either use the Triangle Sum Theorem or Theorem 9-11. We will use the Triangle Sum Theorem. $85^\circ + 45^\circ + m\angle c = 180^\circ, m\angle c = 50^\circ$.

From this example, we see that Theorem 9-8, from the previous section, is also true for angles formed by a tangent and chord with the vertex on the circle. If two angles, with their vertices *on* the circle, intercept the same arc then the angles are congruent.

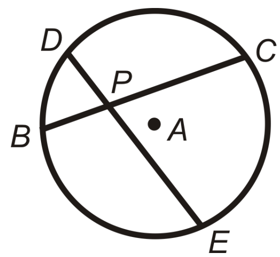
Angles

An angle is considered *inside* a circle when the vertex is somewhere inside the circle, but not on the center. All angles inside a circle are formed by two intersecting chords.

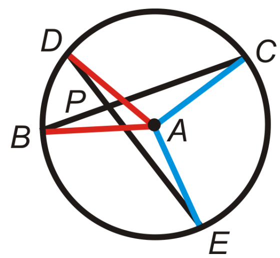
Investigation 9-7: Find the Measure of an Angle *inside* a Circle

Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

a. Draw $\odot A$ with chord $\overline{BC}$ and $\overline{DE}$. Label the point of intersection P.



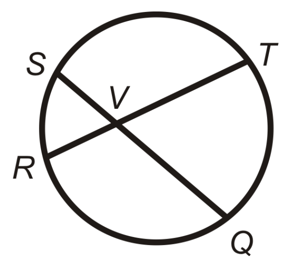
b. Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.



c. Using your protractor, find $m\angle DPB, m\angle DAB$, and $m\angle CAE$. What is $m\widehat{DB}$ and $m\widehat{CE}$?
d. Find $\frac{m\widehat{DB}+m\widehat{CE}}{2}$.
e. What do you notice?

Theorem 9-12: The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

In the picture to the left:

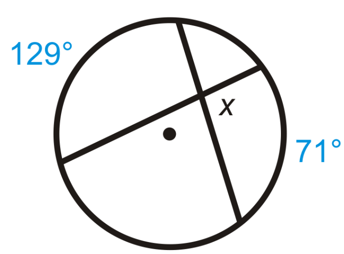


$$m\angle SVR = \frac{1}{2}\left(m\widehat{SR}+m\widehat{TQ}\right) = \frac{m\widehat{SR}+m\widehat{TQ}}{2} = m\angle TVQ$$
$$m\angle SVT = \frac{1}{2}\left(m\widehat{ST}+m\widehat{RQ}\right) = \frac{m\widehat{ST}+m\widehat{RQ}}{2} = m\angle RVQ$$

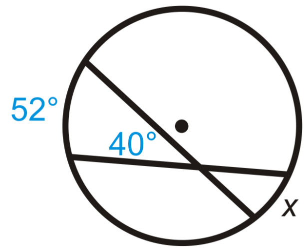
The proof of this theorem is in the review exercises.

Example 3: Find x.

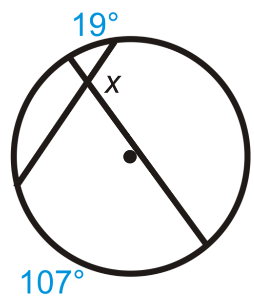
a)



b)



c)



Solution: Use Theorem 9-12 and write an equation.

a) The intercepted arcs for x are 129° and 71°.

$$x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$$

b) Here, x is one of the intercepted arcs for 40°.

$$\begin{aligned} 40^\circ &= \frac{52^\circ + x}{2} \\ 80^\circ &= 52^\circ + x \\ 38^\circ &= x \end{aligned}$$

c) x is supplementary to the angle that the average of the given intercepted arcs. We will call this supplementary angle y.

$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ$ This means that $x = 117^\circ; 180^\circ - 63^\circ$

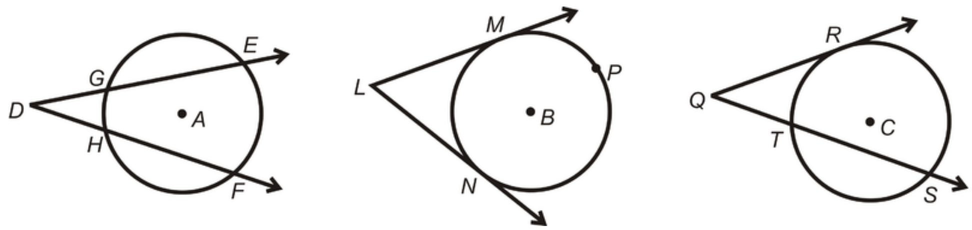
Angles

An angle is considered to be outside a circle if the vertex of the angle is outside the circle and the sides are tangents or secants. There are three types of angles that are outside a circle: an angle formed by two tangents, an angle formed by a tangent and a secant, and an angle formed by two secants. Just like an angle inside or on a circle, an angle outside a circle has a specific formula, involving the intercepted arcs.

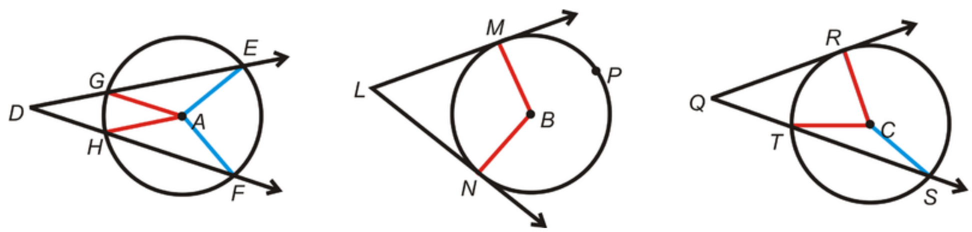
Investigation 9-8: Find the Measure of an Angle outside a Circle

Tools Needed: pencil, paper, ruler, compass, protractor, colored pencils (optional)

a. Draw three circles and label the centers A, B, and C. In $\odot A$ draw two secant rays with the same endpoint, $\overrightarrow{DE}$ and $\overrightarrow{DF}$. In $\odot B$, draw two tangent rays with the same endpoint, $\overrightarrow{LM}$ and $\overrightarrow{LN}$. In $\odot C$, draw a tangent ray and a secant ray with the same endpoint, $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Label the points of intersection with the circles like they are in the pictures below.



b. Draw in all the central angles: $\angle GAH, \angle EAF, \angle MBN, \angle RCT, \angle RCS$. Then, find the measures of each of these angles using your protractor. Use color to differentiate.



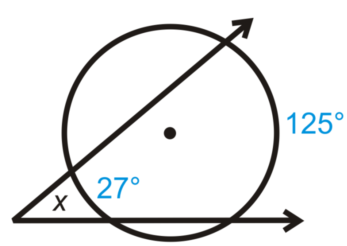
c. Find $m\angle EDF, m\angle MLN$, and $m\angle RQS$.

d. Find $\frac{m\widehat{EF}-m\widehat{GH}}{2}, \frac{m\widehat{MPN}-m\widehat{MN}}{2}$, and $\frac{m\widehat{RS}-m\widehat{RT}}{2}$. What do you notice?

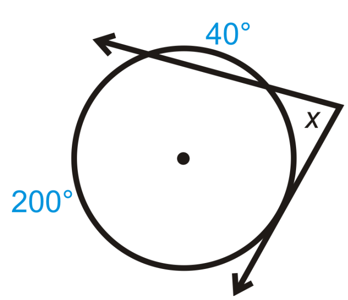
Theorem 9-13: The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.

Example 4: Find the measure of x.

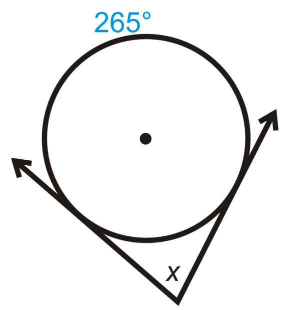
a)



b)



c)



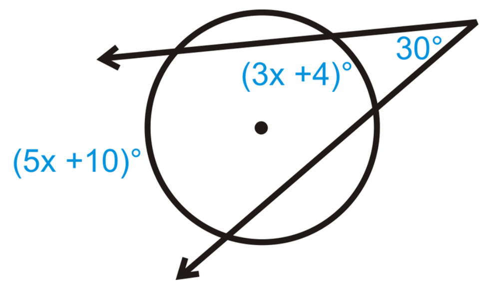
Solution: For all of the above problems we can use Theorem 9-13.

a) $x = \frac{125^\circ - 27^\circ}{2} = \frac{98^\circ}{2} = 49^\circ$

b) 40° is not the intercepted arc. Be careful! The intercepted arc is 120°, $(360^\circ - 200^\circ - 40^\circ)$. Therefore, $x = \frac{200^\circ - 120^\circ}{2} = \frac{80^\circ}{2} = 40^\circ$.

c) First, we need to find the other intercepted arc, $360^\circ - 265^\circ = 95^\circ$. $x = \frac{265^\circ - 95^\circ}{2} = \frac{170^\circ}{2} = 85^\circ$

Example 5: ***Algebra Connection*** Find the value of x. You may assume lines that look tangent, are.



Solution: Set up an equation using Theorem 9-13.

$$\begin{aligned}
\frac{(5x+10)^\circ - (3x+4)^\circ}{2} &= 30^\circ \\
(5x+10)^\circ - (3x+4)^\circ &= 60^\circ \\
5x + 10^\circ - 3x - 4^\circ &= 60^\circ \\
2x + 6^\circ &= 60^\circ \\
2x &= 54^\circ \\
x &= 27^\circ
\end{aligned}$$

Know What? Revisited If 178° of the Earth is exposed to the sun, then the angle at which the sun's rays hit the Earth is 2°. From Theorem 9-13, these two angles are supplementary. From this, we also know that the other 182° of the Earth is not exposed to sunlight and it is probably night time.

Review Questions

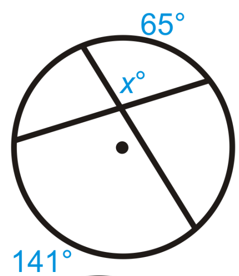
1. Draw two secants that intersect:
    a. inside a circle.
    b. on a circle.
    c. outside a circle.

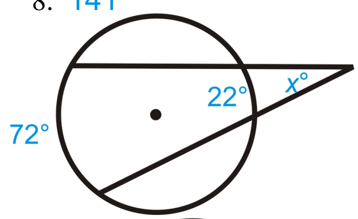
2. Can two tangent lines intersect inside a circle? Why or why not?
3. Draw a tangent and a secant that intersect:
   a. on a circle.
   b. outside a circle.

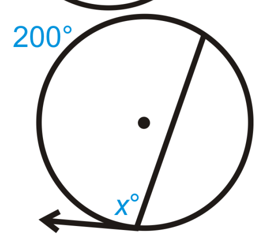
Fill in the blanks.

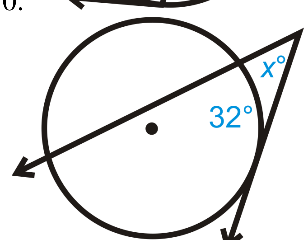
4. If the vertex of an angle is on the ________________ of a circle, then its measure is ________________ to the intercepted arc.
5. If the vertex of an angle is ________________ a circle, then its measure is the average of the ______________-______ arcs.
6. If the vertex of an angle is _________ a circle, then its measure is _______________ the intercepted arc.
7. If the vertex of an angle is _____________ a circle, then its measure is ____________ the difference of the intercepted arcs.

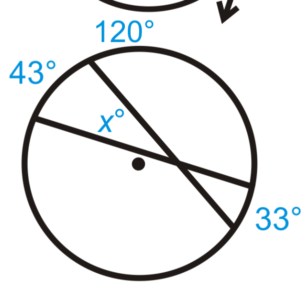
For questions 8-19, find the value of the missing variable(s).

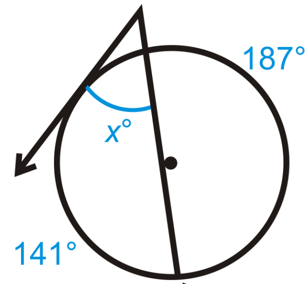
8. 

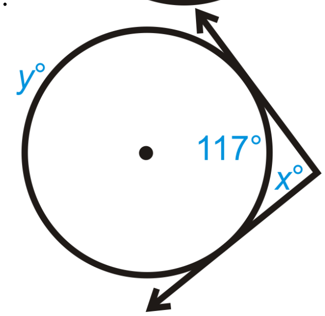
9. 

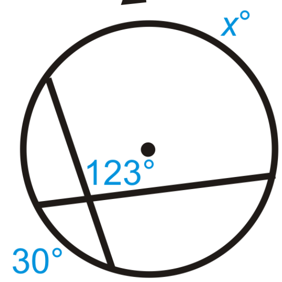
10. 

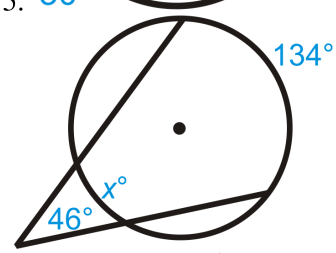
11. 

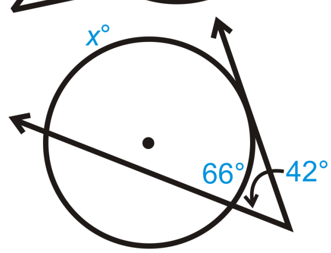
12. 

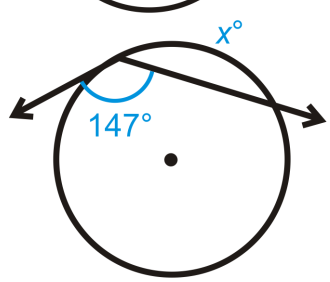
13. 

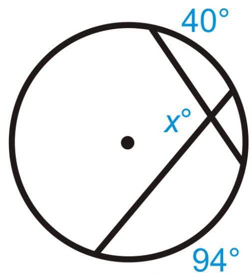
14. 

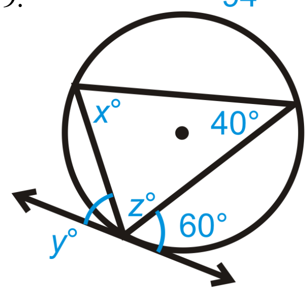
15. 

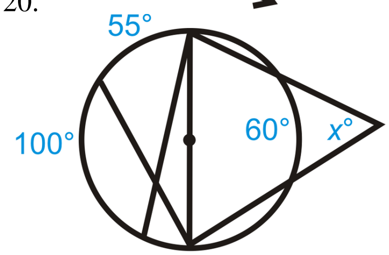
16. 

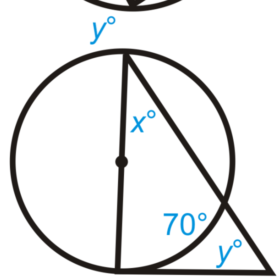
17. 

18. 

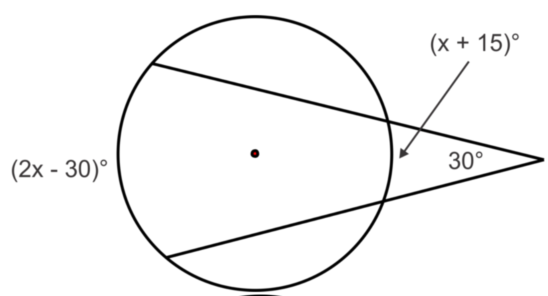
19. 

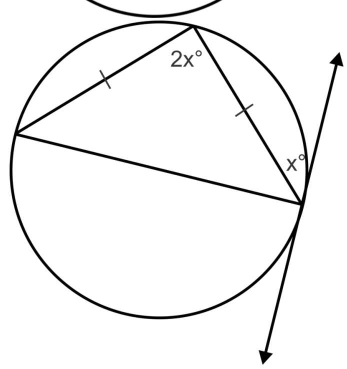
20. 

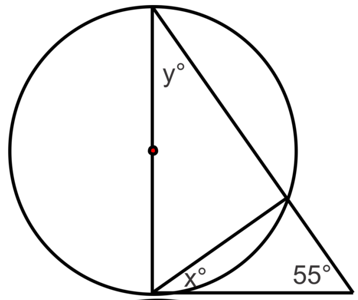
21. 

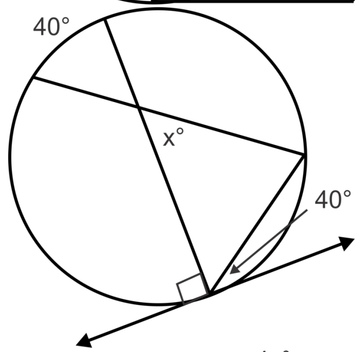
22. 

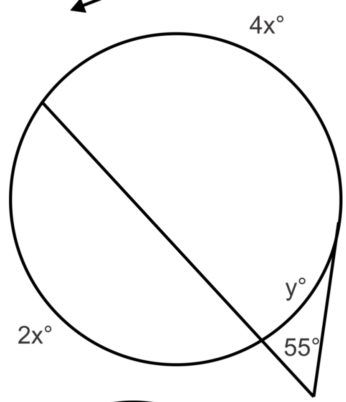
Algebra Connection Solve for the variable(s).

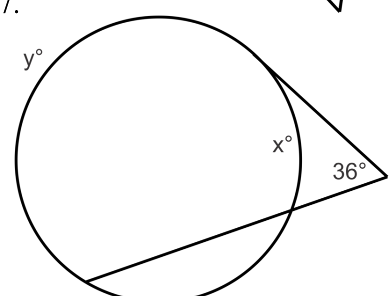
23. 

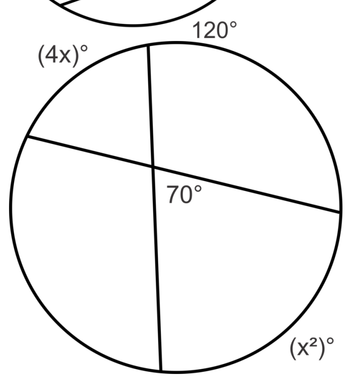
24. 

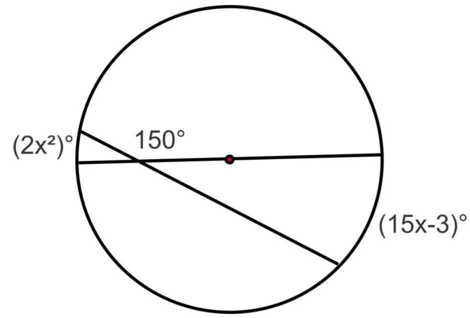
25. 

26. 

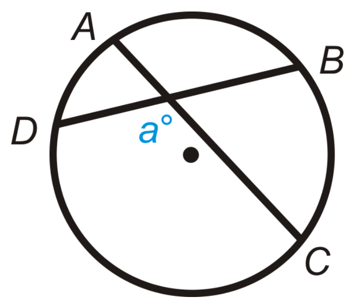
27. 

28. 

29. 

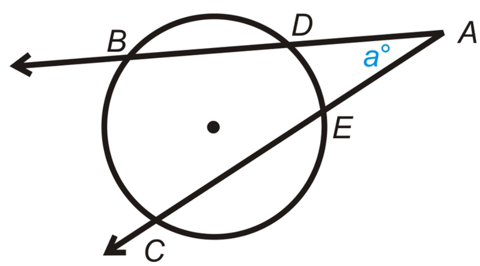
30. 

31. Prove Theorem 9-12.



Given: Intersecting chords $\overline{AC}$ and $\overline{BD}$. Prove: $m\angle a = \frac{1}{2}\left(m\widehat{DC} + m\widehat{AB}\right)$ *HINT:* Draw $\overline{BC}$ and use inscribed angles.

32. Prove Theorem 9-13.



Given: Secant rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$Prove: $m\angle a = \frac{1}{2}\left(m\widehat{BC} - m\widehat{DE}\right)$ *HINT:* Draw $\overline{BE}$ and use inscribed angles.

Review Queue Answers

a. $m\angle OML = m\angle OPL = 90^\circ$ because a tangent line and a radius drawn to the point of tangency are perpendicular.

b. $$\begin{aligned} 165^\circ + m\angle OML + m\angle OPL + m\angle MLP &= 360^\circ \\ 165^\circ + 90^\circ + 90^\circ + m\angle MLP &= 360^\circ \\ m\angle MLP &= 15^\circ \end{aligned}$$

c. $m\widehat{MNP} = 360^\circ - 165^\circ = 195^\circ$

d. $\frac{195^\circ - 165^\circ}{2} = \frac{30^\circ}{2} = 15^\circ$, this is the same as $m\angle MLP$.

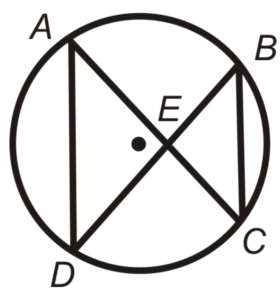
12.6 Segments of Chords, Secants, and Tangents

Learning Objectives

- Find the lengths of segments associated with circles.

Review Queue

a. What can you say about $m\angle DAC$ and $m\angle DBC$? What theorem do you use?

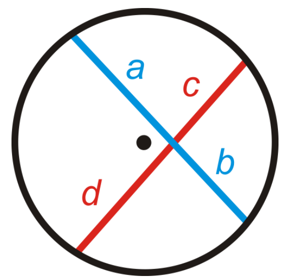


b. What do you know about $m\angle AED$ and $m\angle BEC$? Why?
c. Is $\triangle AED \sim \triangle BEC$? How do you know?
d. If $AE = 8, ED = 7$, and $BE = 6$, find EC.
e. If $\overline{AD}$ and $\overline{BC}$ are not in the circle, would the ratios from #4 still be valid?

Know What? As you know, the moon orbits the earth. At a particular time, the moon is 238,857 miles from Beijing, China. On the same line, Yukon is 12,451 miles from Beijing. Drawing another line from the moon to Cape Horn (the southernmost point of South America), we see that Jakarta, Indonesia is collinear. If the distance from Cape Horn to Jakarta is 9849 miles, what is the distance from the moon to Jakarta?

Segments from Chords

In the Review Queue above, we have two chords that intersect inside a circle. The two triangles are similar, making the sides of each triangle in proportion with each other. If we remove $\overline{AD}$ and $\overline{BC}$ the ratios between $\overline{AE}, \overline{EC}, \overline{DE}$, and $\overline{EB}$ will still be the same. This leads us to our first theorem.

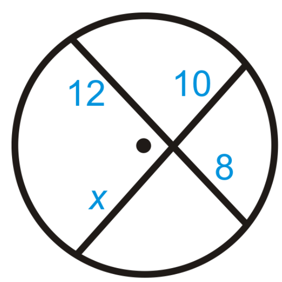


Theorem 9-14: If two chords intersect inside a circle so that one is divided into segments of length a and b and the other into segments of length c and d then $ab = cd$.

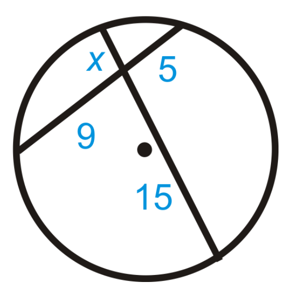
The product of the segments of one chord is equal to the product of segments of the second chord.

Example 1: Find x in each diagram below.

a)



b)



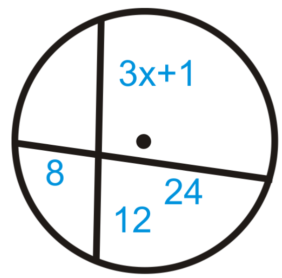
Solution: Use the ratio from Theorem 9-13. The product of the segments of one chord is equal to the product of the segments of the other.

a) $12 \cdot 8 = 10 \cdot x$
$96 = 10x$
$9.6 = x$

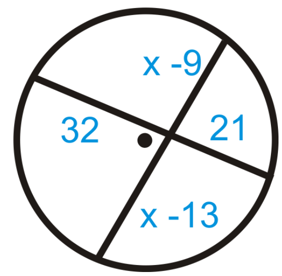
b) $x \cdot 15 = 5 \cdot 9$
$15x = 45$
$x = 3$

Example 2: ***Algebra Connection*** Solve for x.

a)



b)



Solution: Again, we can use Theorem 9-13. Set up an equation and solve for x.

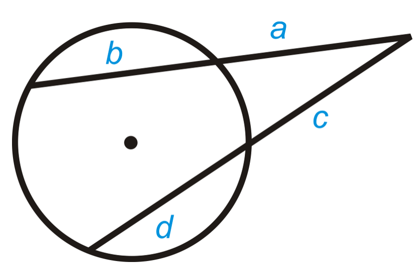
a) $8 \cdot 24 = (3x+1) \cdot 12$
$192 = 36x + 12$
$180 = 36x$
$5 = x$

b) $32 \cdot 21 = (x-9)(x-13)$
$672 = x^2 - 22x + 117$
$0 = x^2 - 22x - 555$
$0 = (x-37)(x+15)$
$x = 37, -15$

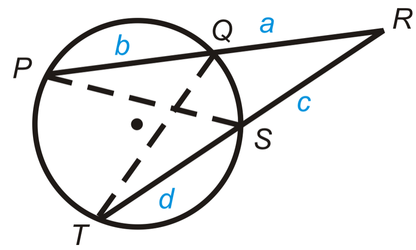
However, $x \neq -15$ because length cannot be negative, so $x = 37$.

Segments from Secants

In addition to forming an angle outside of a circle, the circle can divide the secants into segments that are proportional with each other.



If we draw in the intersecting chords, we will have two similar triangles.



From the inscribed angles and the Reflexive Property $(\angle R \cong \angle R), \triangle PRS \sim \triangle TRQ$.

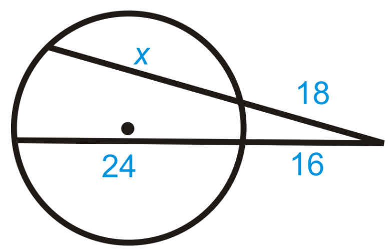
Because the two triangles are similar, we can set up a proportion between the corresponding sides. Then, cross-multiply. $\frac{a}{c+d} = \frac{c}{a+b} \Rightarrow a(a+b) = c(c+d)$

Theorem 9-15: If two secants are drawn from a common point outside a circle and the segments are labeled as above, then $a(a+b) = c(c+d)$.

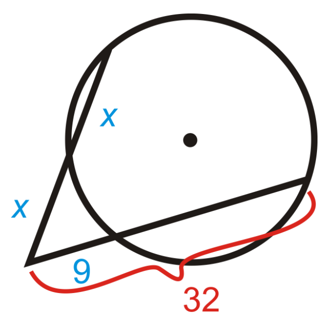
In other words, the product of the outer segment and the whole of one secant is equal to the product of the outer segment and the whole of the other secant.

Example 3: Find the value of the missing variable.

a)



b)



Solution: Use Theorem 9-15 to set up an equation. For both secants, you multiply the outer portion of the secant by the whole.

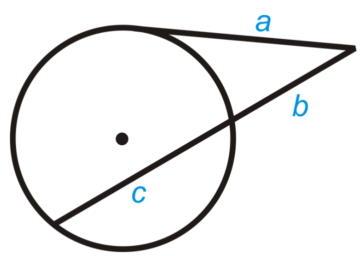
a) $18 \cdot (18+x) = 16 \cdot (16+24)$
$324 + 18x = 256 + 384$
$18x = 316$
$x = 17\frac{5}{9}$

b) $x \cdot (x+x) = 9 \cdot 32$
$2x^2 = 288$
$x^2 = 144$
$x = 12$

$x \neq -12$ because length cannot be negative.

Segments from Secants and Tangents

If a tangent and secant meet at a common point outside a circle, the segments created have a similar relationship to that of two secant rays in Example 3. Recall that the product of the outer portion of a secant and the whole is equal to the same of the other secant. If one of these segments is a tangent, it will still be the product of the outer portion and the whole. However, for a tangent line, the outer portion and the whole are equal.

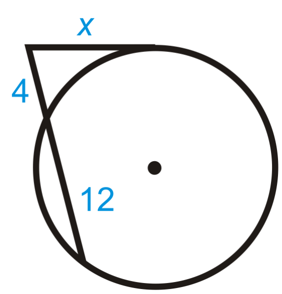


Theorem 9-16: If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture to the left), then $a^2 = b(b+c)$.

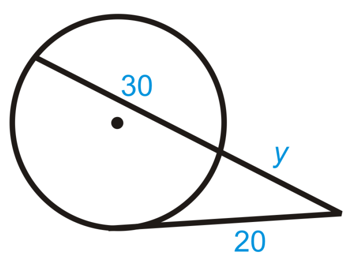
This means that the product of the outside segment of the secant and the whole is equal to the square of the tangent segment.

Example 4: Find the value of the missing segment.

a)



b)



Solution: Use Theorem 9-16. Square the tangent and set it equal to the outer part times the whole secant.

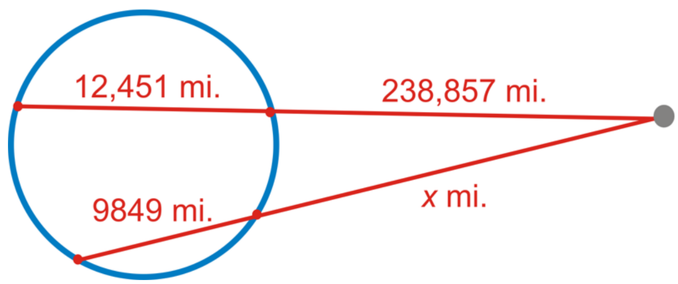
a) $x^2 = 4(4+12)$
$x^2 = 4 \cdot 16 = 64$
$x = 8$

b) $20^2 = y(y+30)$
$400 = y^2 + 30y$
$0 = y^2 + 30y - 400$

$0 = (y+40)(y-10)$
$y = \cancel{-40}, 10$

When you have to factor a quadratic equation to find an answer, always eliminate the negative answer(s). Length is never negative.

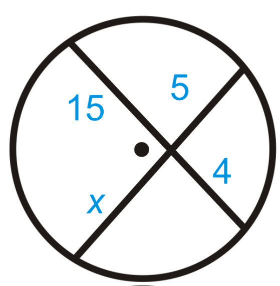
Know What? Revisited The given information is to the left. Let's set up an equation using Theorem 9-15.

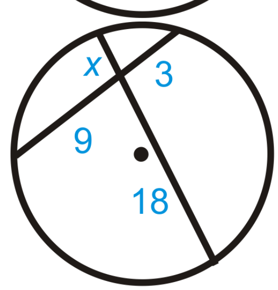


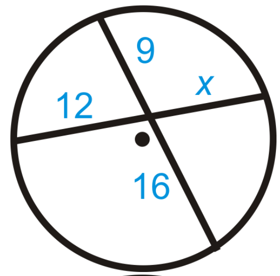
$$\begin{aligned}
238857 \cdot 251308 &= x \cdot (x+9849) \\
60026674956 &= x^2 + 9849x \\
0 &= x^2 + 9849x - 60026674956 \\
\textit{Use the Quadratic Formula } x &\approx \frac{-9849 \pm \sqrt{9849^2 - 4(-60026674956)}}{2} \\
x &\approx 240128.4 \textit{ miles}
\end{aligned}$$

Review Questions

Find x in each diagram below. Simplify any radicals.

1. 

2. 



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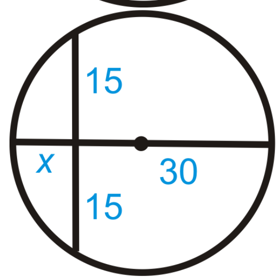

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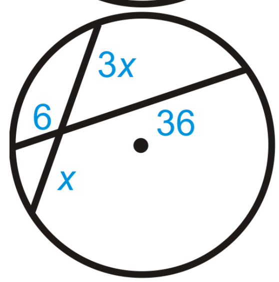

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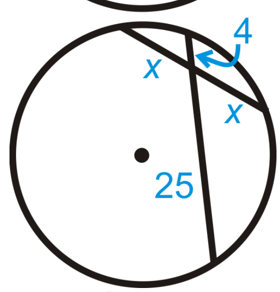

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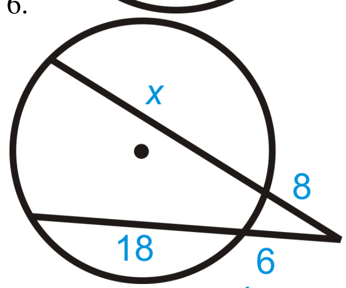

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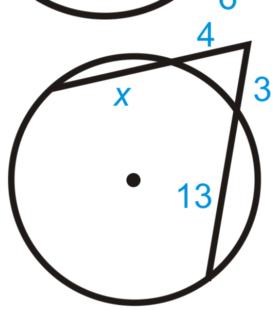

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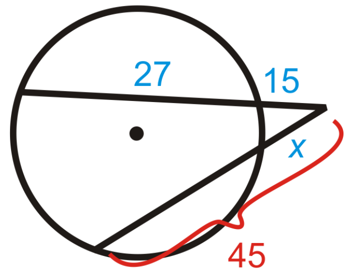
9.



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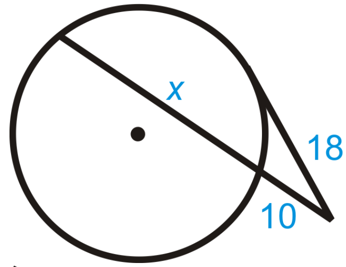

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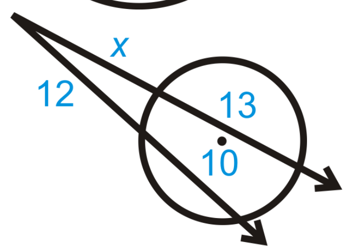

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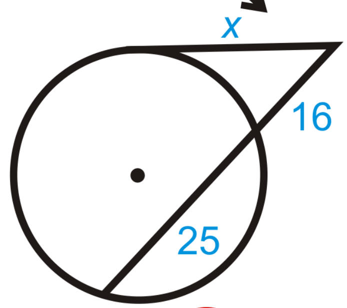

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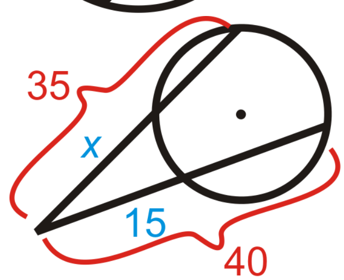

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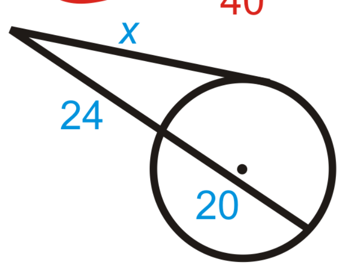

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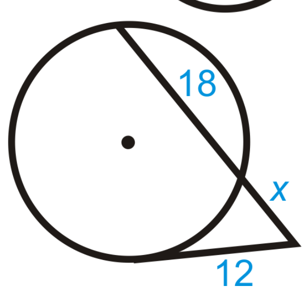
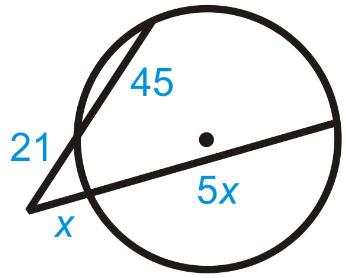
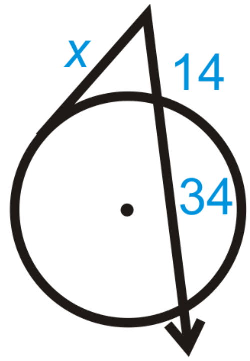
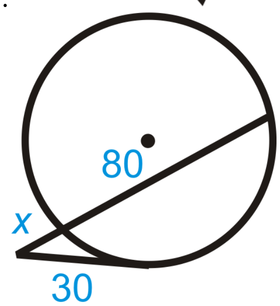
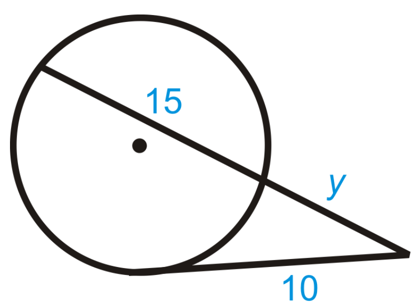


16. 

17. 

18. 

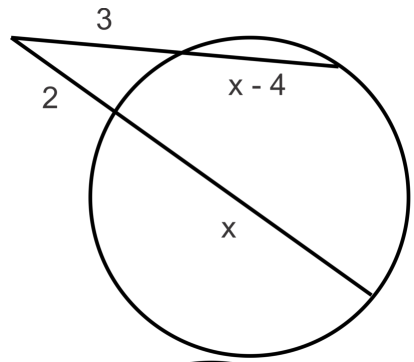
19. ***Error Analysis*** Describe and correct the error in finding y.



$$\begin{aligned} 10 \cdot 10 &= y \cdot 15y \\ 100 &= 15y^2 \\ \frac{20}{3} &= y^2 \\ \frac{2\sqrt{15}}{3} &= y \quad \leftarrow y \text{ is } \underline{\text{not}} \text{ correct} \end{aligned}$$

20. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is found to be 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is found to be 1 inch. Find the diameter of the plate.

Algebra Connection For problems 21-30, solve for x.



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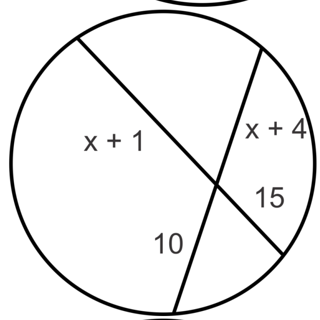

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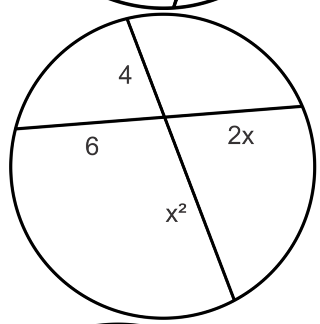

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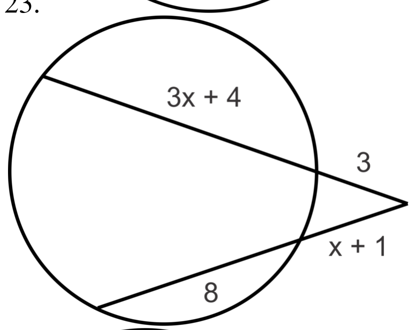

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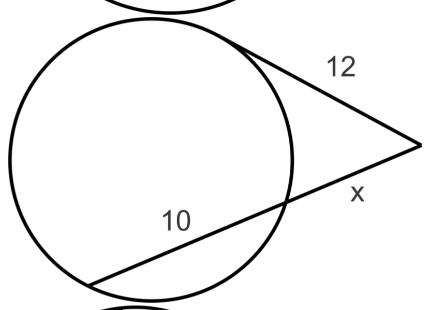

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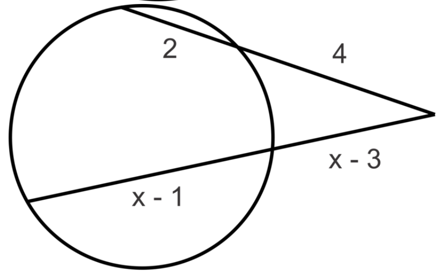

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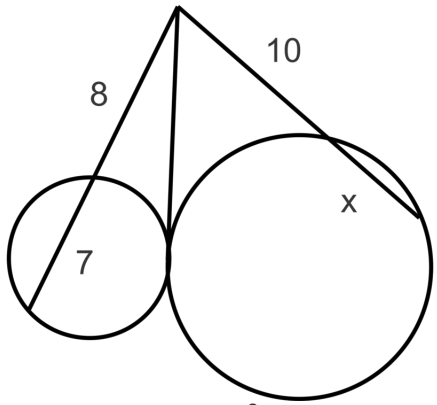
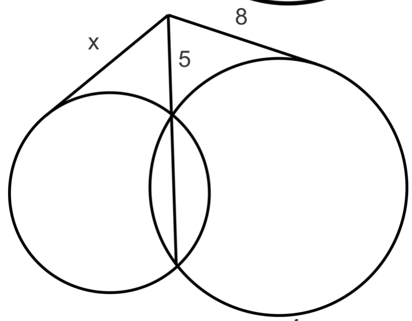
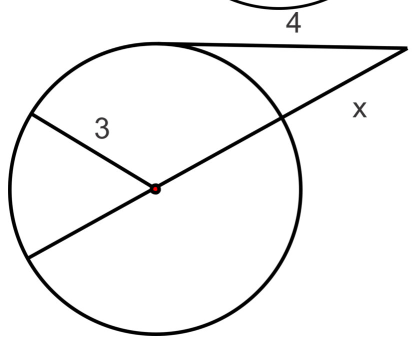
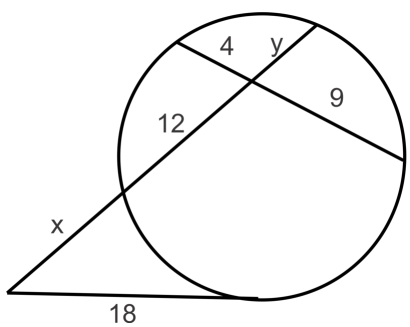
27. 

28. 

29. 

30. Find x and y.



Review Queue Answers

a. $m\angle DAC = m\angle DBC$ by Theorem 9-8, they are inscribed angles and intercept the same arc.
b. $m\angle AED = m\angle BEC$ by the Vertical Angles Theorem.
c. Yes, by AA Similarity Postulate.
d. $\frac{8}{6} = \frac{7}{EC}$
$8 \cdot EC = 42$
$EC = \frac{21}{4} = 5.25$
e. Yes, the EC would be the same and the ratio would still be valid.

12.7 Circles in the Coordinate Plane

Here you'll learn how to find the standard equation for circles given their radius and center. You'll also graph circles in the coordinate plane.

What if you were given the length of the radius of a circle and the coordinates of its center? How could you write the equation of the circle in the coordinate plane? After completing this Concept, you'll be able to write the standard equation of a circle.

Watch This

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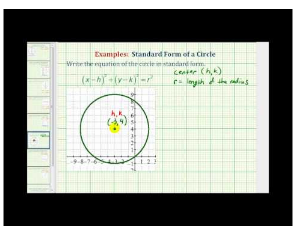
Graphing Circles CK-12



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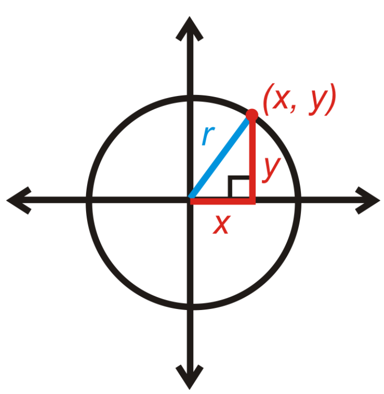
James Sousa: Write the Standard Form of a Circle

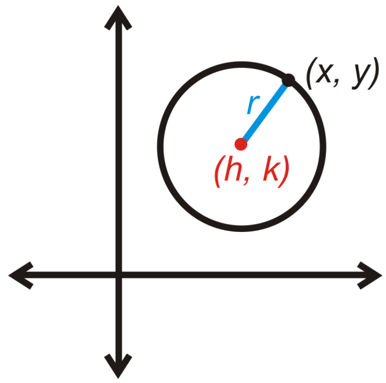
Guidance

Recall that a circle is the set of all points in a plane that are the same distance from the center. This definition can be used to find an equation of a circle in the coordinate plane.



Let's start with the circle centered at (0, 0). If (x, y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance and y is the vertical distance. This forms a right triangle. From the Pythagorean Theorem, the equation of a circle *centered at the origin* is $x^2 + y^2 = r^2$.

The center does not always have to be on (0, 0). If it is not, then we label the center (h, k). We would then use the Distance Formula to find the length of the radius.



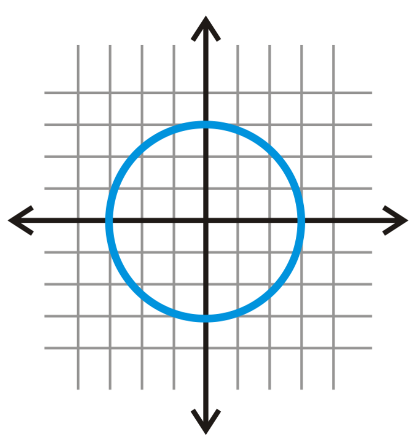
$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

If you square both sides of this equation, then you would have the standard equation of a circle. **The standard equation of a circle with center (h, k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.**

Example A

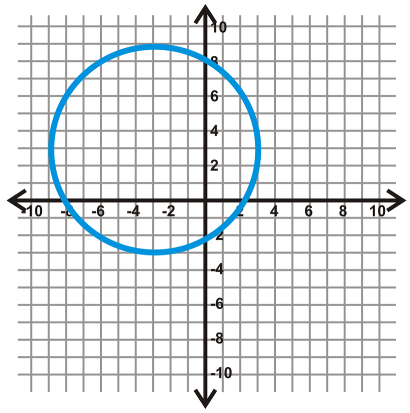
Graph $x^2 + y^2 = 9$.

The center is (0, 0). Its radius is the square root of 9, or 3. Plot the center, plot the points that are 3 units to the right, left, up, and down from the center and then connect these four points to form a circle.

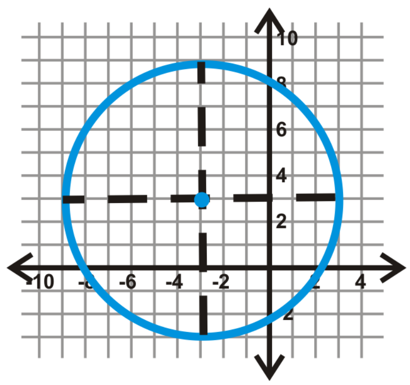


Example B

Find the equation of the circle below.



First locate the center. Draw in the horizontal and vertical diameters to see where they intersect.



From this, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find $r = 6$. Plugging this into the equation of a circle, we get: $(x-(-3))^2 + (y-3)^2 = 6^2$ or $(x+3)^2 + (y-3)^2 = 36$.

Example C

Determine if the following points are on $(x+1)^2 + (y-5)^2 = 50$.

a) (8, -3)

b) (-2, -2)

Plug in the points for x and y in $(x+1)^2 + (y-5)^2 = 50$.

a)

$$\begin{aligned}(8+1)^2 + (-3-5)^2 &= 50\\ 9^2 + (-8)^2 &= 50\\ 81 + 64 &\neq 50\end{aligned}$$

(8, -3) is *not* on the circle

b)

$$\begin{aligned}(-2+1)^2 + (-2-5)^2 &= 50\\ (-1)^2 + (-7)^2 &= 50\\ 1 + 49 &= 50\end{aligned}$$

(-2, -2) is on the circle

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Click image to the left for more content.

Graphing Circles CK-12

Guided Practice

Find the center and radius of the following circles.

1. $(x-3)^2+(y-1)^2=25$
2. $(x+2)^2+(y-5)^2=49$

3. Find the equation of the circle with center (4, -1) and which passes through (-1, 2).

Answers:

1. Rewrite the equation as $(x-3)^2+(y-1)^2=5^2$. The center is (3, 1) and $r=5$.

2. Rewrite the equation as $(x-(-2))^2+(y-5)^2=7^2$. The center is (-2, 5) and $r=7$.

Keep in mind that, due to the minus signs in the formula, the coordinates of the center have the ***opposite signs*** of what they may initially appear to be.

3. First plug in the center to the standard equation.

$$\begin{aligned}(x-4)^2+(y-(-1))^2&=r^2\\(x-4)^2+(y+1)^2&=r^2\end{aligned}$$

Now, plug in (-1, 2) for x and y and solve for r.

$$\begin{aligned}(-1-4)^2+(2+1)^2&=r^2\\(-5)^2+(3)^2&=r^2\\25+9&=r^2\\34&=r^2\end{aligned}$$

Substituting in 34 for r^2, the equation is $(x-4)^2+(y+1)^2=34$.

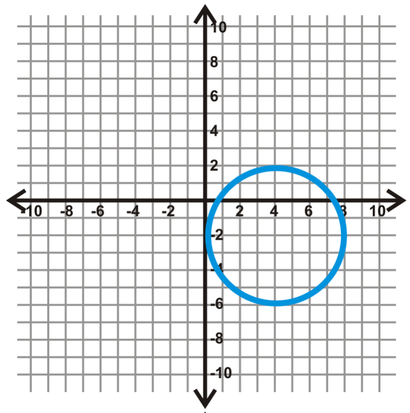
Practice

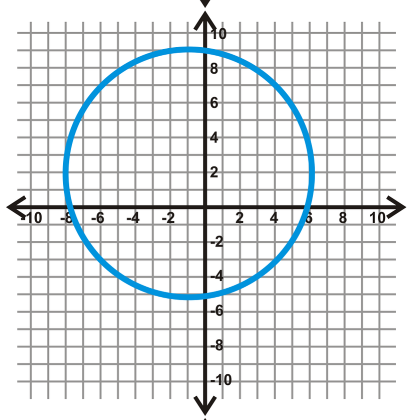
Find the center and radius of each circle. Then, graph each circle.

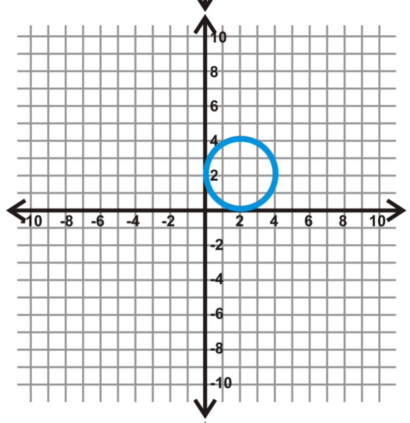
1. $(x+5)^2+(y-3)^2=16$
2. $x^2+(y+8)^2=4$
3. $(x-7)^2+(y-10)^2=20$

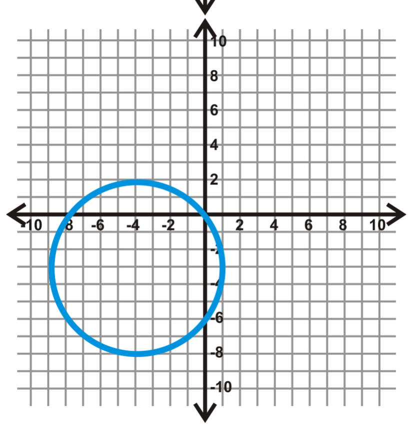
4. $(x+2)^2+y^2=8$

Find the equation of the circles below.

5. 

6. 

7. 

8. 

9. Is (-7, 3) on $(x+1)^2+(y-6)^2=45$?
10. Is (9, -1) on $(x-2)^2+(y-2)^2=60$?

11. Is (-4, -3) on $(x+3)^2+(y-3)^2=37$?
12. Is (5, -3) on $(x+1)^2+(y-6)^2=45$?

Find the equation of the circle with the given center and point on the circle.

13. center: (2, 3), point: (-4, -1)
14. center: (10, 0), point: (5, 2)
15. center: (-3, 8), point: (7, -2)
16. center: (6, -6), point: (-9, 4)

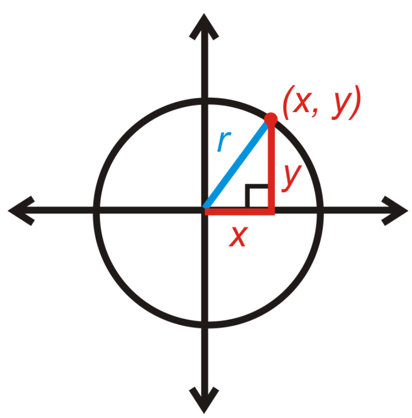
12.8 Extension: Writing and Graphing the Equations of Circles

Learning Objectives

- Graph a circle.
- Find the equation of a circle in the coordinate plane.
- Find the radius and center, given the equation of a circle and vice versa.
- Find the equation of a circle, given the center and a point on the circle.

Graphing a Circle in the Coordinate Plane

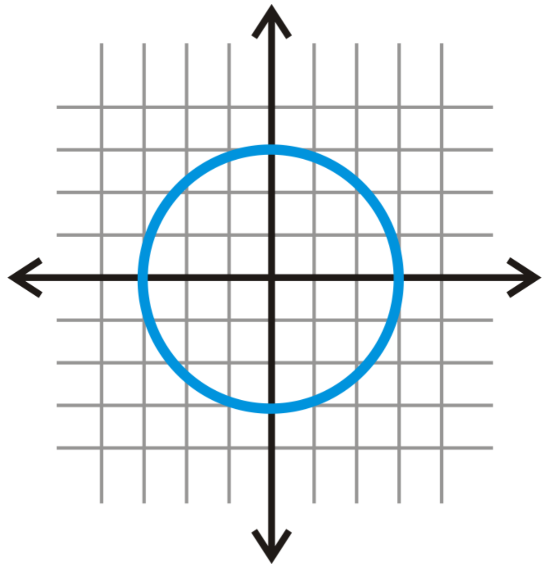
Recall that the definition of a circle is the set of all points that are the same distance from a point, called the center. This definition can be used to find an equation of a circle in the coordinate plane.



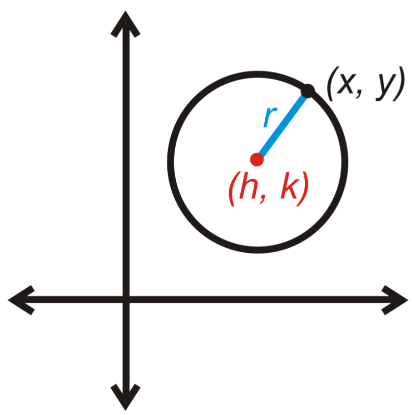
Let's start with the circle centered at the origin, (0, 0). If (x,y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance of the coordinate and y is the vertical distance. Drawing those in, we form a right triangle. Therefore, the equation of a circle, *centered at the origin* is $x^2+y^2=r^2$, by the Pythagorean Theorem.

Example 1: Graph $x^2+y^2=9$.

Solution: This circle is centered at the origin. It's radius is the square root of 9, or 3. The easiest way to graph a circle is to plot the center, and then go out 3 units in every direction and connect them to form a circle.



The center does not always have to be on (0, 0). If it is not, then we label the center (h,k) and would use the distance formula to find the length of the radius.



$$r = \sqrt{(x-h)^2+(y-k)^2}$$

If you square both sides of this equation, then we would have the standard equation of a circle.

Standard Equation of a Circle: The standard equation of a circle with center (h,k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.

Example 2: Find the center and radius of the following circles.

a) $(x-3)^2+(y-1)^2 = 25$

b) $(x+2)^2+(y-5)^2 = 49$

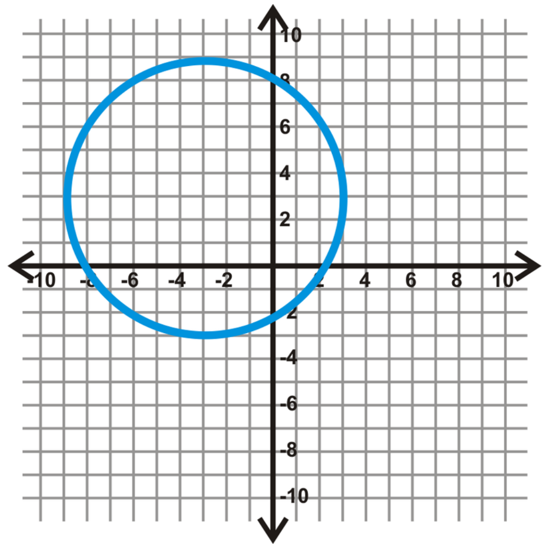
Solution:

a) Rewrite the equation as $(x-3)^2+(y-1)^2 = 5^2$. Therefore, the center is (3, 1) and the radius is 5.

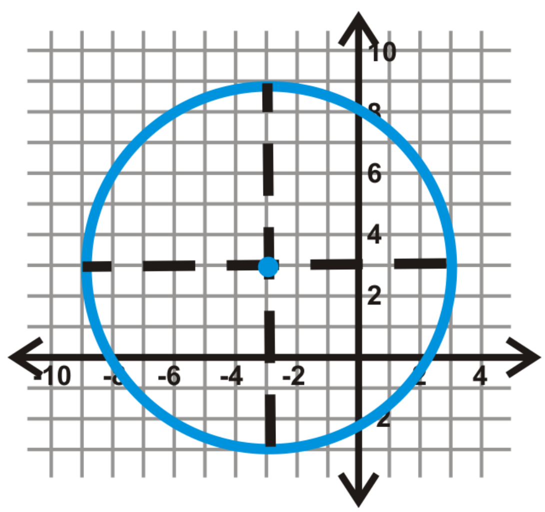
b) Rewrite the equation as $(x-(-2))^2+(y-5)^2 = 7^2$. From this, the center is (-2, 5) and the radius is 7.

When finding the center of a circle always take the ***opposite sign*** of what the value is in the equation.

Example 3: Find the equation of the circle below.



Solution: First locate the center. Draw in a couple diameters. It is easiest to use the horizontal and vertical diameters.



From the intersecting diameters, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find that the radius is 6. Plugging this information into the equation of a circle, we get $(x-(-3))^2+(y-3)^2=6^2$ or $(x+3)^2+(y-3)^2=36$.

Finding the Equation of a Circle

Example 4: Find the equation of the circle with center (4, -1) and passes through (-1, 2).

Solution: To find the equation, first plug in the center to the standard equation.

$(x-4)^2+(y-(-1))^2=r^2$ or $(x-4)^2+(y+1)^2=r^2$

Now, plug in (-1, 2) for x and y and solve for r.

$$\begin{aligned}(-1-4)^2+(2+1)^2 &= r^2\\ (-5)^2+(3)^2 &= r^2\\ 25+9 &= r^2\\ 34 &= r^2\end{aligned}$$

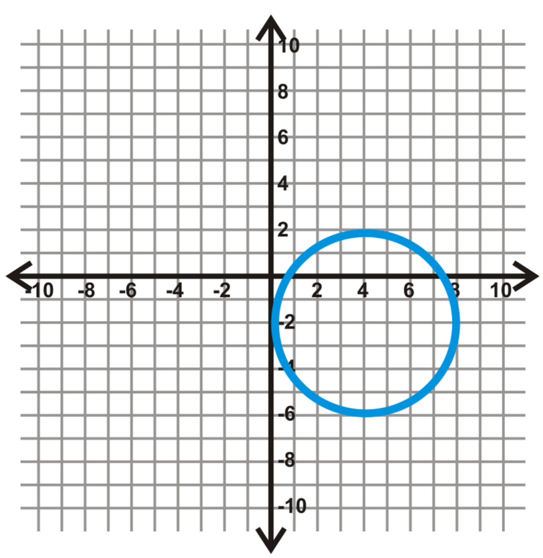
At this point, we don't need to solve for r because r^2 is what is in the equation. Substituting in 34 for r^2, we have $(x-4)^2+(y+1)^2=34$.

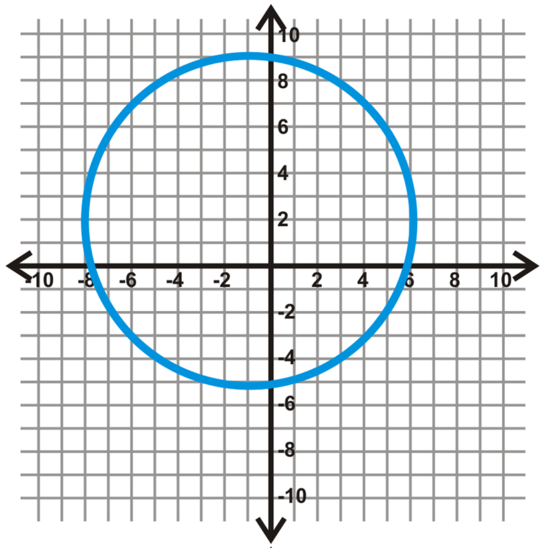
Review Questions

Find the center and radius of each circle. Then, graph each circle.

1. $(x+5)^2+(y-3)^2=16$
2. $x^2+(y+8)^2=4$
3. $(x-7)^2+(y-10)^2=20$
4. $(x+2)^2+y^2=8$

Find the equation of the circles below.

5. 



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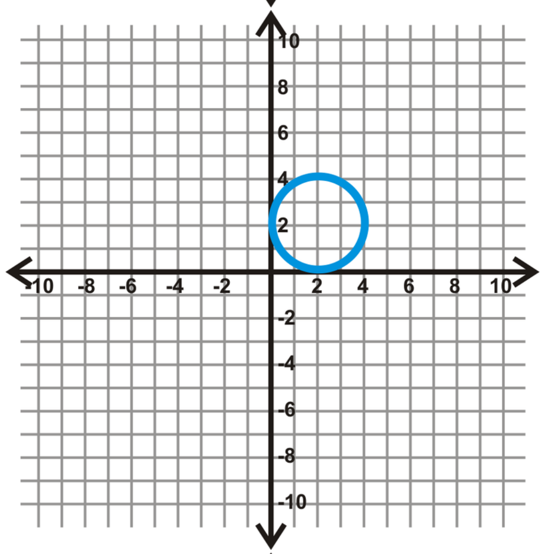

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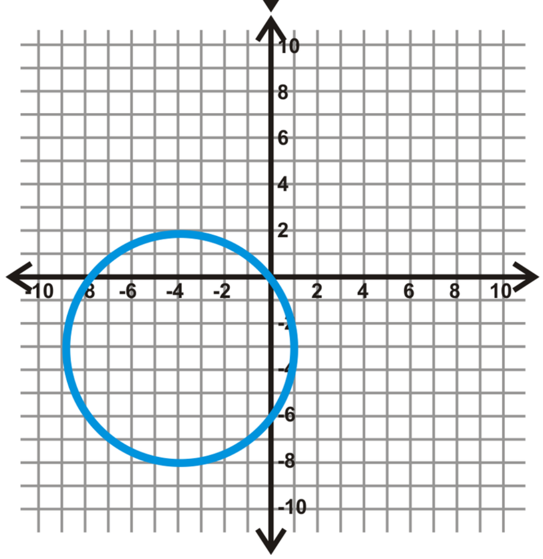


8.

9. Determine if the following points are on $(x+1)^2+(y-6)^2=45$.
   a. (2, 0)
   b. (-3, 4)
   c. (-7, 3)

Find the equation of the circle with the given center and point on the circle.

10. center: (2, 3), point: (-4, -1)
11. center: (10, 0), point: (5, 2)
12. center: (-3, 8), point: (7, -2)
13. center: (6, -6), point: (-9, 4)
14. Now let's find the equation of a circle using three points on the circle. Do you remember how we found the center and radius of a circle given three points on the circle in problem 30 of Section 9-3? We used the fact that the perpendicular bisector of any chord in the circle will pass through the center. By finding the perpendicular bisectors of two different chords and their intersection we can find the center of the circle. Then we can use the distance formula with the center and a point on the circle to find the radius. Finally, we will write the equation. Given the points $A(-12,-21), B(2,27)$ and $C(19,10)$ on the circle (an arc could be drawn through these points from A to C), follow the steps below.
    a. Since the perpendicular bisector passes through the midpoint of a segment we must first find the midpoint between A and B.
    b. Now the perpendicular line must have a slope that is the opposite reciprocal of the slope of $\overleftrightarrow{AB}$. Find the slope of $\overleftrightarrow{AB}$ and then its opposite reciprocal.
    c. Finally, you can write the equation of the perpendicular bisector of $\overline{AB}$ using the point you found in part a and the slope you found in part b.
    d. Repeat steps a-c for chord $\overline{BC}$.
    e. Now that we have the two perpendicular bisectors of the chord we can find their intersection. Solve the system of linear equations to find the center of the circle.
    f. Find the radius of the circle by finding the distance from the center (point found in part *e*) to any of the three given points on the circle.
    g. Now, use the center and radius to write the equation of the circle.

Find the equations of the circles which contain three points in problems 15 and 16.

15. $A(-2,5), B(5,6)$ and $C(6,-1)$
16. $A(-11,-14), B(5,16)$ and $C(12,9)$

12.9 Chapter 12 Review

Keywords Theorems

Circle
The set of all points that are the same distance away from a specific point

Center
The set of all points that are the same distance away from a specific point, called the **center.**

Radius
The distance from the center to the circle.

Chord
A line segment whose endpoints are on a circle.

Diameter
A chord that passes through the center of the circle.

Secant
A line that intersects a circle in two points.

Tangent
A line that intersects a circle in exactly one point.

Point of Tangency
The point where the tangent line touches the circle.

Congruent Circles
Two circles with the same radius, but different centers.

Concentric Circles
When two circles have the same center, but different radii.

Tangent to a Circle Theorem
A line is tangent to a circle if and only if the line is perpendicular to the radius drawn to the point of tangency.

Theorem 9-2
If two tangent segments are drawn from the same external point, then the segments are equal.

Central Angle
The angle formed by two radii of the circle with its vertex at the center of the circle.

Arc
A section of the circle.

Semicircle
An arc that measures 180°.

Minor Arc
An arc that is less than 180°.

Major Arc
An arc that is greater than 180°. ***Always*** use 3 letters to label a major arc.

Congruent Arcs
Two arcs are congruent if their central angles are congruent.

Arc Addition Postulate
The measure of the arc formed by two adjacent arcs is the sum of the measures of the two

Theorem 9-3
In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.

Theorem 9-4
The perpendicular bisector of a chord is also a diameter.

Theorem 9-5
If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.

Theorem 9-6
In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.

Inscribed Angle
An angle with its vertex is the circle and its sides contain chords.

Intercepted Arc
The arc that is on the interior of the inscribed angle and whose endpoints are on the angle.

Inscribed Angle Theorem
The measure of an inscribed angle is half the measure of its intercepted arc.

Theorem 9-8
Inscribed angles that intercept the same arc are congruent.

Theorem 9-9
An angle that intercepts a semicircle is a right angle.

Inscribed Polygon
A polygon where every vertex is on a circle.

Theorem 9-10
A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

Theorem 9-11

The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

Theorem 9-12

The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

Theorem 9-13

The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.

Theorem 9-14

The product of the segments of one chord is equal to the product of segments of the second chord.

Theorem 9-15

If two secants are drawn from a common point outside a circle and the segments are labeled as above, then $a(a+b) = c(c+d)$.

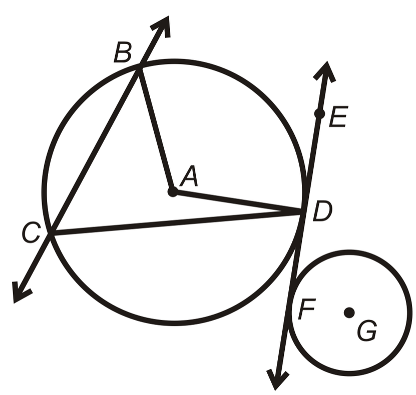
Theorem 9-16

If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture to the left), then $a^2 = b(b+c)$.

Standard Equation of a Circle

The standard equation of a circle with center (h,k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.

Vocabulary



Match the description with the correct label.

1. minor arc - A. $\overline{CD}$
2. chord - B. $\overline{AD}$
3. tangent line - C. $\overleftrightarrow{CB}$
4. central angle - D. $\overleftrightarrow{EF}$
5. secant - E. A
6. radius - F. D
7. inscribed angle - G. $\angle BAD$
8. center - H. $\angle BCD$
9. major arc - I. $\widehat{BD}$
10. point of tangency - J. $\widehat{BCD}$

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9694 .

CHAPTER 13 Probability

Chapter Outline

This chapter introduces the concepts of probability. We will begin looking at sample spaces (the number of ways in which multiple events can occur) using the fundamental counting principle, factorials, permutations, combinations and binomial expansion. Next, we will learn how to calculate probability based on these sample spaces for singular and multiple events. Finally, we will exam the difference between independent, mutually exclusive and dependent events.

13.1 Introduction to Probability

Objective

Develop an understanding of the probability of a particular outcome of an event and to calculate this probability.

Review Queue

1. Simplify $\binom{5}{3}(2x)^{5-3}(-y)^3$.

2. Expand $(m-n)^4$.

3. Find the term containing x^5 in the expansion of $(x-4)^7$.

Finding the Probability of an Event

Objective

Determine the probability of an event with a known sample space.

Guidance

The **probability** of a particular outcome of an event occurring is a measure of how likely the desired outcome is to occur. In this concept we will calculate the probability of an event using the ratio of the number of ways the desired outcome can occur to the number of items in the **sample space**. The sample space is essentially a list of all the possible outcomes. For example, when we roll a single die, the sample space is $\{1,2,3,4,5,6\}$ because these are all the possible outcomes. So, the probability of rolling a 3 is $\frac{1}{6}$ because there is one way to roll a 3 and there are 6 elements in the sample space. We can write a rule for probability when all the outcomes in the sample space are equally likely:

$$P(\text{event}) = \frac{\text{number of desirable outcomes}}{\text{number of outcomes in the sample space}}$$

Example A

What is the probability of rolling a single die and obtaining a prime number?

Solution: In this example there are exactly 3 prime numbers on the die $\{2,3,5\}$ and there are six elements in the sample space $\{1,2,3,4,5,6\}$ so $P(\text{prime}) = \frac{3}{6} = \frac{1}{2}$.

Example B

What is the probability of rolling a four and a three when two dice are rolled? How about a sum of six when two dice are rolled? What is the most likely sum to roll?

Solution: In this case it is useful to make a diagram of the sample space when two dice are rolled.

1,1	2,1	3,1	4,1	5,1	6,1
1,2	2,2	3,2	4,2	5,2	6,2
1,3	2,3	3,3	4,3	5,3	6,3
1,4	2,4	3,4	4,4	5,4	6,4
1,5	2,5	3,5	4,5	5,5	6,5
1,6	2,6	3,6	4,6	5,6	6,6

From this diagram we can see that there are 36 possible outcomes when two dice are rolled.

- To answer the first part of the question, we can observe in the table that there are two ways (shown in red) that can we roll a 4 and a 3 so the probability of rolling a 4 and a 3 is $\frac{2}{36} = \frac{1}{18}$.

- For the second part, there are 5 ways (shown in blue) that a sum of six can be rolled. Therefore the probability of rolling a sum of six is $\frac{6}{36} = \frac{1}{6}$.

More Guidance

The examples above are more accurately described as theoretical probabilities because the theory is that if all the outcomes have an equal likelihood of occurring then the probability is the ratio described about. Does this mean however that when we flip a coin four times we will get 2 heads and 2 tails? Theoretically, this is the most likely outcome, but it is possible that in an experiment we get very different results. When we use the results of an experiment to determine probabilities they are referred to as experimental probabilities. Each flip of the coin Complete the table below to investigate the connection between theoretical and experimental probabilities.

TABLE 13.1:

Number of flips of a coin	Number of Heads	Number of Tails	Probability of flipping a Head	Probability of flipping a Tail
5				
10				
50				
100				
1000*				
Theoretical			$\frac{1}{2}$	$\frac{1}{2}$

* You may wish to use a probability simulator to investigate how many heads and tails are achieved with 1000 flips or combine your results for the 100 flips with 9 other classmates.

Is the experimental probability the same as the theoretical probability? What do you notice as the number of flips increases?

Example C

In a case study of an experimental drug, there were 80 participants. Of the 80 participants, 65 of them experienced no significant side effects from the treatment. What is the probability of a person taking the drug to experience significant side effects? How accurate do you think this probability is? Justify your answer.

Solution: If 65 of the 80 participants did not experience significant side effects, then 15 of them did. So the likelihood of someone in the future experiencing a significant side effect to the drug is $\frac{15}{80} = \frac{3}{16}$. This is experimental probability and as we learned in the investigation, the accuracy of this type of probability will increase as the number of trials in the study increases. Also, as individuals and their general health vary, so will the likelihood of a particular person to experience side effects from a drug vary.

Guided Practice

1. What is the probability of selecting a red chip from a bag containing 10 red chips, 12 blue chips and 15 white chips?

2. What is the probability of rolling doubles when two dice are rolled?

3. Over the course of a month, Sally and Stan recorded how many times their cell phones dropped a call. During this time Sally made 55 phone calls and 4 of them were dropped while Stan made 36 calls and 3 were dropped. What is the probability that Sally's cell phone drops a call? How about Stan's? Who appears to have to more reliable service?

Answers

1. $P(\text{red}) = \frac{10}{10+12+15} = \frac{10}{37}$.

2. There are six ways to roll doubles $(1,1),(2,2)$, etc (refer back to the diagram in Example B). So $P(\text{doubles}) = \frac{6}{36} = \frac{1}{6}$.

3. Sally, $P(\text{dropped call}) = \frac{4}{55} \approx 0.0727$; Stan, $P(\text{dropped call}) = \frac{3}{36} = \frac{1}{12} \approx 0.0833$; Sally appears to have the more reliable service.

Vocabulary

Probability
: The measurement of the likelihood of the desired outcome of an event.

Sample Space
: The set of all the possible outcomes of an event

Theoretical Probability
: The ratio of the number of ways a desired outcome can be achieved to the number of outcomes in the sample space.

Experimental Probability
: The ratio of the number of ways a desired outcome can be achieved to the number of outcomes in the sample space determined by an experiment with multiple trials.

Trial
: Each flip of a coin, roll of the dice or other event which results in one of a set number of possible results.

Problem Set

Determine the following probabilities.

In a standard deck of cards there are 4 suits (two black suits: spades and clubs, and two red suits: hearts and diamonds) and in each suit there are cards numbered 2 through 10, a jack, a queen, a king and an ace. Use this information to answer questions 1-5.

1. What is the probability of randomly drawing a queen?
2. What is the probability of randomly drawing a black card?
3. What is the probability of randomly drawing a face card (jack, queen or king)?
4. What is the probability of randomly drawing a red five?
5. What is the probability of drawing an even numbered card?

Use the table of outcomes for rolling two fair dice (Example B) to answer questions 6-10.

6. What is the probability of rolling a sum greater than 8?
7. What is the probability of rolling doubles?
8. What is the probability of rolling two prime numbers?
9. What is the probability of rolling a sum that is prime?
10. What is the probability of rolling an even sum?

In a bag of goodies at a party there are 8 gum balls, 5 gobstoppers and 10 fireball candies. When children win a party game they get to reach in the bag and pull out a prize. All three candies are spherical and the same size and thus indistinguishable to the touch ensuring a random selection.

11. What is the probability of selecting a gobstopper?
12. What is the probability of selecting a fireball?
13. You are the third person to select a candy from the bag and the first two party goers selected a gum ball and a gobstopper, respectively. What is the probability that you will get a gumball?

Calculating Probabilities of Combined Events

Objective

Calculate the probability of events which require the using of permutations or combinations to determine the number of elements in the sample space and the number of desired outcomes in the sample space.

Guidance

When multiple events occur we can calculate the probability of these combined events by finding their product if the events are independent. **Independent Events** are events for which the outcome of one event does not affect the outcome of a second event. For example, if we roll a die and then roll it again, the outcome of the second roll is independent from the outcome of the first event.

To determine the probability of two independent events, A and B, both occurring, we multiply the probabilities of each of the two events together: $P(A) \times P(B) = P(A \text{ and } B)$.

*Note, the use of set notation will be introduced in the concept "Union and Intersection of Sets".

In some cases, the outcome of one event affects the outcome of a second event. For example, when a hand of cards is dealt in a game of poker, the probability of receiving a particular card changes based on what cards have already been dealt. This is an example of **Conditional Probability**. We will introduce conditional probability here for situations in which we can manipulate the subsequent probabilities to make independent events as shown in Example C.

Example A

Given a fair die and a fair coin, find the following probabilities.

1. rolling a 5 and getting tails.
2. rolling an odd number and getting heads.

Solution: Since the outcome of rolling the die does not affect the outcome of flipping the coin, these are independent events. Thus we can determine their individual probabilities and multiply them together.

1. $P(5) \times P(T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.
2. $P(1,3,5) \times P(H) = \frac{3}{6} \times \frac{1}{2} = \frac{3}{12} = \frac{1}{4}$

Example B

What is the probability of rolling doubles twice in a row? Three times in a row?

Solution: Each of these rolls are independent events. It is human nature to think that just because we have rolled doubles once or twice already that we are unlikely to roll them another time. It is true that that probability of rolling doubles three times in a row is smaller than rolling doubles once but this is not because the probability changes for each roll. Let's look at why this occurs:

$P(\text{doubles}) = \frac{6}{36} = \frac{1}{6}$, since there are six ways to roll doubles.

$P(\text{doubles twice}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$P(\text{doubles three times}) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

Example C

What is the probability that you draw an ace from a deck of cards three times if each card is replaced before the next card is drawn? What if each card is not replaced?

Solution: There are 4 aces in a deck of cards, so there is a $\frac{4}{52} = \frac{1}{13}$ chance of drawing an ace each time a card is selected. For the first part of the question which requires each card chosen to be replaced, the probability of selecting an ace does not chance so the events are independent of one another.

$P(\text{three aces, with replacement}) = \left(\frac{1}{13}\right)^3 = \frac{1}{2197}$.

The second part of the question does not require replacement. Now the events are not strictly independent. We can, however, determine the probability of an independent event and use multiplication to find the probability of the combined events. For the first selection, there are 4 aces in the 52 card deck. After an ace is selected, how many cards remain? Well, to determine the probability of selecting three aces, we must assume the first selected was an ace so now there are 3 aces remaining in a deck of 51 cards. After the second ace is selected, there are 2 aces in a deck of 50 cards. Now we can find the product of these probabilities.

$P(\text{three aces, without replacement}) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{1}{13} \times \frac{1}{17} \times \frac{1}{25} = \frac{1}{5525}$.

Notice that the probability of selecting an ace diminishes with each selection in this situation because the number of aces in the deck is being reduced.

Guided Practice

Determine the following probabilities.

1. Rolling snake eyes (double ones) and then rolling a sum of seven on a pair of dice.

2. Turning over a face card (jack, queen or king) followed by an ace from a full, shuffled deck of cards.

3. Drawing a hand of five cards which contains exactly 2 jacks, from a full, shuffled deck of cards.

4. Randomly selecting a pair of black socks followed by a pair of navy blue socks and then a pair of white socks from a drawer containing 5 black pairs, 4 navy blue pairs and 8 white pairs if each selection is replaced before the next pair is chosen. What if each pair is not replaced?

Answers

1. $P(\text{snake eyes}) \times P(\text{seven}) = \frac{1}{36} \times \frac{1}{6} = \frac{1}{216}$.

2. $P(\text{jack, queen, king}) \times P(\text{ace}) = \frac{12}{52} \times \frac{4}{51} = \frac{4}{221}$

3. $P(\text{jack}) \times P(\text{jack}) \times P(\text{non jack}) \times P(\text{non jack}) \times P(\text{non jack}) = \frac{4}{52} \times \frac{3}{51} \times \frac{48}{50} \times \frac{47}{49} \times \frac{46}{48} = \frac{1081}{270725} \approx 0.00399298$

We aren't quite finished, however, because what we have determined here is the probability that the first two cards are the jacks and the last three are the non jacks. These cards could have been dealt in any order so we need to determine the number of permutations of these cards and multiply by that value. Keep in mind that the jacks are "non distinguishable" and the non jacks are "non distinguishable". The permutations are: $\frac{5!}{2!3!} = 10$. This could also be described as the number of combinations of selecting two jacks from a set of five cards: ${}_5C_2 = \frac{5!}{2!(5-2)!} = 10$. These are just two ways to find the same result. Now that we have the number of combinations or arrangements, we can multiple our probability by this value:

$\frac{1081}{270725} \times 10 \approx 0.0399298181$.

4. For the first part of the question, the total number of socks in the drawer remains the same for each selection, 17. So, $P(\text{black, then blue, then white}) = \frac{5}{17} \times \frac{4}{17} \times \frac{8}{17} = \frac{160}{4913} \approx 0.032567$. Now, if we do not replace the socks, the number of socks decreases with each sock selected: $P(\text{black, then blue, then white}) = \frac{5}{17} \times \frac{4}{16} \times \frac{8}{15} = \frac{160}{4080} = \frac{2}{51} \approx 0.039216$. This probability is slightly higher because removing one pair of black socks makes it more likely that we will select a different color pair of socks for the next pair and so on.

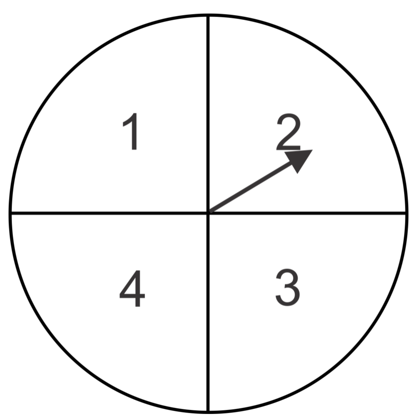
Vocabulary

Independent Events
: Multiple events for which the outcome of one event does not affect the outcome of a second or subsequent event. The probability of multiple independent events occurring is the product of the probabilities of the individual events.

Conditional Events
: Events for which the outcome of one event is influenced by the outcome of a previous event. In this concept, we can manipulate the probabilities of the subsequent events to reflect the desired outcome of previous events and thus create pseudo-independent events.

Problem Set

Given a spinner as shown in the illustration below, two fair six sided dice and a standard deck of 52 playing cards, calculate the probability of each compound event below.



1. Spin a 4 and roll doubles.
2. Spin an odd number and roll an odd sum.
3. Draw four red cards in a row without replacement.
4. Draw three face cards (jack, queen, king) without replacement.
5. Rolling a sum of eight and spinning a 2.
6. Spin a three, three times in a row.
7. Roll an even sum and spin a prime number.
8. Draw a five card hand containing exactly 3 red and 2 black cards.
9. Draw a five card flush (a hand in which all cards are the same suit).
10. Draw a hand of five cards containing exactly two hearts.
11. Draw a hand of three cards that contains at least one spade.
12. Roll a sum of 7 or 11 and draw three cards in which at least one is a face card.

Tree Diagrams and Probability Distributions

Objective

Use Tree Diagrams, combinations or permutations to determine the probabilities of multiple events and probability distributions.

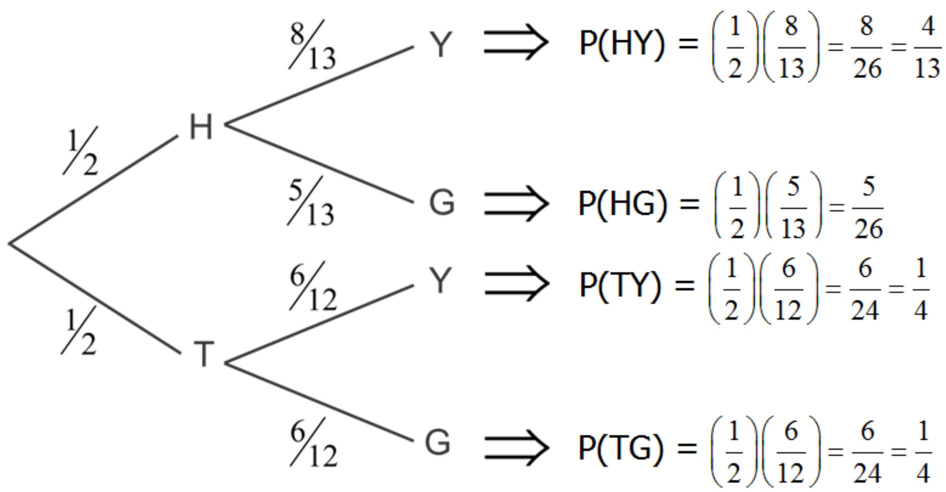
Guidance

Sometimes it is useful to create a tree diagram to illustrate the possible outcomes of multiple events and their individual probabilities, calculate the probabilities of the combined events and the sample space. In other cases, we might use combinations or permutations to create a Probability Distribution table. A **Probability Distribution** is a table which includes all possible outcomes (sample space) and their respective probabilities.

Example A

A game of chance involves flipping a coin and selecting a chip from one of two urns. If the coin toss results in heads, then you select from urn A which contains 8 yellow chips and 5 green chips. If the coin toss results in tails, then you select from urn B which contains 6 yellow chips and 6 green chips. Use this information to create a tree diagram illustrating the possible outcomes and their probabilities and then determine the probability of selecting a yellow chip.

Solution: First, we need to make a tree diagram. The first branches of the diagram show the coin toss results and the second sets of branches show the chip selection results. Notice that each set of branches in the tree diagram has probabilities which sum to 1. This happens because one of the outcomes must occur. In other words, you either select a yellow or a green chip (there is no other outcome in the sample space) so the sum of the probabilities will be 1. These types of events are called **Complimentary Events**.



By multiplying "across" the branches, we can determine the probabilities of the combined events. Now, look at the sum of the probabilities on the far right: $\frac{4}{13}+\frac{5}{26}+\frac{1}{4}+\frac{1}{4}=1$. The entire sample space is shown here so the sum of the probabilities of all the possible outcomes should be 1. This is an excellent way to check for accuracy in your tree diagram calculations.

Now, to answer the question: What is the probability of selecting a yellow chip? Looking at the diagram, there are two ways to select a yellow chip. One, we could toss the coin and get heads and then select a yellow chip from urn A and this probability is $\frac{4}{13}$. Two, we could toss the coin and get tails and then select a yellow chip and this probability is $\frac{1}{4}$. We can add the probabilities of these two "paths" to the same end result and get $\frac{4}{13}+\frac{1}{4}=\frac{29}{52}\approx 0.5577$.

Example B

In a box of 20 candies, there are 8 which contain nuts. If 5 pieces are randomly selected and consumed, create a probability distribution table to show the probability of selecting 0, 1, 2, 3, 4, or 5 candies which contain nuts in the sample.

Solution: First, let's create a formula for determining the probability of each of the outcomes. We can use combinations to help us do this. First, how many ways are there to select 5 pieces of candy from a box of 20 pieces? This is a combination, so ${}_{20}C_5$ or $\binom{20}{5}$. This value will be the total number of possible outcomes and thus the denominator of our probability ratio. Now, how many ways are there to select 0 of the 8 candies with nut and 5 of the 12 candies without nuts? Again, we have combinations and their product can be found for the numerator of our probability ratio: $\binom{8}{0}\binom{12}{5}$. Now we can put it all together and find the probability of selecting 0 candies with nuts: $\frac{\binom{8}{0}\binom{12}{5}}{\binom{20}{5}} = \frac{33}{646} \approx 0.05108$.

Similarly, for 1 candy containing nuts: $\frac{\binom{8}{1}\binom{12}{4}}{\binom{20}{5}} = \frac{165}{646} \approx 0.25542$.

For 2 candies with nuts we get: $\frac{\binom{8}{2}\binom{12}{3}}{\binom{20}{5}} = \frac{385}{969} \approx 0.39732$, and so on.

The table to the right shows all of the final probabilities for each outcome in the sample space. This is called a Probability Distribution Table.

What happens if we add up all of the probabilities in this table?

$0.05108 + 0.25542 + 0.39732 + 0.23839 + 0.05418 + 0.00361 = 1$

This means that the probability of getting one of these outcomes is 100%. Also, this shows that our probability distribution is correct because we have included all of the possible outcomes and the sum of their probabilities is 1. In other words, this illustrates that there are no other possible outcomes since there is a 100% chance of getting one of these results.

TABLE 13.2:

Number of Candies Selected Containing Nuts	Probability
0	$\frac{33}{646} \approx 0.05108$
1	$\frac{165}{646} \approx 0.25542$
2	$\frac{385}{969} \approx 0.39732$
3	$\frac{77}{323} \approx 0.23839$
4	$\frac{35}{646} \approx 0.05418$
5	$\frac{7}{1938} \approx 0.00361$

Example C

Over time, Ronald has shown that in 2 of 5 attempts he makes a bulls eye with a bow and arrow. Create a probability distribution table which shows the possible outcomes and the associated probabilities when Ronald shoots three arrows.

Solution: First, we will consider each bulls eye a success and each non bulls eye a failure. So, the probability of a success is $\frac{2}{5}$ and the probability of a failure is $\frac{3}{5}$. The probability of zero successes is $\left(\frac{3}{5}\right)^3 = \frac{27}{125} = 0.216$.

Similarly, the probability of three successes is $\left(\frac{2}{5}\right)^3 = \frac{8}{125} = 0.064$. With three shots, there are two other possibilities to consider. Ronald could also have one success and two failures or two successes and one failure. In these cases, we must consider that any one or two of the shots could be successes so we will multiply by the number of "combinations" that are possible. For example, if Ronald has one success, then there are $\binom{3}{1}$ or 3 ways this could occur: SFF, FSF or FFS. So we will multiple the combinations by the probability of one success and two failures: $\binom{3}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^2 = \frac{54}{125} = 0.432$. For two successes and one failure we have $\binom{3}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^1 = \frac{36}{125} = 0.288$. Now, make a probability distribution table:

TABLE 13.3:

Number of Bulls Eyes	Probability
0	0.216
1	0.432
2	0.288
3	0.064

This is an example of a special type of probability called a Binomial Probability because its rule resembles the Binomial Theorem. In order for a problem to be a binomial probability it must consist of multiple independent trials, called Bernoulli Trials, in which there is either a success or a failure. In other words, $P(\text{success}) + P(\text{failure}) = 1$ and the result of each trial is independent of the result of a previous trial.

If we let $n =$ number of trials, $p =$ probability of a success and $r =$ number of successes, we can use the following formula to determine the probability of any number of successes.

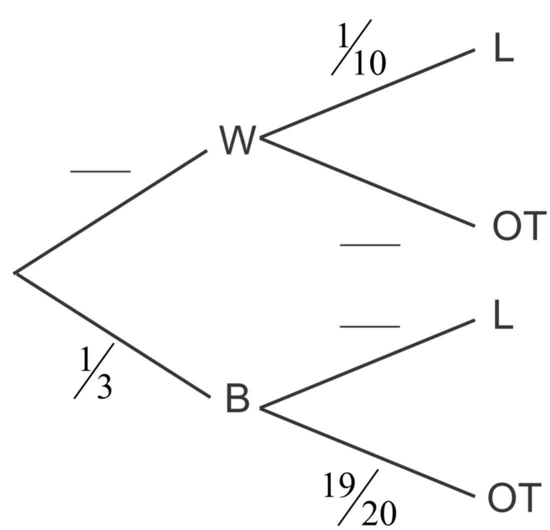
$$P(r \text{ successes}) = \binom{n}{r}(p)^r(1-p)^{n-r}.$$

Notice that this formula is exactly what we did to find the probability of Ronald shooting two bulls eyes:

For two bulls eyes, $n = 3$, $p = \frac{2}{5}$, and $r = 2$: $\binom{3}{2}\left(\frac{2}{5}\right)^2\left(1-\frac{2}{5}\right)^{3-2} = \binom{3}{2}\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^1 = \frac{36}{125} = 0.288$.

Guided Practice

1. Sarah either walks or rides the bus to school. When she walks she is more likely to be late to school than when she rides the bus. Complete the tree diagram and find the probability that Sarah is late for school.

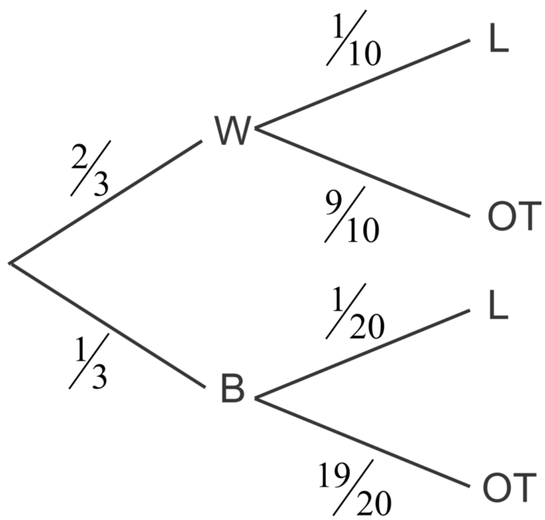


2. In a case of 15 light bulbs there are 2 defective bulbs. Create a probability distribution which illustrates the possible outcomes and their respective probabilities if we randomly select 3 bulbs from the box. Show that the sum of the probabilities is 1.

3. On any given workday, there is a 15% chance that Professor Calculus will cause an explosion in his laboratory. Use the Binomial Probability formula to determine the probability that Professor Calculus will cause less than three explosions in a five day work week.

Answers

1.



To find the probability that Sarah is late we need to add the probabilities of the two different ways she can be late. She can walk and be late or she can ride the bus and be late:

$P(\text{walk and late}) + P(\text{bus and late}) = \left(\frac{2}{3}\right)\left(\frac{1}{10}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{20}\right) = \frac{1}{12} \approx 0.08333$

2.

TABLE 13.4:

Number of defectiteve bulbs	Probability
0	$\frac{\binom{2}{0}\binom{13}{3}}{\binom{15}{3}} = 0.628571$
1	$\frac{\binom{2}{1}\binom{13}{2}}{\binom{15}{3}} = 0.342857$
2	$\frac{\binom{2}{2}\binom{13}{1}}{\binom{15}{3}} = 0.028571$

$0.62857 + 0.342857 + 0.028571 = 0.999998$

Since we rounded the individual probabilities, the sum may reflect this inaccuracy. For all intents and purposes, this answer is 1.

Note that it is not possible to select a sample containing 3 defective bulbs because there are only 2 defective bulbs in the box.

3. Less than three explosions implies that Professor Calculus could cause 0, 1 or 2 explosions in the work week. We can add these individual probabilities to determine the probability. For the Binomial Probability, $n = 5$ (since there are 5 days in the work week), $p = 0.15$ and r takes on the values 0, 1 and 2.

$$\binom{5}{0}(0.15)^0(0.85)^5+\binom{5}{1}(0.15)^1(0.85)^4+\binom{5}{2}(0.15)^2(0.85)^3 = 0.443705+0.391505+0.138178 = 0.973388$$

Vocabulary

Probability Distribution
: A summary, usually a table, of the possible outcomes of an experiment or series of events and all the corresponding probabilities.

Complimentary Events
: A pair of events such that if one event does not occur the other one will. As a result, the sum of their probabilities must be equal to 1.

Problem Set

Jamie and Olivia are best friends and neighbors. As such they often eat dinner together at one of their houses. About 30% of the time, they eat at Jamie's house and her mother makes a non vegetarian meal 65% of the time. The rest of the time, they eat at Olivia's house and her mother serves a vegetarian meal 55% of the time.

1. Make a tree diagram, including the appropriate probabilities to illustrate this scenario.
2. What is the probability of going to Olivia's house and eating a meal containing meat?
3. What is the probability that the girls will eat a vegetarian dinner?

Tommy has 20 water balloons in bucket. There is a $\frac{1}{8}$ chance of each balloon exploding on him before he even has a chance to throw it. Use a Binomial Probability Distribution to answer the following questions.

4. What is the probability that Tommy throws eight without any breaking on him?
5. What is the probability that exactly five of the eight balloons he attempts to throw explode on him?
6. What is the probability that more than two of the eight explode on him?

A vaccine has a 92% success rate. The vaccine is given to 50 patients in a medical practice. Use a Binomial Probability Distribution to answer the following questions.

7. What is the probability that it works for all patients?
8. What is the probability that is fails for exactly 9 patients?
9. What is the probability that it fails for more than 1 patient?

Five cards are randomly selected from a deck of cards.

10. Create a probability distribution table for the number of "high" cards (jack, queen, king or ace) in a 5 card hand chosen at random.
11. What is the probability of getting at least one high card?
12. What is the probability of getting at least two?

Basic Geometric Probabilities

Objective

Determine the probabilities of events occurring based on Geometric properties.

Guidance

Sometimes we need to use our knowledge of geometry to determine the likelihood of an event occurring. We may use areas, volumes, angles, polygons or circles.

Example A

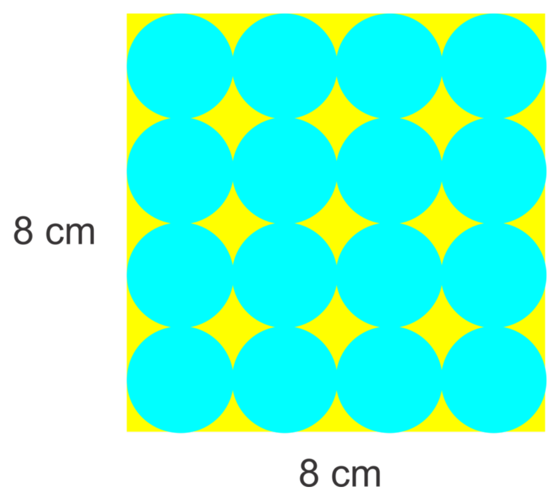
A game of pin-the-tale-on-the-donkey has a rectangular poster that is 2ft by 2ft. The area in which the tale should be pinned is shown as a circle with radius 1 inch. Assuming that the pinning of the tale is completely random and that it will be pinned on the poster (or the player gets another try), find the probability of pinning the tale in the circle?

Solution: This probability can be found by dividing the area of the circle target by the area of the poster. We must have the same units of measure for each area so we will convert the feet to inches.

$$\frac{1^2\pi}{24^2} \approx 0.005454 \textit{ or } \text{about } 0.5\% \text{ chance.}$$

Example B

In a game of chance, a pebble is dropped onto the board shown below. If the radius of each of blue circle is 1 cm, find the probability that the pebble will land in a blue circle.



Solution: The area of the square is 16 cm^2. The area of each of the 16 circles is $1^2\pi = \pi$. The probability of the pebble landing in a circle is the sum of the areas of the circles divided by the area of the square.

$P(\text{blue circle}) = \frac{16\pi}{64} \approx 0.785$

Example C

What is the probability that a randomly thrown dart will land in a red area on the dart board shown? What is the probability that exactly two of three shots will land in the red? The radius of the inner circle is 1 unit and the radius of each annulus is 1 unit as well.

Solution: First we need to determine the probability of landing in the red. There are four rings of width 1 and the radius of the center circle is also 1 so the total radius is 5 units. The area of the whole target is thus 25π square units. Now, we need to find the areas of the two red rings and the red circular center. The center circle area is π square units. The outside ring area can be found by subtracting the area inside from the entire circle's area. The inside circle will have a radius of 4 units, the area of the outer ring is $25\pi - 16\pi = 9\pi$ square units. This smaller red ring's area can be found similarly. The circle with this red ring on the outside has a radius of 3 and the circle inside has a radius of 2 so, $9\pi - 4\pi = 5\pi$ square units. Finally, we can add them together to get the total red area and divide by the area of the entire target. $\frac{9\pi+5\pi+\pi}{25\pi} = \frac{15\pi}{25\pi} = \frac{3}{5}$. So the probability of hitting the red area is $\frac{3}{5}$ or 60%.

For the second part of the problem we will use a binomial probability. There are 3 trials, 2 successes and the probability of a success is 0.6: $\binom{3}{2}(0.6)^2(0.4) = 0.432$

Guided Practice

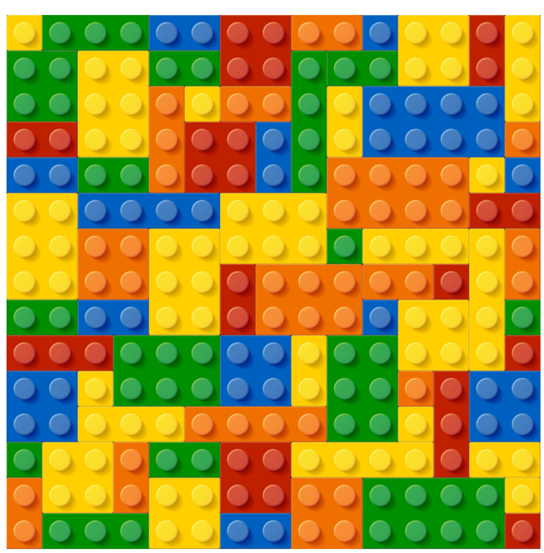
1. Consider the picture below. If a "circle" is randomly chosen, what is the probability that it will be:

a. red

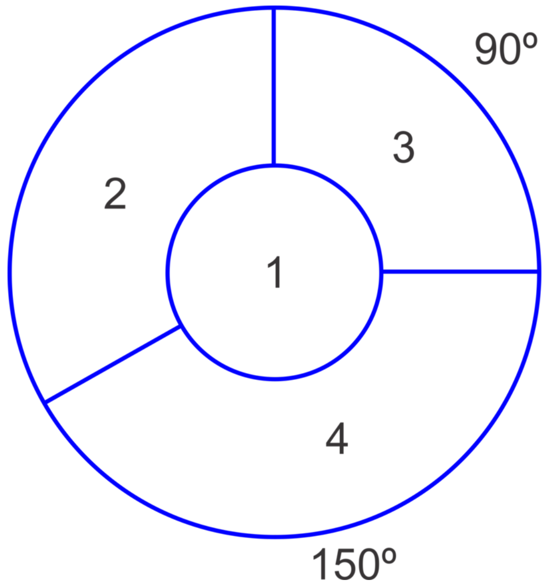
b. yellow

c. blue or green

d. not orange



2. If a dart is randomly thrown at the target below, find the probability of the dart hitting in each of the regions and show that the sum of these probabilities is 1. The diameter of the center circle is 4 cm and the diameter of the outer circle is 10 cm. What is the probability that in 5 shots, at least two will land in the 4 region?



Answers

1. a. $\frac{29}{225}$

b. $\frac{69}{225}$

c. $\frac{84}{225}$

d. $\frac{182}{225}$

2.

$$\begin{aligned}
P(1) &= \frac{2^2\pi}{5^2\pi} = \frac{4}{25} \\
P(2) &= \frac{120}{360}\left(\frac{5^2\pi - 2^2\pi}{5^2\pi}\right) = \frac{1}{3} \times \frac{21\pi}{25\pi} = \frac{7}{25} \\
P(3) &= \frac{90}{360}\left(\frac{5^2\pi - 2^2\pi}{5^2\pi}\right) = \frac{1}{4} \times \frac{21\pi}{25\pi} = \frac{21}{100} \\
P(4) &= \frac{150}{360}\left(\frac{5^2\pi - 2^2\pi}{5^2\pi}\right) = \frac{5}{12} \times \frac{21\pi}{25\pi} = \frac{35}{100}
\end{aligned}$$

Now add them up:

$$\begin{aligned}
&P(1) + P(2) + P(3) + P(4) = \\
&= \frac{4}{25} + \frac{7}{25} + \frac{21}{100} + \frac{35}{100} \\
&= \frac{16}{100} + \frac{28}{100} + \frac{21}{100} + \frac{35}{100} \\
&= \frac{100}{100} = 1
\end{aligned}$$

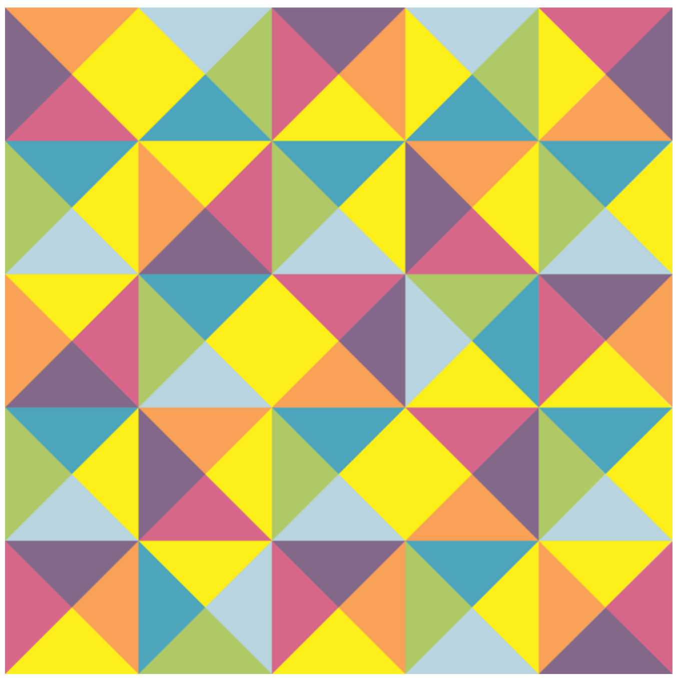
The probability of landing in region 4 at least twice in five shots is equivalent to $1 - [P(0) + P(1)]$.

Use binomial probability to determine these probabilities:

$$1-\left[\binom{5}{0}\left(\frac{35}{100}\right)^0\left(\frac{65}{100}\right)^5+\binom{5}{1}\left(\frac{35}{100}\right)^1\left(\frac{65}{100}\right)4\right]$$
$$=1-(0.116029+0.062477)$$
$$\approx 0.821$$

Problem Set

Use the diagram below to find the probability that a randomly dropped object would land in a triangle of a particular color.



1. yellow
2. green
3. plum
4. not yellow
5. not yellow or light blue

The dart board to the right has a red center circle (bull's eye) with area $\pi\ cm^2$. Each ring surrounding this bull's eye has a width of 2 cm. Use this information to answer the following questions.

6. Given a random throw of a dart, what is the probability that it will land in a white ring?
7. What is the probability of a bull's eye?
8. What is the probability that in 10 throws, exactly 6 land in the black regions?
9. What is the probability that in 10 throws, at least one will land in the bull's eye?
10. How many darts must be thrown to have a 95% chance of making a bull's eye?

13.2 Permutations and Combinations

Objective

To define and use factorials to determine the numbers of permutations or combinations that can be made of multiple items.

Review Queue

1. The summer reading requirement for an English course is to read one of 6 novels, one of 3 biographies and one of 4 other nonfiction selections. How many different assignments might be completed?

2. How many six-digit identification numbers can be formed if digits may be repeated but none can start with zero?

3. How many ways can seven DVD's be arranged on a shelf?

Define and Apply Permutations and Factorials

Objective

To define and use factorials to determine the number of permutations or arrangements of objects.

Guidance

The number of **permutations** of objects is the number of possible arrangements of the objects. Consider question three in the review queue: How many ways can seven DVD's be arranged on a shelf? This is an example of a permutation. We used the Fundamental Counting Principle without repetition to determine the permutations of DVD's.

Example A

How many ways can 5 students sit in a row?

Solution: If we consider the students sitting in one of five seats, then we have 5 students to choose from for the first seat, four remaining to choose from for the second seat and so on until all the seats are filled.

$\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 120$ So there are 120 ways to seat the students.

More Guidance

The way we just wrote out $\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$ can also be expressed as a **factorial**. A factorial is the product of a number with each number less than itself. We use the notation, 5!, which is read as "five factorial" to represent the expression $\underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$. It is important to note that $0! = 1! = 1$. Students are often perplexed that both zero and one factorial are equal to one but think back to the context for illustration. If you want to arrange zero items, how many ways can you do it? If you want to arrange one item, how many ways can you do it? There is only one way to "arrange" zero or one item.

To evaluate a factorial on the TI-83 or 84 graphing calculator, type in the number, then press MATH→NUM, 4!. Press ENTER to evaluate.

Example B

Evaluate $\frac{10!}{6!}$

Solution: We should expand the numerator and the denominator to see which common factors the numerator and denominator that we can cancel out to simplify the expression.

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!}} = 10 \times 9 \times 8 \times 7 = 5040$$

Example C

On a shelf there are 6 different math books, 4 different science books and 8 novels. How many ways can the books be arranged if the groupings are maintained (meaning all the math books are together, the science books are together and the novels are together).

Solution: There are 6 math books so if we think of filling 6 slots with the six books, then we start with 6 books for the first slot, then 5, then 4, etc: $\underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 720$ ways.

There are 4 science books so we can arrange them in $\underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 24$ ways.

There are 8 novels so we can arrange them in $\underline{8} \times \underline{7} \times \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 40,320$ ways.

Now, if each type of book can be arranged in so many ways and there are three types of books which can be displayed in $\underline{3} \times \underline{2} \times \underline{1} = 6$ ways, then there are:

$720 \times 24 \times 40320 \times 6 = 4,180,377,600$ total ways to arrange the books.

Guided Practice

Evaluate the following expressions with factorials.

1. $\frac{12!}{9!}$

2. $\frac{4 \times 8!}{3!5!}$

3. How many ways can nine children line up?

4. How many ways can 3 cookbooks, 5 textbooks, 7 novels and 4 nonfiction books be arranged on a shelf if the groupings are maintained?

Answers

1. $\frac{12 \times 11 \times 10 \times \cancel{9!}}{\cancel{9!}} = 1320$

2. $\frac{4 \times 8 \times 7 \times 6 \times \cancel{5!}}{3 \times 2 \times 1 \times \cancel{5!}} = \frac{4 \times 8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1} = 224$

3. $9! = 362,880$

4. $3! \times 5! \times 7! \times 4! \times 4! = 2,090,188,800$

Vocabulary

Permutations
The number of ways items in a set can be arranged or ordered.

Factorial
Operation in which a number is multiplied by each positive number less than itself.

Problem Set

Evaluate the following factorial expressions.

1. $\frac{5!}{2!3!}$
2. $\frac{10!}{2!7!}$
3. $\frac{4!8!}{9!}$
4. $\frac{5!10!}{12!}$

5. How many ways can a baseball team manager arrange nine players in a lineup?
6. How many ways can the letters in the word FACTOR be arranged?
7. How many ways can 12 school buses line up?
8. How many ways can eight girls sit together in a row?
9. If the two of the eight girls in problem eight must sit together, how many ways can the 8 girls be arranged in the row such that the two girls sit together?
10. How many ways can seven diners sit around a circular table. (Hint: It is not 7!, consider how a circular seating arrangement is different than a linear arrangement.)
11. How many ways can three cookbooks, four novels and two nonfiction books be arranged on a shelf if the groupings are maintained?
12. How many ways can two teachers, four male students, five female students and one administrator be arranged if the teachers must sit together, the male students must sit together and the female students must sit together?

Permutations of Subsets and Permutations with Repetition

Objective

To calculate the permutations of a subset of items and to calculate permutations of a set with indistinguishable objects, i.e. two or more objects are identical.

Guidance

Sometimes we want to order a select subset of a group. For example, suppose we go to an ice cream shop that offers 15 flavors. If we want to layer 3 scoops of different flavors on an ice cream cone, how many arrangements are possible? Here, the order matters so a chocolate, strawberry and vanilla cone is different than a strawberry, vanilla and chocolate cone. This is an example of permutations of a subset. We don't need to know how many ways we can order all 15 flavors, just three choices. You have actually solved problems like this already using the Fundamental Counting Principle. There are 15 choices for scoop one, 14 choices for scoop two and 13 choices for scoop 3, so $15 \times 14 \times 13 = 2730$.

We can use factorials to solve this as well. Consider the expression: $\frac{15 \times 14 \times 13 \times 12!}{12!} = \frac{15!}{(15-3)!} = \frac{n!}{(n-r)!}$, where n represents the total number of elements in the set and r represents the number of elements in the subset we are selecting. Mathematically, this can be written using the notation ${}_{15}P_3$ or ${}_nP_r$. To summarize, if we wish to find the number of permutations of r elements selected from a larger set containing n elements, we can use the formula: ${}_nP_r = \frac{n!}{(n-r)!}$.

* Note that some textbooks use the notation P^n_r to represent ${}_nP_r$.

We can evaluate this expression easily on the calculator as well. First, enter the value of $n(15)$, then go to MATH$\rightarrow$PRB, select 2: ${}_nP_r$. Now enter the value of $r(3)$ to get the expression 15 ${}_nP_r$ 3 on your screen. Press ENTER one more time and the result is 2730.

Example A

How many ways can a President, Vice President, Secretary and Treasurer be selecting from a club with ten members?

Solution: In the selection process here, the order matters so we are calculating the number of permutations of a subset of 4 members of the 10 member club. So,

$${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!} = 10 \times 9 \times 8 \times 7 = 5040.$$

Example B

Consider the word, VIRGINIA. How many unique ways can these letters be arranged?

Solution: There are eight letters which can be arranged 8! ways. However, some of these arrangements will not be unique because there are multiple I's in the word VIRGINIA. For example, if we let the three I's be different colors, then we can see that there are several indistinguishable ways the I's can be arranged.

$$\text{VIRGINIA, VIRGINIA, VIRGINIA, VIRGINIA, VIRGINIA, VIRGINIA}$$

In fact, there are 3! or 6 ways that the three I's can be arranged that are indistinguishable when the arrangement of the remaining letters is constant. To figure out the number of unique arrangements of the 8 letters with 3 that are indistinguishable we can find the permutations of the 8 letters and divide by the permutations of the indistinguishable items.

$$\frac{8!}{3!} = \frac{8\times 7\times 6\times 5\times 4\times 3!}{3!} = 8\times 7\times 6\times 5\times 4 = 6720$$

Example C

Consider the word PEPPERS. How many unique arran gements can be made of these letters?

Solution: There are seven letters in total, three of which are P and two of which are E. We can expand upon what we did in the last example and divide by the number of ways each of these letters can be arranged.

$$\frac{7!}{3!2!} = \frac{7\times 6\times 5\times 4\times 3!}{3!\times 2\times 1} = \frac{7\times 6\times 5\times (2\times \cancel{2})}{\cancel{2}} = 7\times 6\times 5\times 2 = 420$$

More Guidance

We can generalize the rule used in Examples B and C as follows. In a set with n elements, in which $n_1, n_2, n_3, \ldots$ are indistinguishable, we can find the number of unique permutations of the n elements using the formula:

$$\frac{n!}{n_1!\times n_2!\times n_3!\times \ldots}$$

Guided Practice

1. Find ${}_{10}P_6$

2. On a team of 12 players, how many ways can the coach select players to receive on of each of the following awards (one award per player): most valuable player, best sportsmanship, most improved player.

3. How many ways can 5 yellow, 4 red and 3 green balls be arranged in a row?

Answers

1. ${}_{10}P_6 = \frac{10!}{(10-6)!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4!}{4!} = 151,200$

2. ${}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12\times 11\times 10\times 9!}{9!} = 1,320$

3. $\frac{12!}{5!\times 4!\times 3!} = \frac{12\times 11\times 10\times 9\times 8\times 7\times 6\times 5!}{4!\times 4!\times 3!} = \frac{12\times 11\times 10\times 9\times (4\times 2)7\times 6}{(4\times 3)\times 2\times 3\times 2} = 11\times 10\times 9\times 4\times 7 = 27,720$

Problem Set

Evaluate the following expressions.

1. ${}_{8}P_5$

2. ${}_{11}P_8$
3. Evaluate and explain the results of each of the following: ${}_5P_5,\ {}_5P_0,\ {}_5P_1$
4. Sarah needs to go to five different stores. How many ways can she go to two of them before lunch?
5. In a race there are eleven competitors in a particular age group. How many possible arrangements are there for the top five finishers in this age group?
6. How many ways can eight distinct raffle prizes be awarded to fifteen ticket holders.
7. In a class of 24 students, there are six groups of four students. How many ways can a teacher select one group for each of three classroom maintenance responsibilities?
8. At a birthday party there are 6 unique prizes to be given randomly to the 6 quests such that no one guest receives more than one prize. How many ways can this be done?
9. How many unique ways can the letters in MISSISSIPPI be arranged?
10. How many ways can two Geometry books, eight Algebra books and three Pre-Calculus books be arranged on a shelf if all the books of each respective subject are identical.
11. At a math department luncheon, the department chair has three \$20 gift certificates to the local coffee shop, five \$25 gift certificates to a local bookstore and two state of the art calculators to give as prizes amongst the 10 department members. If each teacher receives one prize, how many unique distributions of prizes can be made?

Define and Apply Combinations

Objective

To define and calculate combinations of objects and to recognize the difference between a combination and a permutation.

Guidance

Combinations of a subset of a larger set of objects refer to the number of ways we can choose items in any order. For comparison, look at the table below to see when order matters and when order doesn't matter.

TABLE 13.5:

Combinations	Permutations
• Ways to select the members of a committee from a larger population	• Ways to select specific officers in a club-president, vice president, treasurer, etc.
• Ways to select a set number of pizza toppings from a larger list of choices	• Ways to select and arrange scoops of ice cream on a cone
• Ways to select books from a reading list	• Ways to select and order in which to read books selected from a reading list.

The simplest way to describe the difference between a combination and a permutation is to say that in a combination the order doesn't matter. The members of a committee could be selected in any order but the officers in a club are assigned a specific position and therefore the order does matter. Be careful of the use of these words in the real world as they are sometimes misused. For example, a locker combination. The ways to select and order the three numbers for a locker combination is not actually a combination, but a permutation since the order does matter.

Example A

How many ways can we choose three different flavors of ice cream from a selection of 15 flavors to place in a bowl?

Solution: First, does order matter in the bowl? When we created an ice cream cone with three scoops in an earlier concept the order did matter but here it does not. Let's work from the example of the ice cream cone. We determined the number of permutations of a subset of three flavors from the total 15 flavors using the formula: $\frac{n!}{(n-r)!} = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15\times14\times13\times\cancel{12!}}{\cancel{12!}} = 2730$. Now that order doesn't matter, this number includes the 3! ways to arrange each combination of 3 flavors. We can divide 2,730 by 3! to determine the number of combinations: $\frac{2730}{3\times2\times1} = \frac{2730}{6} = 455$.

The notation and formula for combinations can be written as: $\binom{n}{r} = {}_nC_r = C_r^n = \frac{n!}{r!(n-r)!}$, where n represents the number of elements in the set and r represents the number of elements in the subset.

Example B

Evaluate the following expressions:

1. $\binom{8}{5}$
2. ${}_8C_0$
3. ${}_8C_8$
4. C_7^{10}
5. Explain why the answers to 2 and 3 are the same.

Solution: All of the notations in problems 1-4 indicate that we should use the formula for a combination. We can use the graphing calculator to evaluate these as well. Problems 2 and 3 are set up in the form of the calculator notation so we will use the calculator to evaluate those two and the formula for the other two.

1. $\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8\times7\times6\times\cancel{5!}}{\cancel{5!}\times3\times2\times1} = \frac{8\times7}{1} = 56$.

2. Type in 8 on the TI-83 Graphing calculator, then MATH→PRB, select 3: ${}_nC_r$. Now type in 0 and your screen should read 8 ${}_nC_r$ 0 before your press ENTER to get the answer 1.

3. Type in 8 on the TI-83 Graphing calculator, then MATH→PRB, select 3: ${}_nC_r$. Now type in 8 and your screen should read 8 ${}_nC_r$ 8 before your press ENTER to get the answer 1.

4. $C_7^{10} = \frac{10!}{7!(10-7)!} = \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}\times3!} = \frac{10\times\cancel{3}\times3\times4\times\cancel{2}}{\cancel{3}\times\cancel{2}} = 120$

5. In problem 2, we are looking at the ways to choose 0 items from 8 choices. There is only one way to do this. In problem 3 we are looking at the ways to choose 8 items from 8 choices. Well, the only way to do that is to choose all 8 items. So, there is only 1 way to choose zero items or all the items from a set.

Example C

How many ways can a team of five players be selected from a class of 20 students?

Solution: We can express this problem using the notation $\binom{20}{5}$ and then use the formula to evaluate.

$$\binom{20}{5} = \frac{20!}{5!\times15!} = \frac{\cancel{20}\times19\times\cancel{6}\times3\times17\times16\times\cancel{15!}}{\cancel{5\times4}\times\cancel{3\times2}\times\cancel{15!}} = 15,504.$$

Guided Practice

1. Evaluate the following using the formula for combinations of the calculator.

a. $\binom{7}{5}$

b. ${}_{20}C_{12}$

c. C_7^{15}

2. How many ways can a committee of three students be formed from a club of fifteen members?

3. How many three-topping pizzas can be made if there are 10 topping choices?

Answers

1. Using the calculator for each of these we get:

a. $7 \ {}_nC_r \ 5 = 21$

b. $20 \ {}_nC_r \ 12 = 125,970$

c. $15 \ {}_nC_r \ 7 = 6,435$

2. $\binom{15}{3} = \frac{15!}{3!(15-3)!} = \frac{\cancel{3} \times 5 \times \cancel{2} \times 7 \times 13 \times \cancel{12!}}{\cancel{3 \times 2} \times \cancel{12!}} = 455.$

3. $C_3^{10} = 10 \ {}_nC_r \ 3 = 120.$

Vocabulary

Combinations
The number of ways a subset of items can be selected from a larger set disregarding order of selection.

Problem Set

Evaluate the following combinations with or without a calculator.

1. ${}_{13}C_{10}$
2. C_6^{10}
3. $\binom{18}{10}$
4. Explain why ${}_9C_5 = {}_9C_4 = 126$.
5. Decide whether the following situations are permutations and which are combinations.
    1. Ways to arrange students in a row.
    2. Ways to select a group of students.
    3. Ways to organize books on a shelf.
    4. Ways to select books to read from a larger collection.
    5. Ways to select three different yogurt flavors from a collection of ten flavors.

In each scenario described below, use either a combination or permutation as appropriate to answer the question.

6. There are seven selections for appetizers on a caterer's menu. How many ways can you select three of them?
7. You only have time for seven songs on your workout playlist. If you have 10 favorites, how many ways can you select seven of them for the list? Now, how many ways can you select them in a particular order?
8. How many ways can you select two teams of five players each from a group of ten players?
9. At the local frozen yogurt shop a sundae comes with your choice of three toppings. If there are 12 choices for toppings, how many combinations of toppings are possible?
10. How many ways can four people be selected from a group of 30 to serve on a committee? What if each of the four people was selected to fill a specific position on the committee?
11. A soccer team has 20 players, but only 11 play at any one time.

1. How many ways can the coach select a group of eleven players to start (disregard positions)?
2. Now, of the eleven players on the field, one is a goalie, four play defense, three play midfield and three play offense. How many ways are there to assign the eleven players to these positions?
3. Considering your answers to parts a and b, how many ways can the coach select eleven players and assign them positions on the field? Assume all players can play each position.

13.3 The Fundamental Counting Principle

Objective

To develop an understanding of and use the Fundamental Counting Principle to determine the number of unique arrangements that can be made of multiple items.

Review Queue

Write the n^{th} term rules for the following sequences.

1. $21, 26, 31, 36, 41, \dots$
2. $-16, 8, -4, 2, -1, \dots$

Find the sums of the following series.

3. $\sum_{n=1}^{6} (n+1)^2$
4. $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$

Applying the Fundamental Counting Principle

Objective

Define and apply the Fundamental Counting Principle.

Guidance

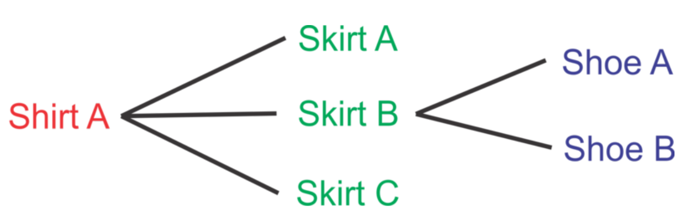
Sometimes we want to know how many different combinations can be made of a variety of items. The **fundamental counting principle** which states that the number of ways in which multiple events can occur can be determined by multiplying the number of possible outcomes for each event together. In other words, if events A, B, and C have 5, 3 and 4 possible outcomes respectively, then the possible combinations of outcomes is $5 \times 3 \times 4 = 60$.

The following examples will aid in developing an understanding of this concept and its application.

Example A

Sofia works in a clothing store. He has been given the task of setting up a mannequin with a skirt, a shirt and a pair of shoes from a display of coordinating skirts, shirts and shoes. Since they all coordinate she can pick any shirt, any skirt and any pair of shoes and the outfit will work. If there are 3 skirts, 5 shirts and 2 pairs of shoes, how many ways can she dress the mannequin?

Solution: Let's use a tree to help us visualize the possibilities. If we start with Shirt A, we get the following possibilities for the remainder of the outfit:



So we could have the following 6 combinations with Shirt A:

Shirt A, Skirt A, Shoe A

Shirt A, Skirt A, Shoe B

Shirt A, Skirt B, Shoe A

Shirt A, Skirt B, Shoe B

Shirt A, Skirt C, Shoe A

Shirt A, Skirt C, Shoe B

Consider that there are four other shirts that will also have 6 combinations of skirts and shirts that will go with them. Now, there are 5×6 total combinations which is 30 ways that Sofia could dress the mannequin.

Example B

Ralph is trying to purchase a new car. The salesperson tells him that there are 8 different possible interior colors, 5 exterior colors and 3 car models to choose from. How many different unique cars does he have to choose from?

Solution: Instead of making a tree diagram this time, let's look at a more efficient method for determining the number of combinations. If we consider what happens in the tree diagram, the 8 different interior colors would each be matched with each of the 5 exterior colors and those combinations would then be linked to the 3 different models, we can see that:

$$8 \text{ interior colors} \times 5 \text{ exterior colors} \times \ 3 \text{ models} = 8 \times 5 \times 3 = 120 \text{ combinations}$$

Example C

Monique is having a 5 course dinner in the dining room on a cruise. The menu consists of 2 appetizers, 3 soups, 2 salads, 4 entrees and 3 desserts. How many different meals could be configured if she chooses one of each course?

Solution: Following the method described in example B, we can multiply the number of chooses for each course together to determine the total combinations:

$2 \times 3 \times 2 \times 4 \times 3 = 144$ unique 5 course meals.

Guided Practice

1. A coffee shop offers a special espresso deal. You choose one of three sizes, one of 5 flavored syrups and whole, nonfat or soy milk. How many drink combinations can be made?

2. Sarah goes to a local deli which offers a soup, salad and sandwich lunch. There are 3 soups, 3 salads and 6 sandwiches from which to choose. How many different lunches can be formed?

3. A design your own t-shirt website offers 5 sizes, 8 colors and 25 designs for their shirts. How many different t-shirts can be designed?

Answers

1. There are 3 sizes, 5 syrups and 3 kinds of milk from which to choose. So, $3 \times 5 \times 3 = 45$ drinks.

2. 3 soups $\times$ 3 salads $\times$ 6 sandwiches $=$ 54 lunch combos.

3. 5 sizes $\times$ 8 colors $\times$ 25 designs $=$ 1000 shirts.

Vocabulary

The Fundamental Counting Principle
: states that the number of ways in which multiple events can occur can be determined by multiplying the number of possible outcomes for each event together.

Problem Set

Use the Fundamental Counting Principle to answer the following questions.

1. A frozen yogurt shop has a half price Sunday Sundae special. Customers can get one of four flavors, one of three syrups and one of twelve toppings on their sundae. How many possible sundae combinations can be made?
2. At a neighborhood restaurant wings are the specialty. The restaurant offers 3 sizes of wings, 4 levels of heat and ranch or blue cheese dipping sauce. How many different orders are possible?
3. A noodle restaurant offers five types of noodles to choose from and each dish comes with a choice of one of four meats and six different sauces. How many combinations can be made?
4. Charlie flips a coin and then rolls a die. How many different outcomes are possible?
5. On a one week cruise, the ship stops in four ports. At each port there are six different excursions to choose from. If a passenger chooses one excursion at each port, how many different vacation experiences can be created?
6. Samuel wants to know if he can go a whole month without wearing the exact same outfit twice. He has three pairs of pants, six shirts and two pairs of shoes. Can he make a unique outfit for each day of the month?
7. A car dealership has four different models to choose from in six exterior colors. If there are three different interior colors to choose from, how many different vehicles can be designed?
8. A burrito bar offers a lunch special burrito. Customers can choose a flour or corn tortilla; chicken, steak or carnitas; white or brown rice; black beans or pinto beans; cheese, guacamole, or sour cream; and one of four salsas for a special price. How many different burritos can be made?
9. Maria rolls a die, spins a spinner with four numbers and then flips a coin. How many possible outcomes are there?
10. A local restaurant offers a dinner special. Diners can choose one of six entrees, one of three appetizers and one of 3 desserts. How many different meals can be formed?

Using the Fundamental Counting Principle with and without Repetition

Objective

To determine the number of possible combinations in situations where elements may be repeated.

Guidance

Consider a phone number. A phone number consists entirely of numbers or repeated items. In this concepts we will look at how to determine the total number of possible combinations of items which may be repeated.

Example A

A license plate consists of three letters and four numbers in the state of Virginia. If letters and numbers can be repeated, how many possible license plates can be made?

Solution: If we consider the three slots for the letters, how many letters can be chosen to place in each slot? How about the four slots for the numbers? If there are no restrictions, i.e. letter and numbers can be repeated, the total number of license plates is:

$$\underline{26} \times \underline{26} \times \underline{26} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 175,760,000$$

Now, what if letters or numbers could not be repeated? Well, after the first letter is chosen, how many letters could fill the next spot? Since we started with 26, there would be 25 unused letters for the second slot and 24 for the third slot. Similarly with the numbers, there would be one less each time:

$$\underline{26} \times \underline{25} \times \underline{24} \times \underline{10} \times \underline{9} \times \underline{8} \times \underline{7} = 78,624,000$$

Example B

How many unique five letter passwords can be made? How many can be made if no letter is to be repeated?

Solution: Since there are 26 letters from which to choose for each of 5 slots, the number of unique passwords can be found by multiplying 26 by itself 5 times or $(26)^5 = 11,881,376$. If we do not repeat letters, then we need to subtract one each time we multiply: $26 \times 25 \times 24 \times 23 \times 22 = 7,893,600$.

Example C

How many unique 4 digit numbers can be made? What if no digits can be repeated?

Solution: For the first part, consider that in order for the number to be a four digit number, the first digit cannot be zero. So, we start with only 9 digits for the first slot. The second slot could be filled with any of the ten digits and so on:

$$\underline{9} \times \underline{10} \times \underline{10} \times \underline{10} = 9000.$$

For the second part, in which digits cannot be repeated, we would still have 9 possible digits for the first slot, then we'd have 9 again for the second slot (we cannot repeat the first digit, but we can add 0 back into the mix), then 8 for the third slot and 7 for the final slot:

$$\underline{9} \times \underline{9} \times \underline{8} \times \underline{7} = 4536.$$

Guided Practice

1. How many unique passwords can be made from 6 letters followed by 1 number or symbol if there are ten possible symbols? No letters or numbers can be repeated.

2. If a license plate has three letters and three numbers, how many possible combinations can be made?

3. In a seven digit phone number, the first three digits represent the exchange. If, within a particular area code, there are 53 exchanges, how many phone numbers can be made?

Answers

1. $\underline{26} \times \underline{25} \times \underline{24} \times \underline{23} \times \underline{22} \times \underline{21} \times \underline{20} = 3,315,312,000$

2. $\underline{26} \times \underline{26} \times \underline{26} \times \underline{10} \times \underline{10} \times \underline{10} = 17,576,000$

3. $\underline{53} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 530,000$

Problem Set

Use the Fundamental Counting Principle to answer the following questions. Refer back to the examples and guided practice for help.

1. How many six digit numbers can be formed if no digits can be repeated?
2. How many five digit numbers can be formed that end in 5?
3. How many license plates can be formed of 4 letters followed by 2 numbers?
4. How many seven digit phone numbers can be made if there are 75 exchanges in the area?
5. How many four letter pins (codes) can be made?
6. How many four number/letter pins can be made if no number or letter can be repeated?

7. How many different ways can nine unique novels be arranged on a shelf?
8. How many different three scoop cones can be made from 12 flavors of ice cream allowing for repetition? What if no flavors can be repeated?
9. How many different driver's license numbers can be formed by 2 letters followed by 6 numbers?
10. How many student ID numbers can be made by 4 random digits (zero cannot come first) followed by the student's grade (9, 10, 11 or 12). Example: 5422-12 for a 12^{th} grader.

13.4 The Binomial Theorem

Objective

To recognize the connection between the elements in Pascal's Triangle and the expansion of Binomials and to use the Binomial Theorem to expand binomials.

Review Queue

1. How many ways can seven different floats be ordered in a parade?
2. How many unique ways can the letters in the word COMMITTEE be arranged?
3. How many combinations of 4 sundae toppings can be made from a selection of 12?

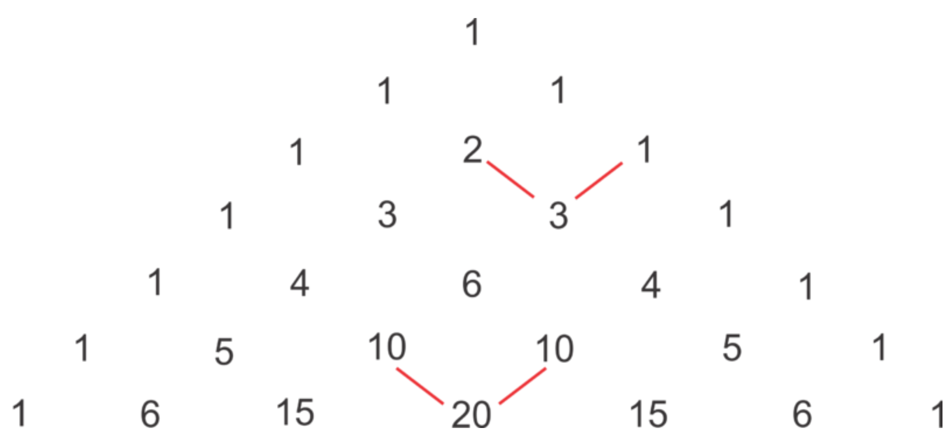
Pascal's Triangle and the Coefficients in the Expansion of Binomials

Objective

Observe and use the connection between Pascal's Triangle and expanded binomials to assist in expanding binomials.

Guidance

Pascal's Triangle:



Each row begins and ends with a one. Each "interior" value in each row is the sum of the two numbers above it. For example, $2+1=3$ and $10+10=20$. This pattern produces the symmetry in the triangle.

Another pattern that can be observed is that the row number is equal to the number of elements in that row. Row 1, for example has 1 element, 1. Row 2 has 2 elements, 1 and 1. Row 3 has 3 elements, 1, 2 and 1.

A third pattern is that the second element in the row is equal to one less than the row number. For example, in row 5 we have 1, 4, 6, 4 and 1.

Example A

Continue the triangle to determine the elements in the 9^{th} row of Pascal's Triangle.

Solution: Following the pattern of adding adjacent elements to get the elements in the next row, we find hat the eighth row is: 1 7 21 35 35 21 7 1

Now, continue the pattern again to find the 9^{th} row: 1 8 28 56 70 56 28 8 1

Example B

Expand the binomial $(a+b)^4$ and discuss the pattern within the exponents of each variable as well as the pattern found in the coefficients of each term.

Solution:

$$\begin{gathered}(a+b)(a+b)(a+b)(a+b)\\(a^2+2ab+b^2)(a^2+2ab+b^2)\\a^4+2a^3b+a^2b^2+2a^3b+4a^2b^2+2ab^3+a^2b^2+2ab^3+b^4\\a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{gathered}$$

1. Take two binomials at a time and square them using $(a+b)^2=a^2+2ab+b^2$

2. Next, distribute each term in the first trinomial over each term in the second trinomial and collect like terms.

We can see that the powers of a start with 4 (the degree of the binomial) and decrease by one each term while the powers of b start with zero and increase by one each term. The number of terms is 5 which is one more than the degree of the binomial. The coefficients of the terms are 1 4 6 4 1, the elements of row 5 in Pascal's Triangle.

Example C

Use what you discovered in the previous example to expand $(x+y)^6$.

Solution: The degree of this expansion is 6, so the powers of x will begin with 6 and decrease by one each term until reaching 0 while the powers of y will begin with zero and increase by one each term until reaching 6. We can write the variables in the expansion (leaving space for the coefficients) as follows:

$$\_\_x^6\ +\_\_x^5y\ +\_\_x^4y^2\ +\_\_x^3y^3\ +\_\_x^2y^4\ +\_\_xy^5\ +\_\_y^6$$

In the previous example we observed that the coefficients for a fourth degree binomial were found in the fifth row of Pascal's Triangle. Here we have a 6^{th} degree binomial, so the coefficients will be found in the 7^{th} row of Pascal's Triangle. Now we can fill in the blanks with the correct coefficients.

$$x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$$

Example D

Use Pascal's Triangle to expand $(x-2)^5$.

Solution: We will use the 6^{th} row of Pascal's Triangle for the coefficients and the powers observed in the exponents to begin the expansion as follows:

$$1x^5+5x^4(-2)+10x^3(-2)^2+10x^2(-2)^3+5x(-2)^4+1(-2)^5$$

Now, compute the powers of -2 and multiply these by the coefficients of each terms to simplify:

$$\begin{gathered}1x^5+5x^4(-2)+10x^3(4)+10x^2(-8)+5x(16)+1(-32)\\x^5-10x^4+40x^3-80x^2+80x-32\end{gathered}$$

Guided Practice

1. Write out the elements in row 10 of Pascal's Triangle.

2. Expand $(a+4)^3$

3. Write out the coefficients in the expansion of $(2x-3)^4$

Answers

1. The 9^{th} row was determined in Example A to be: 1 8 28 56 70 56 28 8 1

Subsequently, the 10^{th} row is: 1 9 36 84 126 126 84 36 9 1

2.

$$a^3+3a^2(4)+3a(4)^2+(4)^3$$
$$a^3+12a^2+48a+64$$

3.

$$(2x)^4+4(2x)^3(-3)+6(2x)^2(-3)^2+4(2x)(-3)^3+(-3)^4$$
$$16x^4-96x^3+216x^2-216x+81$$

Problem Set

1. Write out the elements in row 7 of Pascal's Triangle.
2. Write out the elements in row 13 of Pascal's Triangle.

Use Pascal's Triangle to expand the following binomials.

3. $(x-6)^4$
4. $(2x+5)^6$
5. $(3-x)^7$
6. $(x^2-2)^3$
7. $(x+4)^5$
8. $(2-x^3)^4$
9. $(a-b)^6$
10. $(x+1)^{10}$

Using the Binomial Theorem

Objective

Define and apply the binomial theorem to determine the expansions of binomials.

Guidance

Using Pascal's Triangle and the patterns within it are only one way to expand binomials. The Binomial Theorem can also be used to expand binomials and is sometimes more efficient, particularly for higher degree binomials. The Binomial Theorem is given by:

$$(a+b)^n=\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n-1}a^1b^{n-1}+\binom{n}{n}a^0b^n$$

It can be seen in this rule that the powers of a and b decrease and increase, respectively, as we observed in the previous concept. Recall that the notation $\binom{n}{r}$ refers to the calculation of the number of combinations of r elements selected from a set of n elements and that $\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}$.

As it turns out, $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3} \cdots \binom{n}{n-1}, \binom{n}{n}$, are the elements in the $(n+1)^{st}$ row of Pascal's Triangle. If we let $n = 5$, then we can find the coefficients as follows:

$$\binom{5}{0} = \frac{5!}{0!5!} = 1; \binom{5}{1} = \frac{5!}{1!4!} = 5; \binom{5}{2} = \frac{5!}{2!3!} = 10; \binom{5}{3} = \frac{5!}{3!2!} = 10; \binom{5}{4} = \frac{5!}{4!1!} = 5; \binom{5}{5} = \frac{5!}{5!0!} = 1$$

These are the elements of the 6^{th} row of Pascal's Triangle: 1 5 10 10 5 1

The Binomial Theorem allows us to determine the coefficients of the terms in the expansion without having to extend the triangle to the appropriate row.

Example A

Use the Binomial Theorem to expand $(x+2y)^6$

Solution: First, in this example, $a = x$, $b = 2y$ and $n = 6$. Now we can substitute into the rule.

$$(x+2y)^6 = \binom{6}{0}x^6(2y)^0 + \binom{6}{1}x^5(2y)^1 + \binom{6}{2}x^4(2y)^2 + \binom{6}{3}x^3(2y)^3 + \binom{6}{4}x^2(2y)^4 + \binom{6}{5} = x^1(2y)^5 + \binom{6}{6}x^0(2y)^6$$

Now we can simplify:

$$\begin{aligned} &= (1)x^6(1) + (6)x^5(2y) + (15)x^4(4y^2) + (20)x^3(8y^3) + (15)x^2(16y^4) + (6)x(32y^5) + (1)(1)(64y^6) \\ &= x^6 + 12x^5y + 30x^4y^2 + 160x^3y^3 + 240x^2y^4 + 192xy^5 + 64y^6 \end{aligned}$$

More Guidance

What if we just want to find a single term in the expansion? We can use the following rule to represent the $(r+1)^{st}$ term in the expansion: $\binom{n}{r}a^{n-r}b^r$. The rule is for the $(r+1)^{st}$ term because if we want the 1^{st} term, then $r = 0$ (refer back to the Binomial Theorem expansion rule). The value of r in the expansion is always one less than the term number.

Example B

Find the 4^{th} term in the expansion of $(3x-5)^8$.

Solution: Since we want the 4^{th} term, $r = 3$. Now we can set up the formula with $a = 3x$, $b = -5$, $n = 8$ and $r = 3$ and evaluate:

$$\binom{8}{3}(3x)^{8-3}(-5)^3 = (56)(243x^5)(-125) = -1,701,000x^5$$

Example C

Find the coefficient of the term containing y^5 in the expansion of $(4+y)^9$.

Solution:

This time, think about the rule, $\binom{n}{r}a^{n-r}b^r$, and that we know that $b^r = y^5$ and thus $r = 5$. We also know that $n = 9$ and $a = 4$. Now we can fill in the rest of the rule:

$$\binom{9}{5}(4)^{9-5}y^5 = (126)(256)y^5 = 32,256y^5$$

Guided Practice

1. Use the Binomial Theorem to show that $(a+b)^2 = a^2 + 2ab + b^2$.

2. Find the coefficient of the 5^{th} term in the expansion of $(1-3x)^{10}$.

3. Find the constant term in the expansion of $\left(4x^3 + \frac{1}{x}\right)^4$.

Answers

1.

$$\begin{aligned}(a+b)^2 &= \binom{2}{0}a^2b^0 + \binom{2}{1}a^1b^1 + \binom{2}{2}a^0b^2 \\ &= (1)a^2(1) + (2)ab + (1)(1)b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

2. $r = 4$ in the 5^{th} term so, $\binom{10}{4}(1)^{10-4}(-3x)^4 = 210(1)(81x^4) = 17,010x^4$. Since only the coefficient is required, we can drop the variable for the final answer: 17,010.

3. The constant term occurs when the power of x is zero. Let r remain unknown for the time being: $\binom{4}{r}(4x^3)^{4-r}\left(\frac{1}{x}\right)^r$. Now isolate the variables to determine when the power of x will be zero as shown:

We can set the variable portion of the expanded term rule equal to x^0.

Then simplify using the rules of exponents on the left hand side of the equation until we have like bases, x, on both sides and can drop the bases to set the exponents equal to each other and solve for r.

$$\begin{aligned}(x^3)^{4-r}(x^{-1})^r &= x^0 \\ x^{12-3r} \cdot x^{-r} &= x^0 \\ x^{12-3r-r} &= x^0 \\ x^{12-4r} &= x^0 \\ 12 - 4r &= 0 \\ 12 &= 4r \\ r &= 3\end{aligned}$$

Now, plug the value of r into the rule to get the constant term in the expansion.

$$\binom{4}{3}(4x^3)^{4-3}\left(\frac{1}{x}\right)^3 = 4(4x^3)\left(\frac{1}{x}\right)^3 = 16.$$

Problem Set

Expand the following binomials using the Binomial Theorem.

1. $(x-a)^7$
2. $(2a+3)^4$

Find the n^{th} term in the expansions of the following binomials.

3. $(7x-2)^5;\ n=4$
4. $\left(6x+\frac{1}{2}\right)^7;\ n=3$
5. $(5-a)^9;\ n=7$
6. $\left(\frac{2}{3}x+9y\right)^6;\ n=4$
7. Find the term with x^5 in the expansion of $(3x-2)^7$.
8. Find the term with y^6 in the expansion of $(5-y)^8$.
9. Find the term with a^3 in the expansion of $(2a-b)^{10}$.
10. Find the term with x^4 in the expansion of $(8-3x)^5$.
11. Find the constant term in the expansion of $\left(x^2+\frac{3}{x}\right)^6$.
12. Find the constant term in the expansion of $\left(\frac{5}{x^3}-x\right)^8$.

13.5 Venn Diagrams and Independence

Objective

To use Venn diagrams and the related set notation to show the relationships between sets.

Review Queue

1. A fair pair of dice is rolled twice. What is the probability of obtaining a sum of six on both rolls?

2. A drawer contains 4 pairs of brown socks and 5 pairs of black socks. If two pairs are randomly selected without replacement, what is the probability of selecting at least one pair of black socks?

3. Three cards are selected randomly from a deck of 52 playing cards. What is the probability of selecting cards of the same suit (all three are either red or black) if the cards are replaced between each drawing? What if the cards are not replaced?

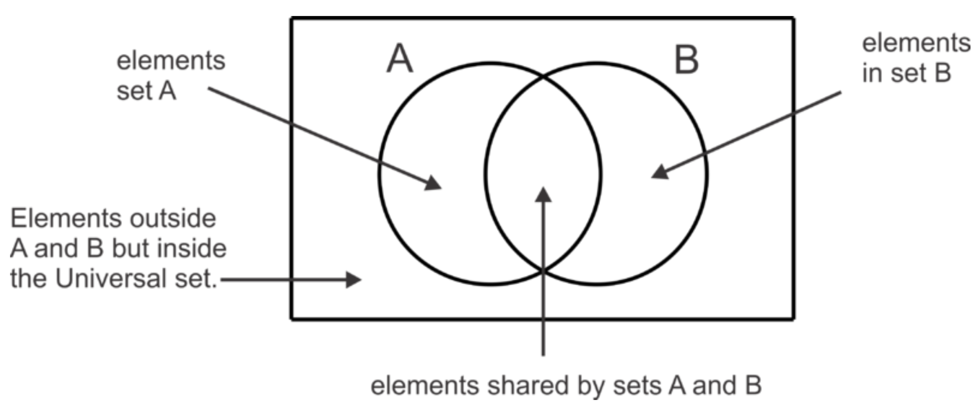
Union and Intersection of Sets

Objective

Define and apply the notations for union and intersection of sets and the compliment of a set.

Guidance

A Venn diagram is shown below.

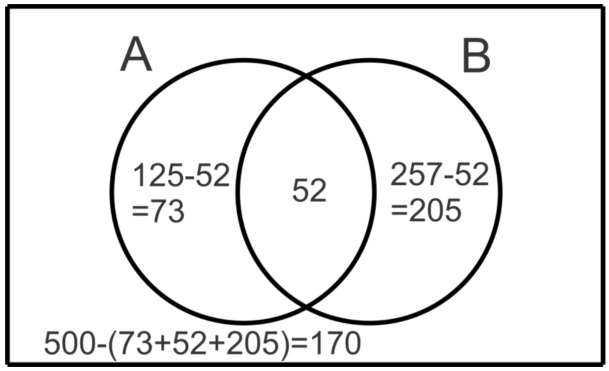


The diagram illustrates that within some universe of data, there are two subsets, A and B, which have some elements in common. The following example relates the use of a Venn diagram to a real world situation.

Example A

At a school of 500 students, there are 125 students enrolled in Algebra II, 257 students who play sports and 52 students that are enrolled in Algebra II and play sports. Create a Venn diagram to illustrate this information.

Solution: First, let's let set A represent the students enrolled in Algebra II and set B represent the students who play sports. Generally speaking, it is easiest to start in the center or "intersection" of the Venn diagram. Once we place 52 in the intersection, then we can subtract it from the total number of students who play sports and the total number of student who take Algebra II to determine how many just do one or the other. Finally, we can subtract this total from 500 to figure out how many are outside the circles altogether.



There symbols that can be used to describe the number of elements in each region in the diagram as well.

TABLE 13.6:

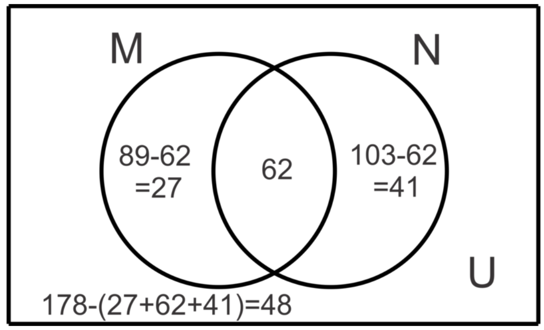
Symbol	Description	Value for this Problem
$n(A)$	The number of elements in set A	125
$n(A \cap B)$	The number of elements in the intersection of sets A and B (all the elements that are in both sets-the overlap)	52
$n(A \cup B)$	The number of elements in the union of sets A and B (all the elements that are in one or both of the sets)	330
$n(A')$	The number of elements in the compliment of A (the number of elements outside set A)	375
$n((A \cup B)')$	The number of elements in the compliment of $A \cup B$ (everything outside the union of A and B)	170
$n((A \cap B)')$	The number of elements in the compliment of $A \cap B$ (everything outside the intersection of A and B)	448
$n(A \cap B')$	The number of elements in the intersection of A and B's compliment (the number of elements in A but not in B)	73

Example B

Create a Venn diagram to illustrate the following information regarding the subsets M and N in universe U:

$$n(M) = 89;\ n(N) = 103;\ n(M \cup N) = 130;\ n(U) = 178$$

Solution: Again, we will start in the middle or intersection. We must determine how many elements are in the intersection. Let's consider that when we add the elements in M to the elements in N, we are adding the elements in the intersection twice. This happens because they are counted in set M and counted again in set N. Did you notice that $n(M) + n(N) = 89 + 103 = 192$ while the $n(M \cup N) = 130$? We have double counted the 62 (192-130) elements in $M \cap N$. Now we can put this number in the Venn diagram and work our way out as we did in the previous example.



In general, for two sets, M and N, we can use the formula: $n(M)+n(N)-n(M\cap N)=n(M\cup N)$ to represent the relationship between the regions in the Venn diagram and to solve problems. In this case, substituting in the given information we can determine the $n(M\cap N)$ as shown below:

$$\begin{aligned}89+103-n(M\cap N)&=130\\192-n(M\cap N)&=130\\-n(M\cap N)&=-62\\n(M\cap N)&=62\end{aligned}$$

Example C

Create a Venn diagram to represent the following information and answer the questions that follow.

In a survey of 150 high school students it was found that:

80 students have laptops

110 students have cell phones

125 students have iPods

62 students have both a laptop and a cell phone

58 students have both a laptop and iPod

98 students have both a cell phone and an iPod

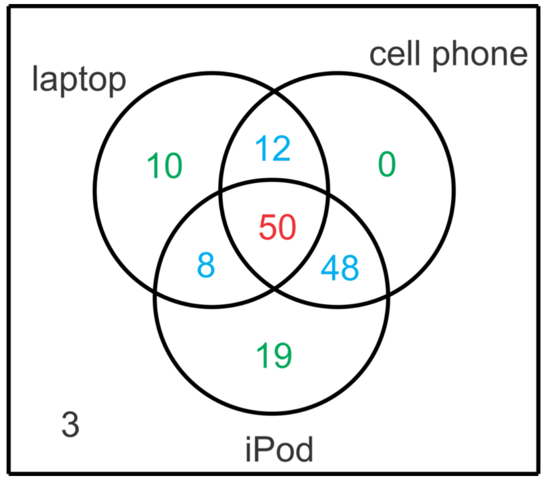
50 students have all three items

a. How many students have just a cell phone?

b. How many students have none of the mentioned items?

c. How many students have an iPod and laptop but not a cellphone?

Solution: First we will use the given information to construct the Venn diagram as shown.



We can start by putting 50 in the center where students have all three items. Next we can find the values in blue by subtracting 50 from each of the "overlapping" values. For example, there are 62 students with both a laptop and a cell phone and 50 of them also have an iPod. To find the number that have a laptop and cell phone but no iPod, subtract $62 - 50 = 12$. Once the blue values are found we can find the green values by subtracting the blue and red values in each subset from the total in the subset. For example, the number of students with a cell phone but no other technology item is $110 - (50 + 12 + 48) = 0$. Finally we can add up all the values in the circles and subtract this from 150, the total number of students surveyed to determine that 3 students have none of the items.

Now that the Venn diagram is complete, we can use it to answer the questions.

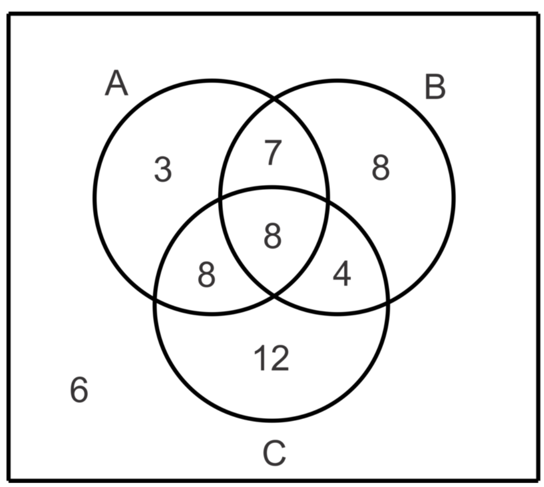
a. There are 0 students that just have a cell phone.

b. There are 3 students with none of the mentioned technology.

c. There are 8 students with an iPod and laptop but no cell phone.

Guided Practice

Use the Venn diagram to determine the number of elements in each set described in the problems.



1. $n(A)$
2. $n(C)$
3. $n(A')$
4. $n(A \cap B)$

5. $n(A \cup B \cup C)$

6. $n(A \cap C')$

7. $n(A \cap B \cap C)$

8. $n(A' \cap B' \cap C')$

Answers

1. $3+7+8+8=26$

2. $8+8+4+12=32$

3. $8+4+12+6=30$

4. $7+8=15$

5. $3+7+8+8+8+4+12=50$

6. $3+7=10.$

7. 8

8. 6

Problem Set

Use the information below for problems 1-5.

In a survey of 80 households, it was found that:

30 had at least one dog

42 had at least one cat

21 had at least one "other" pet (fish, turtle, reptile, hamster, etc.)

20 had dog(s) and cat(s)

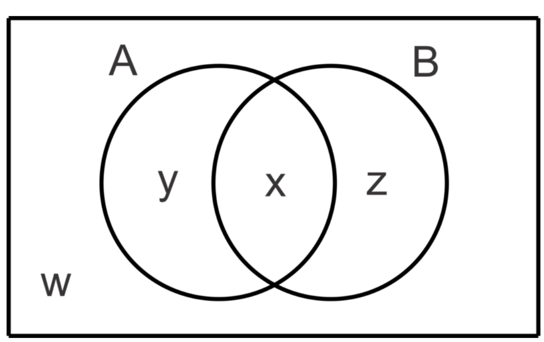
10 had cat(s) and "other" pet(s)

8 had dog(s) and "other" pet(s)

5 had all three types of pets

1. Make a Venn diagram to illustrate the results of the survey.
2. How many have dog(s) and cat(s) but no "other" pet(s)?
3. How many have only dog(s)?
4. How many have no pets at all?
5. How many "other" pet(s) owners also have dog(s) or cat(s) but not both?

Use the letters in the Venn diagram below to describe the region for each of the sets.



6. $A \cap B$

7. A
8. $A \cup B$
9. $A \cap B'$
10. $(A \cap B)'$
11. $(A \cup B)'$
12. A'
13. $B' \cup A$

Probability Using a Venn Diagram and Conditional Probability

Objective

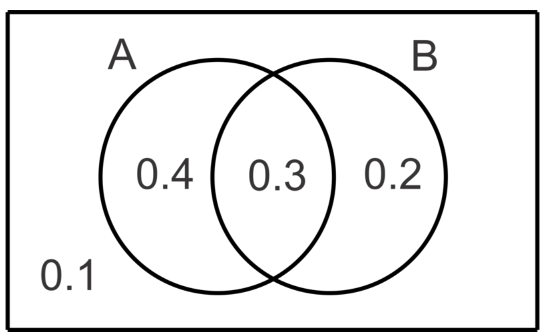
Use Venn diagrams to solve probability problems.

Guidance

It is often useful to use a Venn diagram to visualize the probabilities of multiple events. In Example A we explore the use of a Venn diagram to determine the probabilities of individual events, the intersection of events and the compliment of an event. In Example C we will continue to explore the concept of a conditional probability and how to use a Venn diagram to solve these problems as well as the formula for conditional probability first introduced in the concept "Calculating Probabilities of Combined Events".

Example A

Use the Venn diagram to find the probabilities.



a. $P(A)$

b. $P(B)$

c. $P(A \cap B)$

d. $P(A \cup B)$

e. $P(A' \cap B')$

Solution: Essentially we will do exactly what we did in the previous concept with probabilities rather than whole numbers. Notice that the sum of all the values in the diagram is $0.4+0.3+0.2+0.1=1$. This diagram represents the entire sample space for two events, A and B.

a. To find the $P(A)$, we will add the probability that only A occurs to the probability that A and B occur to get $0.4+0.3=0.7$. So $P(A)=0.7$.

b. Similarly, $P(B)=0.2+0.3=0.5$.

c. Now, $P(A \cap B)$ is the value in the overlapping region 0.3.

d. $P(A \cup B) = 0.4 + 0.3 + 0.2 = 0.9$. Which can also be found using the formula $P(A) + P(B) - P(A \cap B) = 0.7 + 0.5 - 0.3 = 0.9$.

e. $P(A' \cap B')$ needs to be determined by finding where in the diagram everything outside of A overlaps with everything outside of B. That will be the region outside of both circles and that probability is 0.1. Another way to think of this is $P(A \cup B)'$, or $1 - P(A \cup B)$.

More Guidance

There are a couple of equivalent set notations or probabilities and they are called De Morgan's Laws.

$(A \cap B)' = (A' \cup B')$ for sets or $P(A \cap B)' = P(A' \cup B')$ for probabilities.

and

$(A \cup B)' = (A' \cap B')$ for sets or $P(A \cup B)' = P(A' \cap B')$ for probabilities.

We looked at the second one in part e of Example A. We will look at the first one in the next Example.

Example B

Given $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cup B) = 0.7$, find $P(A \cap B)$ and $P(A' \cup B')$.

Solution: First, we can use the formula for the union of two sets to determine the intersection.

$$\begin{aligned}
P(A) + P(B) - P(A \cap B) &= P(A \cup B) \\
0.6 + 0.3 - P(A \cap B) &= 0.7 \\
0.9 - 0.7 &= P(A \cap B) \\
0.2 &= P(A \cap B)
\end{aligned}$$

Now we can use De Morgan's Law to find $P(A' \cup B')$.

$$P(A' \cup B') = P(A \cap B)' = 1 - P(A \cup B) = 1 - 0.2 = 0.8.$$

We could have also created a Venn diagram for the probabilities and interpreted $P(A' \cup B')$ and the regions outside A union with the regions outside B which would be everything in the Venn diagram except the overlap of the two regions or $P(A \cap B)$.

Example C

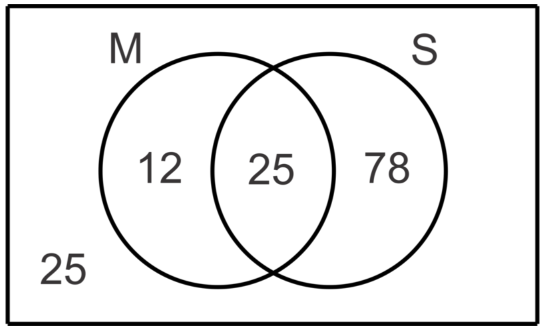
The data from a survey of 140 students showed that 37 study music, 103 play a sport and 25 do neither. Create a Venn diagram to illustrate the data collected and then determine the probability that if a student is selected at random,

a. he or she will study music

b. he or she will study music given that he or she plays a sport.

Solution: Let M represent the set of students who study music and S represent the set of students who play sports. First let's determine the number of students that study music and play a sport to fill in the overlapping region in the diagram and then we can find the other values.

$$\begin{aligned}
n(M) + n(S) - n(M \cap S) &= n(M \cup S) \\
37 + 103 - n(M \cap S) &= 115 \\
n(M \cap S) &= 25
\end{aligned}$$



a. The probability that a randomly selected student studies music is the number of students who study music divided by the total number of students surveyed or $P(M) = \frac{n(M)}{140} = \frac{37}{140} \approx 0.264$.

b. The probability that a randomly selected student will study music given that he/she plays a sport is called a conditional probability. We use the notation $P(M|S)$ to represent the $P(M)$ given that S has already occurred. To find this probability using a Venn diagram, we find the number of student who study music and play a sport and divide by the number of students who play a sport or $P(M|S) = \frac{n(M \cap S)}{n(S)} = \frac{25}{103} \approx 0.243$. Think of it this way, when we say that we know that the student plays a sport, then the numerator is limited to those students who study music and play a sport and the denominator is limited to those who play a sport.

There is also a formula for conditional probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

In the context of our problem, it is: $P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{\frac{25}{140}}{\frac{103}{140}} = \frac{25}{140} \cdot \frac{140}{103} = \frac{25}{103}$.

Notice that the resulting probability is the same as previously determined. Either method can be used.

Guided Practice

1. In a class of 260 seniors, 125 study Spanish, 95 study Chemistry, 165 study Mathematics, 18 study Spanish and Chemistry, 75 study Chemistry and Math, 20 study Math and Spanish and 15 study all three subjects. Make a Venn diagram to illustrate the data and then find the probability that a student selected at random studies:

a. just Spanish

b. Math and Chemistry but not Spanish

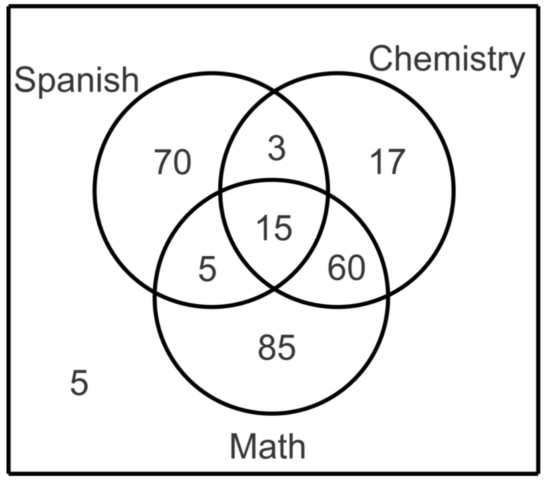
c. none of these subjects

d. Spanish, given that he/she studies Math

2.Given $P(A \cap B) = 0.4$, $P(A \cap B') = 0.2$ and $P(A' \cap B') = 0.3$, find $P(B)$ and $P(A|B)$.

Answers

1.



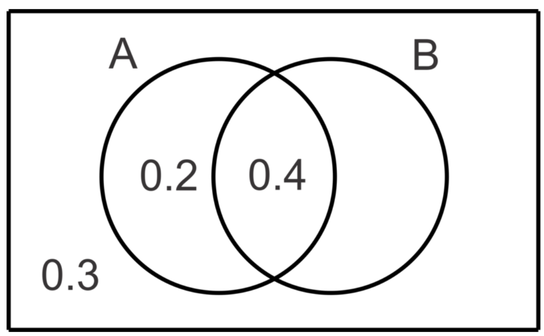
a. $P(S \cap M' \cap C') = \frac{70}{260} = \frac{7}{26} \approx 0.269$

b. $P(M \cap C \cap S') = \frac{60}{260} = \frac{3}{13} \approx 0.231$

c. $P(M' \cap C' \cap S') = \frac{5}{260} = \frac{1}{52} \approx 0.0192$

d. $P(S|M) = \frac{P(S \cap M)}{P(M)} = \frac{\frac{20}{260}}{\frac{165}{260}} = \frac{4}{33} \approx 0.121$

2. The information gives us the Venn diagram:



The missing value, $P(B \cap A')$, must be 0.1 in order for the total of the probabilities in the sample space to equal 1. Thus, $P(B) = 0.5$. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.4}{0.5} = \frac{4}{5} = 0.8$.

Problem Set

For questions 1-3, find the indicated probabilities given $P(A) = 0.5$, $P(B) = 0.65$ and $P(A \cup B) = 0.75$.

1. $P(A \cap B)$
2. $P(A' \cap B')$
3. $P(B|A)$

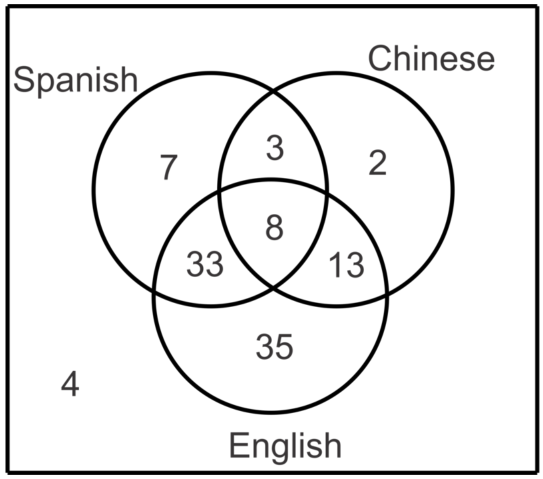
For questions 4-6, find the indicated probabilities given $P(A) = 0.6$, $P(B) = 0.8$ and $P(A \cup B)' = 0.2$.

4. $P(A \cap B')$
5. $P(B|A)$
6. $P(A|B)$

For questions 7-9, find the indicated probabilities given $P(A \cap B') = 0.3$, $P(B \cap A') = 0.2$ and $P(A \cup B) = 0.8$.

7. $P(A \cap B)$

8. $P(A)$
9. $P(B|A)$
10. Given $P(A) = 2P(B)$, $P(A \cup B) = 0.8$ and $P(A \cap B) = 0.1$, find $P(A)$.
11. The international club at a school has 105 members, many of whom speak multiple languages. The most commonly spoken languages in the club are English, Spanish and Chinese. Use the Venn Diagram below to determine the probability of selecting a student who:



1. does not speak English.
2. speaks Spanish given that he/she speaks English.
3. speaks English given that he/she speaks Chinese.
4. speaks Spanish and English but not Chinese.

Independent, Conditional and Mutually Exclusive Events

Objective

Define and apply the appropriate formulas to determine probabilities of combined events which are Independent, Conditional or Mutually Exclusive.

Guidance

We have already discussed the formula for conditional probabilities: $P(A|B) = \frac{P(A \cap B)}{P(B)}$. These events are not independent, they are conditional because the outcome of event B affects the outcome of event A. When events are independent, $P(A|B) = P(A)$, meaning that it doesn't matter that event B has occurred, the result of event B does not affect the result of event A. Now we can replace $P(A|B)$ with $\frac{P(A \cap B)}{P(B)}$ in the previous statement to get $\frac{P(A \cap B)}{P(B)} = P(A)$. Finally, multiply both sides by $P(B)$ to get $P(A \cap B) = P(A) \times P(B)$ for independent events A and B. We can use this rule to determine if events are independent or to find the intersection of known independent events.

It is also possible for two events to have not intersection, or $P(A \cap B) = 0$. When this occurs we say that the events (or sets) are Mutually Exclusive. If one has occurred, then the other cannot occur. Examples of **Mutually Exclusive** sets are Boys and Girls, Juniors and Seniors-it is not possible to be both. It is important to note here that mutually exclusive events cannot be independent unless the probability of one of the events is zero since for independent events $P(A \cap B) = P(A) \times P(B)$ and the only way a product can equal zero is if one of the factors is equal to zero.

Example A

Given two events, A and B, such that $P(A) = 0.3$, $P(B) = 0.5$ and $P(A \cup B) = 0.65$

a. Find $P(A \cap B)$.

b. State with a reason whether the events are independent.

c. State with a reason whether the events are mutually exclusive.

Solution:

a. Since we are not told that the events are independent, we do not know that $P(A \cap B) = P(A) \times P(B)$. However, for all events, independent or otherwise, it is true that $P(A) + P(B) - P(A \cap B) = P(A \cup B)$ so

$$\begin{aligned} 0.3 + 0.5 - P(A \cap B) &= 0.65 \\ P(A \cap B) &= 0.15 \end{aligned}$$

b. To determine if the events are independent we will test the rule $P(A \cap B) = P(A) \times P(B)$.

$$P(A) \times P(B) = 0.3 \times 0.5 = 0.15 = P(A \cap B).$$

Thus, the events are independent since the product of their probabilities is equal to the probability of their intersection.

c. The events are not mutually exclusive because $P(A \cap B) = 0.15 \neq 0$.

Example B

Given that A and B are independent events such that $P(A) = 0.4$ and $P(A \cup B) = 0.76$, find

a. $P(B)$

b. Probability of A or B but **not** both occurring.

Solution:

a. Since we know the two events are independent, we know that $P(A \cap B) = 0.4P(B)$. Now we can use the formula for the probability of the union of the two sets and substitute this product for the probability of the intersection:

$$\begin{aligned} 0.4 + P(B) - 0.4P(B) &= 0.76 \\ 0.6P(B) &= 0.36 \\ P(B) &= 0.6 \end{aligned}$$

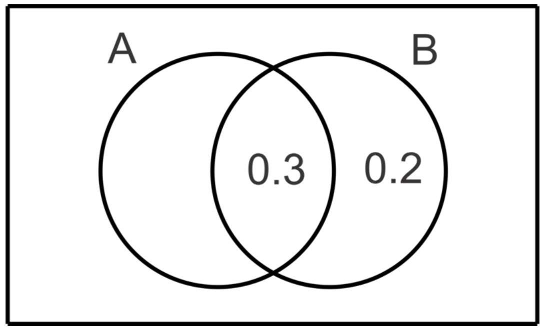
b. To find the probability of either A or B occurring but not both, we need to find $P(A \cap B)$ and subtract this from $P(A \cup B)$.

$$0.76 - (0.4)(0.6) = 0.76 - 0.24 = 0.52$$

Example C

Events A and B are independent such that $P(B \cap A') = 0.2$ and $P(A \cap B) = 0.3$. Find $P(A \cup B)$.

Solution: For this problem, a Venn diagram might be useful to illustrate the given information.



From the diagram we can see $P(B) = 0.5$ and since we know that the events are independent, we know:

$$\begin{aligned} P(A) \times P(B) &= P(A \cap B) \\ P(A) \times 0.5 &= 0.3 \\ P(A) &= \frac{0.3}{0.5} = 0.6 \end{aligned}$$

Now, $P(A \cup B) = 0.6 + 0.5 - 0.3 = 0.8$.

Guided Practice

1. Given two events, A and B, such that $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cup B) = 0.75$

a. Find $P(A \cap B)$.

b. State with a reason whether the events are independent.

c. Find $P(A|B)$.

2. Given that A and B are independent events such that $P(A) = 0.8$ and $P(A \cup B) = 0.9$, find

a. $P(B)$

b. $P(B \cap A')$

3. Events A and B are independent such that $P(A \cap B') = 0.25$ and $P(A \cap B) = 0.25$. Find $P(A \cup B)$.

Answers

1. a.

$$\begin{aligned} 0.4 + 0.5 - P(A \cap B) &= 0.75 \\ P(A \cap B) &= 0.15 \end{aligned}$$

b. If the events are independent, $P(A \cap B) = 0.4 \times 0.5 = 0.2$.

Since $0.2 \neq 0.15$, the events are not independent.

c. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.5} = 0.3$.

2. a.

$$\begin{aligned} 0.8 + P(B) - 0.8P(B) &= 0.9 \\ 0.2P(B) &= 0.1 \\ P(B) &= 0.5 \end{aligned}$$

b. $P(B \cap A') = P(B) - P(B \cap A) = 0.5 - 0.8 \times 0.5 = 0.1$

3. $P(A) = P(A \cap B') + P(A \cap B) = 0.25 + 0.25 = 0.5$

$P(A \cap B) = 0.5P(B) = 0.25$ and thus $P(B) = 0.5$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.5 - 0.25 = 0.75$

Vocabulary

Mutually Exclusive
: Two or more sets which have no common elements or for which the intersection does not exist. In the case of probability, these would be events such that if one occurs, the other one cannot occur

Problem Set

1. Events A and B are mutually exclusive. Describe $P(A|B)$.
2. Events A and B are independent. Show that $P(B) = P(B|A)$.

For problems 3-7, use the given information about events A and B to determine whether or not the events are independent.

3. $P(A) = 0.6$, $P(B) = 0.4$ and $P(A \cup B) = 0.76$
4. $P(A) = 0.5$, $P(A \cap B) = 0.2$ and $P(A \cup B) = 0.8$
5. $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cup B) = 0.55$
6. $P(A) = 0.6$, $P(B \cap A') = 0.28$ and $P(A \cap B) = 0.42$

For problems 7-10, events A and B are independent.

7. Given $P(A) = 0.8$ and $P(A \cup B) = 0.88$, find $P(B)$ and $P(A \textit{ or } B \text{ but not both})$.
8. Given $P(A \cap B') = 0.54$ and $P(A \cap B) = 0.36$, find $P(B)$ and $P(A \cup B)$.
9. Given $P(B) = 0.8$ and $P(A' \cap B') = 0.04$, find $P(A)$ and $P(A' \cup B')$.
10. $P(A \cap B) = 0.28$ and $P(A \cup B) = 0.82$, find $P(A)$ and $P(B)$.

Chapter 14 Algebra Review - Polynomial Functions

Chapter Outline

Polynomial Functions

In this chapter we will continue to explore non-linear functions. This chapter covers any polynomial function, or a function whose greatest exponent is larger than 2. We will add, subtract, multiply, divide and solve these types of functions. Towards the end of the chapter, we will analyze the graph of a polynomial function and find the inverse and composition.

14.1 Properties of Exponents

Objective

Using the properties of exponents to simplify numeric and algebraic expressions.

Review Queue

Simplify the following expressions.

1. 5^2

2. 3^3

3. $2^3 \cdot 2^2$

For questions 4-6, $x = -3, y = 2$, and $z = -4$. Evaluate the following.

4. xy^2

5. $(xy)^2$

6. $\frac{x^2 z}{y^2}$

Product and Quotient Properties

Objective

To use and understand the multiplication and quotient properties of exponents.

Watch This

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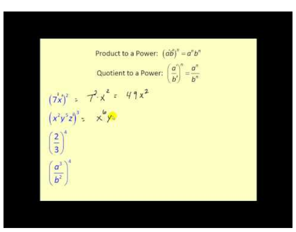

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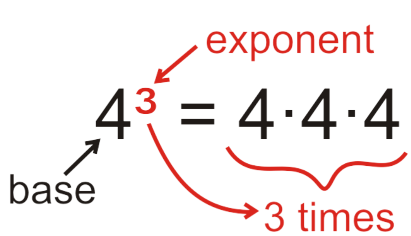
James Sousa: Properties of Exponents

Guidance

To review, the power (or exponent) of a number is the little number in the superscript. The number that is being raised to the power is called the **base**. The **exponent** indicates how many times the base is multiplied by itself.



There are several properties of exponents. We will investigate two in this concept.

Example A

Expand and solve 5^6.

Solution: 5^6 means 5 times itself six times.

$$5^6 = 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$$

Investigation: Product Property

1. Expand $3^4 \cdot 3^5$.

$\underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{3^4} \cdot \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{3^5}$

2. Rewrite this expansion as one power of three.

3^9

3. What is the sum of the exponents?

$4+5=9$

4. Fill in the blank: $a^m \cdot a^n = a^{\_+\_}$

$a^m \cdot a^n = a^{m+n}$

Rather than expand the exponents every time or find the powers separately, we can use this property to simplify the product of two exponents with the same base.

Example B

Simplify:

(a) $x^3 \cdot x^8$

(b) $xy^2x^2y^9$

Solution: Use the Product Property above.

(a) $x^3 \cdot x^8 = x^{3+8} = x^{11}$

(b) If a number does not have an exponent, you may assume the exponent is 1. Reorganize this expression so the x's are together and y's are together.

$$xy^2x^2y^9 = x^1 \cdot x^2 \cdot y^2 \cdot y^9 = x^{1+2} \cdot y^{2+9} = x^3y^{11}$$

Investigation: Quotient Property

1. Expand $2^8 \div 2^3$. Also, rewrite this as a fraction.

$\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2}$

2. Cancel out the common factors and write the answer one power of 2.

$\frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2}} = 2^5$

3. What is the difference of the exponents?

$8-3=5$

4. Fill in the blank: $\frac{a^m}{a^n} = a^{\_-\_}$

$\frac{a^m}{a^n} = a^{m-n}$

Example C

Simplify:

(a) $\frac{5^9}{5^7}$

(b) $\frac{x^4}{x^2}$

(c) $\frac{xy^5}{x^6y^2}$

Solution: Use the Quotient Property from above.

(a) $\frac{5^9}{5^7} = 5^{9-7} = 5^2 = 25$

(b) $\frac{x^4}{x^2} = x^{4-2} = x^2$

(c) $\frac{x^{10}y^5}{x^6y^2} = x^{10-6}y^{5-2} = x^4y^3$

Guided Practice

Simplify the following expressions. Evaluate any numerical answers.

1. $7 \cdot 7^2$
2. $\frac{3^7}{3^3}$
3. $\frac{16x^4y^5}{4x^2y^2}$

Answers

1. $7 \cdot 7^2 = 7^{1+2} = 7^3 = 343$
2. $\frac{3^7}{3^3} = 3^{7-3} = 3^4 = 81$
3. $\frac{16x^4y^5}{4x^2y^2} = 4x^{4-2}y^{5-3} = 4x^2y^2$

Vocabulary

Product of Powers Property
: $a^m \cdot a^n = a^{m+n}$

Quotients of Powers Property
: $\frac{a^m}{a^n} = a^{m-n}; a \neq 0$

Problem Set

Expand the following numbers and evaluate.

1. 2^6
2. 10^3
3. $(-3)^5$
4. $(0.25)^4$

Simplify the following expressions. Evaluate any numerical answers.

5. $4^2 \cdot 4^7$
6. $6 \cdot 6^3 \cdot 6^2$
7. $\frac{8^3}{8}$
8. $\frac{2^4 \cdot 3^5}{2 \cdot 3^2}$

9. $b^6 \cdot b^3$
10. $5^2 x^4 \cdot x^9$
11. $\frac{y^{12}}{y^5}$
12. $\frac{a^8 \cdot b^6}{b \cdot a^4}$
13. $\frac{3^7 x^6}{3^3 x^3}$
14. $d^5 f^3 d^9 f^7$
15. $\frac{2^8 m^{18} n^{14}}{2^5 m^{11} n^4}$
16. $\frac{9^4 p^5 q^8}{9^2 p q^2}$

Investigation Evaluate the powers of negative numbers.

17. Find:
    a. $(-2)^1$
    b. $(-2)^2$
    c. $(-2)^3$
    d. $(-2)^4$
    e. $(-2)^5$
    f. $(-2)^6$
18. Make a conjecture about even vs. odd powers with negative numbers.
19. Is $(-2)^4$ different from -2^4? Why or why not?

Negative and Zero Exponent Properties

Objective

To evaluate and use negative and zero exponents.

Guidance

In this concept, we will introduce negative and zero exponents. First, let's address a zero in the exponent through an investigation.

Investigation: Zero Exponents

1. Evaluate $\frac{5^6}{5^6}$ by using the Quotient of Powers property.

$\frac{5^6}{5^6} = 5^{6-6} = 5^0$

2. What is a number divided by itself? Apply this to #1.

$\frac{5^6}{5^6} = 1$

3. Fill in the blanks. $\frac{a^m}{a^m} = a^{m-m} = a^{-} = \_$

$a^0 = 1$

Investigation: Negative Exponents

1. Expand $\frac{3^2}{3^7}$ and cancel out the common 3's and write your answer with positive exponents.

$\frac{3^2}{3^7} = \frac{\cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{1}{3^5}$

2. Evaluate $\frac{3^2}{3^7}$ by using the Quotient of Powers property.

$\frac{3^2}{3^7} = 3^{2-7} = 3^{-5}$

3. Are the answers from #1 and #2 equal? Write them as a single statement.

$\frac{1}{3^5} = 3^{-5}$

4. Fill in the blanks. $\frac{1}{a^m} = a-$ and $\frac{1}{a^{-m}} = a-$

$\frac{1}{a^m} = a^{-m}$ and $\frac{1}{a^{-m}} = a^m$

From the two investigations above, we have learned two very important properties of exponents. First, anything to the zero power is one. Second, negative exponents indicate placement. If an exponent is negative, it needs to be moved from where it is to the numerator or denominator. We will investigate this property further in the Problem Set.

Example A

Simplify the following expressions. Your answer should only have positive exponents.

(a) $\frac{5^2}{5^5}$

(b) $\frac{x^7yz^{12}}{x^{12}yz^7}$

(c) $\frac{a^4b^0}{a^8b}$

Solution: Use the two properties from above. An easy way to think about where the "leftover" exponents should go, is to look at the fraction and determine which exponent is greater. For example, in b, there are more x's in the denominator, so the leftover should go there.

(a) $\frac{5^2}{5^5} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(b) $\frac{x^7yz^{12}}{x^{12}yz^7} = \frac{y^{1-1}z^{12-7}}{x^{12-7}} = \frac{y^0z^5}{x^5} = \frac{z^5}{x^5}$

(c) $\frac{a^4b^0}{a^8b} = a^{4-8}b^{0-1} = a^{-4}b^{-1} = \frac{1}{a^4b}$

Alternate Method: Part c

$\frac{a^4b^0}{a^8b} = \frac{1}{a^{8-4}b} = \frac{1}{a^4b}$

Example B

Simplify the expressions. Your answer should only have positive exponents.

(a) $\frac{xy^5}{8y^{-3}}$

(b) $\frac{27g^{-7}h^0}{18g}$

Solution: In these expressions, you will need to move the negative exponent to the numerator or denominator and then change it to a positive exponent to evaluate. Also, simplify any numerical fractions.

(a) $\frac{xy^5}{8y^{-3}} = \frac{xy^5y^3}{8} = \frac{xy^{5+3}}{8} = \frac{xy^8}{8}$

(b) $\frac{27g^{-7}h^0}{18g} = \frac{3}{2g^1g^7} = \frac{3}{2g^{1+7}} = \frac{3}{2g^8}$

Example C

Multiply the two fractions together and simplify. Your answer should only have positive exponents.

$$\frac{4x^{-2}y^5}{20x^8} \cdot \frac{-5x^6y}{15y^{-9}}$$

Solution: The easiest way to approach this problem is to multiply the two fractions together first and then simplify.

$$\frac{4x^{-2}y^5}{20x^8} \cdot \frac{-5x^6y}{15y^{-9}} = -\frac{20x^{-2+6}y^{5+1}}{300x^8y^{-9}} = -\frac{x^{-2+6-8}y^{5+1+9}}{15} = -\frac{x^{-4}y^{15}}{15} = -\frac{y^{15}}{15x^4}$$

Guided Practice

Simplify the expressions.

1. $\frac{8^6}{8^9}$
2. $\frac{3x^{10}y^2}{21x^7y^{-4}}$
3. $\frac{2a^8b^{-4}}{16a^{-5}} \cdot \frac{4^3a^{-3}b^0}{a^4b^7}$

Answers

1. $\frac{8^6}{8^9} = 8^{6-9} = \frac{1}{8^3} = \frac{1}{512}$
2. $\frac{3x^{10}y^2}{21x^7y^{-4}} = \frac{x^{10-7}y^{2-(-4)}}{7} = \frac{x^3y^6}{7}$
3. $\frac{2a^8b^{-4}}{16a^{-5}} \cdot \frac{4^3a^{-3}b^0}{a^4b^7} = \frac{128a^{8-3}b^{-4}}{16a^{-5+4}b^7} = \frac{8a^{5+1}}{b^{7+4}} = \frac{8a^6}{b^{11}}$

Vocabulary

Zero Exponent Property
$a^0 = 1, a \neq 0$

Negative Exponent Property
$\frac{1}{a^m} = a^{-m}$ and $\frac{1}{a^{-m}} = a^m, a \neq 0$

Problem Set

Simplify the following expressions. Answers cannot have negative exponents.

1. $\frac{8^2}{8^4}$
2. $\frac{x^6}{x^{15}}$
3. $\frac{7^{-3}}{7^{-2}}$
4. $\frac{y^{-9}}{y^{10}}$
5. $\frac{x^0y^5}{xy^7}$
6. $\frac{a^{-1}b^8}{a^5b^7}$
7. $\frac{14c^{10}d^{-4}}{21c^6d^{-3}}$
8. $\frac{8g^0h}{30g^{-9}h^2}$
9. $\frac{5x^4}{10y^{-2}} \cdot \frac{y^7x}{x^{-1}y}$
10. $\frac{g^9h^5}{6gh^{12}} \cdot \frac{18h^3}{g^8}$
11. $\frac{4a^{10}b^7}{12a^{-6}} \cdot \frac{9a^{-5}b^4}{20a^{11}b^{-8}}$
12. $\frac{-g^8h}{6g^{-8}} \cdot \frac{9g^{15}h^9}{-h^{11}}$
13. Rewrite the following exponential pattern with positive exponents: $5^{-4}, 5^{-3}, 5^{-2}, 5^{-1}, 5^0, 5^1, 5^2, 5^3, 5^4$.
14. Evaluate each term in the pattern from #13.
15. Fill in the blanks.

As the numbers increase, you _______________ the previous term by 5.

As the numbers decrease, you ______________ the previous term by 5.

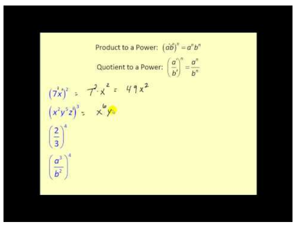
Power Properties

Objective

To discover and use the power properties of exponents.

Watch This

Watch the second part of this video, starting around 3:30.



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James Sousa: Properties of Exponents

Guidance

The last set of properties to explore are the power properties. Let's investigate what happens when a power is raised to another power.

Investigation: Power of a Power Property

1. Rewrite $(2^3)^5$ as 2^3 five times.

$(2^3)^5 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3$

2. Expand each 2^3. How many 2's are there?

$(2^3)^5 = \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} \cdot \underbrace{2 \cdot 2 \cdot 2}_{2^3} = 2^{15}$

3. What is the *product* of the powers?

$3 \cdot 5 = 15$

4. Fill in the blank. $(a^m)^n = a^{\_\cdot\_}$

$(a^m)^n = a^{mn}$

The other two exponent properties are a form of the distributive property.

Power of a Product Property: $(ab)^m = a^m b^m$

Power of a Quotient Property: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example A

Simplify the following.

(a) $(3^4)^2$

(b) $(x^2y)^5$

Solution: Use the new properties from above.

(a) $(3^4)^2 = 3^{4 \cdot 2} = 3^8 = 6561$

(b) $(x^2y)^5 = x^{2 \cdot 5}y^5 = x^{10}y^5$

Example B

Simplify $\left(\frac{3a^{-6}}{2^2a^2}\right)^4$ without negative exponents.

Solution: This example uses the Negative Exponent Property from the previous concept. Distribute the 4^{th} power first and then move the negative power of a from the numerator to the denominator.

$$\left(\frac{3a^{-6}}{2^2a^2}\right)^4 = \frac{3^4a^{-6\cdot 4}}{2^{2\cdot 4}a^{2\cdot 4}} = \frac{81a^{-24}}{2^8a^8} = \frac{81}{256a^{8+24}} = \frac{81}{256a^{32}}$$

Example C

Simplify $\frac{4x^{-3}y^4z^6}{12x^2y} \div \left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2$ without negative exponents.

Solution: This example is definitely as complicated as these types of problems get. Here, all the properties of exponents will be used. Remember that dividing by a fraction is the same as multiplying by its reciprocal.

$$\begin{aligned}
\frac{4x^{-3}y^4z^6}{12x^2y} \div \left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2 &= \frac{4x^{-3}y^4z^6}{12x^2y} \cdot \frac{225x^6z^{-4}}{25x^2y^{-2}} \\
&= \frac{y^3z^6}{3x^5} \cdot \frac{9x^4y^2}{z^4} \\
&= \frac{3x^4y^5z^6}{x^5z^4} \\
&= \frac{3y^5z^2}{x}
\end{aligned}$$

Guided Practice

Simplify the following expressions without negative exponents.

1. $\left(\frac{5a^3}{b^4}\right)^7$
2. $(2x^5)^{-3}(3x^9)^2$
3. $\frac{(5x^2y^{-1})^3}{10y^6} \cdot \left(\frac{16x^8y^5}{4x^7}\right)^{-1}$

Answers

1. Distribute the 7 to every power within the parenthesis.

$$\left(\frac{5a^3}{b^4}\right)^7 = \frac{5^7a^{21}}{b^{28}} = \frac{78,125a^{21}}{b^{28}}$$

2. Distribute the -3 and 2 to their respective parenthesis and then use the properties of negative exponents, quotient and product properties to simplify.

$$(2x^5)^{-3}(3x^9)^2 = 2^{-3}x^{-15}3^2x^{18} = \frac{9x^3}{8}$$

3. Distribute the exponents that are outside the parenthesis and use the other properties of exponents to simplify. Anytime a fraction is raised to the -1 power, it is equal to the reciprocal of that fraction to the first power.

$$\begin{aligned}
\frac{\left(5x^2y^{-1}\right)^3}{10y^6}\cdot\left(\frac{16x^8y^5}{4x^7}\right)^{-1} &= \frac{5^3x^{-6}y^{-3}}{10y^6}\cdot\frac{4x^7}{16x^8y^5}\\
&= \frac{500xy^{-3}}{160x^8y^{11}}\\
&= \frac{25}{8x^7y^{14}}
\end{aligned}$$

Vocabulary

Power of Power Property
$(a^m)^n = a^{mn}$

Power of a Product Property
$(ab)^m = a^m b^m$

Power of a Quotient Property
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Problem Set

Simplify the following expressions without negative exponents.

1. $(2^5)^3$
2. $(3x)^4$
3. $\left(\frac{4}{5}\right)^2$
4. $(6x^3)^3$
5. $\left(\frac{2a^3}{b^5}\right)^7$
6. $(4x^8)^{-2}$
7. $\left(\frac{1}{7^2h^9}\right)^{-1}$
8. $\left(\frac{2x^4y^2}{5x^{-3}y^5}\right)^3$
9. $\left(\frac{9m^5n^{-7}}{27m^6n^5}\right)^{-4}$
10. $\frac{(4x)^2(5y)^{-3}}{(2x^3y^5)^2}$
11. $(5r^6)^4\left(\frac{1}{3}r^{-2}\right)^5$
12. $(4t^{-1}s)^3(2^{-1}ts^{-2})^{-3}$
13. $\frac{6a^2b^4}{18a^{-3}b^4}\cdot\left(\frac{8b^{12}}{40a^{-8}b^5}\right)^2$
14. $\frac{2(x^4y^4)^0}{2^4x^3y^5z}\div\frac{8z^{10}}{32x^{-2}y^5}$
15. $\frac{5g^6}{15g^0h^{-1}}\cdot\left(\frac{h}{9g^{15}j^7}\right)^{-3}$
16. **Challenge** $\frac{a^7b^{10}}{4a^{-5}b^{-2}}\cdot\left[\frac{(6ab^{12})^2}{12a^9b^{-3}}\right]^2\div(3a^5b^{-4})^3$
17. Rewrite 4^3 as a power of 2.
18. Rewrite 9^2 as a power of 3.
19. Solve the equation for x. $3^2\cdot 3^x = 3^8$
20. Solve the equation for x. $(2^x)^4 = 4^8$

14.2 Adding, Subtracting and Multiplying Polynomials

Objective

To add, subtract, and multiply polynomials.

Review Queue

1. Multiply $(3x-1)(x+4)$.
2. Factor x^2-6x+9.
3. Multiply $(2x+5)(2x-11)$.
4. Combine like terms: $3x+15+2x-8-x$

Adding and Subtracting Polynomials

Objective

Adding and subtracting polynomials, as well as learning about the different parts of a polynomial.

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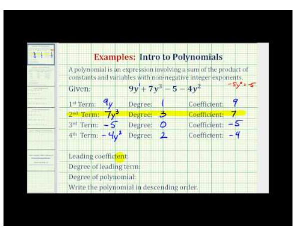

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James Sousa: Ex: Intro to Polynomials in One Variable

Guidance

A **polynomial** is an expression with multiple variable terms, such that the exponents are greater than or equal to zero. All quadratic and linear equations are polynomials. Equations with negative exponents, square roots, or variables in the denominator are not polynomials.

Polynomials	Not Polynomials
$2x^2+6x-9$	$10x^{-1}+6x^2$
$-x^3+9$	$\sqrt{x}-2$
$4x^4+5x^3-8x^2+12x+24$	$\frac{3}{x}+5$

Now that we have established what a polynomial is, there are a few important parts. Just like with a quadratic, a polynomial can have a **constant**, which is a number without a variable. The **degree** of a polynomial is the largest exponent. For example, all quadratic equations have a degree of 2. Lastly, the **leading coefficient** is the coefficient in front of the variable with the degree. In the polynomial $4x^4+5x^3-8x^2+12x+24$ above, the degree is 4 and the leading coefficient is also 4. Make sure that when finding the degree and leading coefficient you have the polynomial in standard form. **Standard form** lists all the variables in order, from greatest to least.

Example A

Rewrite $x^3-5x^2+12x^4+15-8x$ in standard form and find the degree and leading coefficient.

Solution: To rewrite in standard form, put each term in order, from greatest to least, according to the exponent. Always write the constant last.

$$x^3-5x^2+12x^4+15-8x \to 12x^4+x^3-5x^2-8x+15$$

Now, it is easy to see the leading coefficient, 12, and the degree, 4.

Example B

Simplify $(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)$

Solution: To add or subtract two polynomials, combine like terms. **Like terms** are any terms where the exponents of the variable are the same. We will regroup the polynomial to show the like terms.

$$\begin{aligned}
&(4x^3-2x^2+4x+15)+(x^4-8x^3-9x-6)\\
&x^4+(4x^3-8x^3)-2x^2+(4x-9x)+(15-6)\\
&x^4-4x^3-2x^2-5x+9
\end{aligned}$$

Example C

Simplify $(2x^3+x^2-6x-7)-(5x^3-3x^2+10x-12)$

Solution: When subtracting, distribute the negative sign to every term in the second polynomial, then combine like terms.

$$\begin{aligned}
&(2x^3+x^2-6x-7)-(5x^3-3x^2+10x-12)\\
&2x^3+x^2-6x-7-5x^3+3x^2-10x+12\\
&(2x^3-5x^3)+(x^2+3x^2)+(-6x-10x)+(-7+12)\\
&-3x^3+4x^2-16x+5
\end{aligned}$$

Guided Practice

1. Is $\sqrt{2x^3-5x}+6$ a polynomial? Why or why not?

2. Find the leading coefficient and degree of $6x^2-3x^5+16x^4+10x-24$.

Add or subtract.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)$

Answers

1. No, this is not a polynomial because x is under a square root in the equation.

2. In standard form, this polynomial is $-3x^5+16x^4+6x^2+10x-24$. Therefore, the degree is 5 and the leading coefficient is -3.

3. $(9x^2+4x^3-15x+22)+(6x^3-4x^2+8x-14)=10x^3+5x^2-7x+8$

4. $(7x^3+20x-3)-(x^3-2x^2+14x-18)=6x^3+2x^2+6x+15$

Vocabulary

Polynomial
An expression with multiple variable terms, such that the exponents are greater than or equal to zero.

Constant
A number without a variable in a mathematical expression.

Degree(of a polynomial)
The largest exponent in a polynomial.

Leading coefficient
The coefficient in front of the variable with the degree.

Standard form
Lists all the variables in order, from greatest to least.

Like terms
Any terms where the exponents of the variable are the same.

Problem Set

Determine if the following expressions are polynomials. If not, state why. If so, write in standard form and find the degree and leading coefficient.

1. $\frac{1}{x^2}+x+5$
2. $x^3+8x^4-15x+14x^2-20$
3. x^3+8
4. $5x^{-2}+9x^{-1}+16$
5. $x^2\sqrt{2}-x\sqrt{6}+10$
6. $\frac{x^4+8x^2+12}{3}$
7. $\frac{x^2-4}{x}$
8. $-6x^3+7x^5-10x^6+19x^2-3x+41$

Add or subtract the following polynomials.

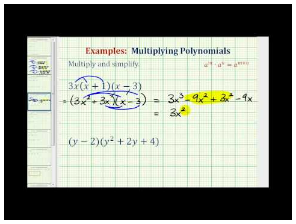
9. $(x^3+8x^2-15x+11)+(3x^3-5x^2-4x+9)$
10. $(-2x^4+x^3+12x^2+6x-18)-(4x^4-7x^3+14x^2+18x-25)$
11. $(10x^3-x^2+6x+3)+(x^4-3x^3+8x^2-9x+16)$
12. $(7x^3-2x^2+4x-5)-(6x^4+10x^3+x^2+4x-1)$
13. $(15x^2+x-27)+(3x^3-12x+16)$
14. $(2x^5-3x^4+21x^2+11x-32)-(x^4-3x^3-9x^2+14x-15)$
15. $(8x^3-13x^2+24)-(x^3+4x^2-2x+17)+(5x^2+18x-19)$

Multiplying Polynomials

Objective

To multiply together several different types of polynomials.

Watch This



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James Sousa: Ex: Polynomial Multiplication Involving Binomials and Trinomials

Guidance

Multiplying together polynomials is very similar to multiplying together factors. You can FOIL or we will also present an alternative method. When multiplying together polynomials, you will need to use the properties of exponents, primarily the Product Property $(a^m \cdot a^n = a^{m+n})$ and combine like terms.

Example A

Find the product of $(x^2-5)(x^3+2x-9)$.

Solution: Using the FOIL method, you need be careful. First, take the x^2 in the first polynomial and multiply it by every term in the second polynomial.

$$(x^2-5)(x^3+2x-9) = x^5+2x^3-9x^2$$

Now, multiply the -5 and multiply it by every term in the second polynomial.

$$(x^2-5)(x^3+2x-9) = x^5+2x^3-9x^2-5x^3-10x+45$$

Lastly, combine any like terms. In this example, only the x^3 terms can be combined.

$$\begin{aligned}(x^2-5)(x^3+2x-9) &= x^5 \boxed{+2x^3} -9x^2 \boxed{-5x^3} -10x+45 \\ &= x^5-3x^3-9x^2-10x+45\end{aligned}$$

Example B

Multiply $(x^2+4x-7)(x^3-8x^2+6x-11)$.

Solution: In this example, we will use the "box" method. Align the two polynomials along the top and left side of a rectangle and make a row or column for each term. Write the polynomial with more terms along the top of the rectangle.

	x^3	$-8x^2$	$+6x$	-11
x^2				
$4x$				
-7				

Multiply each term together and fill in the corresponding spot.

	x^3	$-8x^2$	$+6x$	-11
x^2	x^5	$-8x^4$	$6x^3$	$-11x^2$
$4x$	$4x^4$	$-32x^3$	$24x^2$	$-44x$
-7	$-7x^3$	$56x^2$	$-42x$	77

Finally, combine like terms. The final answer is $x^5-4x^4-33x^3+69x^2-86x+77$. This method presents an alternative way to organize the terms. Use whichever method you are more comfortable with. Keep in mind, no matter which method you use, you will multiply every term in the first polynomial by every term in the second.

Example C

Find the product of $(x-5)(2x+3)(x^2+4)$.

Solution: In this example we have three binomials. When multiplying three polynomials, start by multiplying the first two binomials together.

$$\begin{aligned}(x-5)(2x+3) &= 2x^2+3x-10x-15\\ &= 2x^2-7x-15\end{aligned}$$

Now, multiply the answer by the last binomial.

$$\begin{aligned}(2x^2-7x-15)(x^2+4) &= 2x^4+8x^2-7x^3-28x-15x^2-60\\ &= 2x^4-7x^3-7x^2-28x-60\end{aligned}$$

Guided Practice

Find the product of the polynomials.

1. $-2x^2(3x^3-4x^2+12x-9)$
2. $(4x^2-6x+11)(-3x^3+x^2+8x-10)$
3. $(x^2-1)(3x-4)(3x+4)$

4. $(2x-7)^2$

Answers

1. Use the distributive property to multiply $-2x^2$ by the polynomial.

$$-2x^2(3x^3-4x^2+12x-9) = -6x^5+8x^4-24x^3+18x^2$$

2. Multiply each term in the first polynomial by each one in the second polynomial.

$$\begin{aligned}(4x^2-6x+11)(-3x^3+x^2+8x-10) &= -12x^5+4x^4+32x^3-40x^2\\ &\quad +18x^4-6x^3-48x^2+60x\\ &\quad -33x^3+11x^2+88x-110\\ &= -12x^5+22x^4-7x^3-77x^2+148x-110\end{aligned}$$

3. Multiply the first two binomials together.

$$(x^2-1)(3x-4) = 3x^3-4x^2-3x+4$$

Multiply this product by the last binomial.

$$\begin{aligned}(3x^3-4x^2-3x+4)(3x+4) &= 9x^4+12x^3-12x^3-16x^2-9x^2-12x+12x-16\\ &= 9x^4-25x^2-16\end{aligned}$$

4. The square indicates that there are two binomials. Expand this and multiply.

$$\begin{aligned}(2x-7)^2 &= (2x-7)(2x-7)\\ &= 4x^2-14x-14x+49\\ &= 4x^2-28x+49\end{aligned}$$

Problem Set

Find the product.

1. $5x(x^2-6x+8)$
2. $-x^2(8x^3-11x+20)$
3. $7x^3(3x^3-x^2+16x+10)$
4. $(x^2+4)(x-5)$
5. $(3x^2-4)(2x-7)$
6. $(9-x^2)(x+2)$
7. $(x^2+1)(x^2-2x-1)$
8. $(5x-1)(x^3+8x-12)$
9. $(x^2-6x-7)(3x^2-7x+15)$
10. $(x-1)(2x-5)(x+8)$

11. $(2x^2+5)(x^2-2)(x+4)$
12. $(5x-12)^2$
13. $-x^4(2x+11)(3x^2-1)$
14. $(4x+9)^2$
15. $(4x^3-x^2-3)(2x^2-x+6)$
16. $(2x^3-6x^2+x+7)(5x^2+2x-4)$
17. $(x^3+x^2-4x+15)(x^2-5x-6)$

14.3 Factoring and Solving Polynomial Equations

Objective

To solve and factor polynomials using several different methods.

Review Queue

Factor the following quadratics.

1. $x^2 - 9x - 22$

2. $4x^2 - 25$

3. $6x^2 + 7x - 5$

4. Set #3 equal to zero and solve.

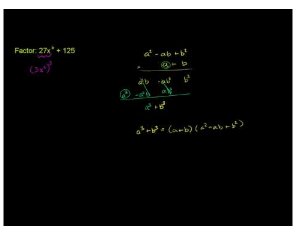
Sum and Difference of Cubes

Objective

To learn the sum and difference of cubes formulas for factoring certain types of polynomials.

Watch This

First watch this video.



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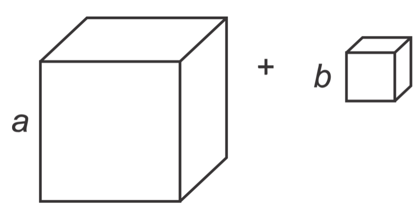
Khan Academy: Factoring Sum of Cubes

Guidance

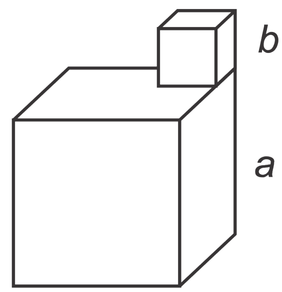
In the previous chapter, you learned how to factor several different types of quadratic equations. Here, we will expand this knowledge to certain types of polynomials. The first is the sum of cubes. The sum of cubes is what it sounds like, the sum of two cube numbers or $a^3 + b^3$. We will use an investigation involving volume to find the factorization of this polynomial.

Investigation: Sum of Cubes Formula

1. Pictorially, the sum of cubes looks like this:



Or, we can put one on top of the other.

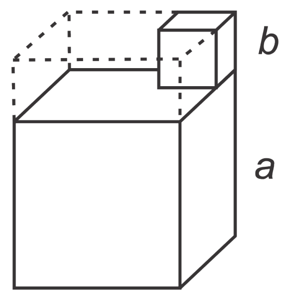


2. Recall that the formula for volume is $length \times width \times depth$. Find the volume of the sum of these two cubes.

$V = a^3 + b^3$

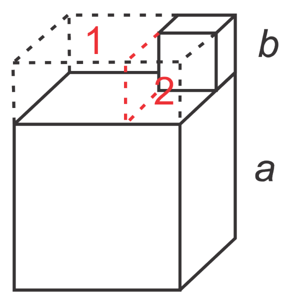
3. Now, we will find the volume in a different way. Using the second picture above, will add in imaginary lines so that these two cubes look like one large prism. Find the volume of this prism.

$$\begin{aligned} V &= a \times a \times (a+b) \\ &= a^2(a+b) \end{aligned}$$



4. Subtract the imaginary portion on top. In the picture, they are prism 1 and prism 2.

$$V = a^2(a+b) - \left[\underbrace{ab(a-b)}_{Prism\ 1} + \underbrace{b^2(a-b)}_{Prism\ 2}\right]$$



5. Pull out any common factors within the brackets.

$V = a^2(a+b) - b(a-b)[a+b]$

6. Notice that both terms have a common factor of $(a+b)$. Pull this out, put it in front, and get rid of the brackets.

$V = (a+b)(a^2 - b(a-b))$

7. Simplify what is inside the second set of parenthesis.

$V = (a+b)(a^2 - ab + b^2)$

In the last step, we found that $a^3 + b^3$ factors to $(a+b)(a^2 - ab + b^2)$. This is the **Sum of Cubes Formula**.

Example A

Factor $8x^3 + 27$.

Solution: First, determine if these are "cube" numbers. A cube number has a cube root. For example, the cube root of 8 is 2 because $2^3 = 8$. $3^3 = 27, 4^3 = 64, 5^3 = 125$, and so on.

$$\begin{aligned} a^3 &= 8x^3 = (2x)^3 & b^3 &= 27 = 3^3 \\ a &= 2x & b &= 3 \end{aligned}$$

In the formula, we have:

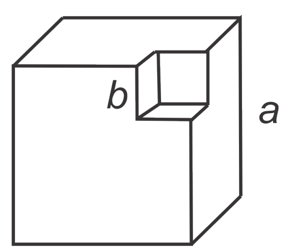
$$\begin{aligned} (a+b)(a^2 - ab + b^2) &= (2x+3)((2x)^2 - (2x)(3) + 3^2) \\ &= (2x+3)(4x^2 - 6x + 9) \end{aligned}$$

Therefore, $8x^3 + 27 = (2x+3)(4x^2 - 6x + 9)$. The second factored polynomial does not factor any further.

Investigation: Difference of Cubes

1. Pictorially, the difference of cubes looks like this:

Imagine the smaller cube is taken out of the larger cube.

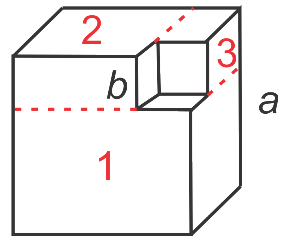


2. Recall that the formula for volume is $length \times width \times depth$. Find the volume of the difference of these two cubes.

$V = a^3 - b^3$

3. Now, we will find the volume in a different way. Using the picture here, will add in imaginary lines so that the shape is split into three prisms. Find the volume of prism 1, prism 2, and prism 3.

$$\begin{aligned} &\text{Prism 1} : a \cdot a \cdot (a-b) \\ &\text{Prism 2} : a \cdot b \cdot (a-b) \\ &\text{Prism 3} : b \cdot b \cdot (a-b) \end{aligned}$$



4. Add the volumes together to get the volume of the entire shape.

$V = a^2(a-b)+ab(a-b)+b^2(a-b)$

5. Pull out any common factors and simplify.

$V = (a-b)(a^2+ab+b^2)$

In the last step, we found that a^3-b^3 factors to $(a-b)(a^2+ab+b^2)$. This is the **Difference of Cubes Formula**.

Example B

Factor x^5-125x^2.

Solution: First, take out any common factors.

$$x^5-125x^2 = x^2(x^3-125)$$

What is inside the parenthesis is a difference of cubes. Use the formula.

$$\begin{aligned} x^5-125x^2 &= x^2(x^3-125) \\ &= x^2(x^3-5^3) \\ &= x^2(x-5)(x^2+5x+25) \end{aligned}$$

Example C

Find the real-number solutions of $x^3-8=0$.

Solution: Factor using the difference of cubes.

$$\begin{aligned} x^3-8 &= 0 \\ (x-2)(x^2+2x+4) &= 0 \\ x &= 2 \end{aligned}$$

In the last step, we set the first factor equal to zero. The second factor, x^2+2x+4, will give imaginary solutions. For both the sum and difference of cubes, this will always happen.

Guided Practice

Factor using the sum or difference of cubes.

1. x^3-1

2. $3x^3+192$

3. $125-216x^3$

4. Find the real-number solution to $27x^3+8=0$.

Answers

1. Factor using the difference of cubes.

$$\begin{aligned} x^3-1 &= x^3-1^3 \\ &= (x-1)(x^2+x+1) \end{aligned}$$

2. Pull out the 3, then factor using the sum of cubes.

$$\begin{aligned} 3x^3+192 &= 3(x^3+64) \\ &= 3(x^3+4^3) \\ &= 3(x+4)(x^2-4x+16) \end{aligned}$$

3. Factor using the difference of cubes.

$$\begin{aligned} 125-216x^3 &= 5^3-(6x)^3 \\ &= (5-6x)(5^2+(5)(6x)+(6x)^2) \\ &= (5-6x)(25+30x+36x^2) \end{aligned}$$

4. Factor using the sum of cubes and then solve.

$$\begin{aligned} 27x^3+8 &= 0 \\ (3x)^3+2^3 &= 0 \\ (3x+2)(9x^2-6x+4) &= 0 \\ x &= -\frac{2}{3} \end{aligned}$$

Vocabulary

Sum of Cubes Formula
$a^3+b^3=(a+b)(a^2-ab+b^2)$

Difference of Cubes Formula
$a^3-b^3=(a-b)(a^2+ab+b^2)$

Problem Set

Factor each polynomial by using the sum or difference of cubes.

1. x^3-27
2. $64+x^3$
3. $32x^3-4$
4. $64x^3+343$
5. $512-729x^3$
6. $125x^4+8x$

7. $648x^3 + 81$
8. $5x^6 - 135x^3$
9. $686x^7 - 1024x^4$

Find the real-number solutions for each equation.

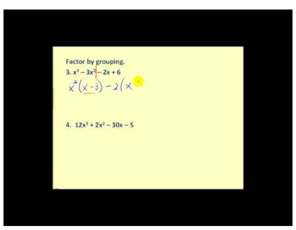
10. $125x^3 + 1 = 0$
11. $64 - 729x^3 = 0$
12. $8x^4 - 343x = 0$
13. **Challenge** Find ALL solutions (real and imaginary) for $5x^5 + 625x^2 = 0$.
14. **Challenge** Find ALL solutions (real and imaginary) for $686x^3 + 2000 = 0$.
15. **Real Life Application** You have a piece of cardboard that you would like to fold up and make an open (no top) box out of. The dimensions of the cardboard are $36'' \times 42''$. Write a factored equation for the volume of this box. Find the volume of the box when $x = 1, 3$, and 5.

Factoring by Grouping

Objective

To factor and solve certain polynomials by grouping.

Watch This



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James Sousa: Factor By Grouping

Guidance

In the *Factoring when* $a \neq 1$ concept (in the previous chapter), we introduced factoring by grouping. We will expand this idea to other polynomials here.

Example A

Factor $x^4 + 7x^3 - 8x - 56$ by grouping.

Solution: First, group the first two and last two terms together. Pull out any common factors.

$$\underbrace{x^4 + 7x^3}_{x^3(x+7)}\underbrace{-8x - 56}_{-8(x+7)}$$

Notice what is inside the parenthesis is *the same*. This should always happen when factoring by grouping. Pull out this common factor.

$$x^3(x+7) - 8(x+7)$$
$$(x+7)(x^3 - 8)$$

Look at the factors. Can they be factored any further? Yes. The second factor is a difference of cubes. Use the formula.

$$(x+7)(x^3-8)$$
$$(x+7)(x-2)(x^2+2x+4)$$

Example B

Factor x^3+5x^2-x-5 by grouping.

Solution: Follow the steps from above.

$$x^3+5x^2-x-5$$
$$x^2(x+5)-1(x+5)$$
$$(x+5)(x^2-1)$$

Look to see if we can factor either factor further. Yes, the second factor is a difference of squares.

$$(x+5)(x^2-1)$$
$$(x+5)(x-1)(x+1)$$

Example C

Find all real-number solutions of $2x^3-3x^2+8x-12=0$.

Solution: Follow the steps from Example A.

$$\begin{aligned}2x^3-3x^2+8x-12&=0\\x^2(2x-3)+4(2x-3)&=0\\(2x-3)(x^2+4)&=0\end{aligned}$$

Now, determine if you can factor further. No, x^2+4 is a sum of squares and not factorable. Setting the first factor equal to zero, we get $x=\frac{3}{2}$.

Guided Practice

Factor the following polynomials by grouping.

1. $x^3+7x^2-2x-14$

2. $2x^4-5x^3+2x-5$

3. Find all the real-number solutions of $4x^3-8x^2-x+2=0$.

Answers

Each of these problems is done in the same way: Group the first two and last two terms together, pull out any common factors, what is inside the parenthesis is the same, factor it out, then determine if either factor can be factored further.

1.

$$\begin{aligned}&x^3+7x^2-2x-14\\&x^2(x+7)-2(x+7)\\&(x+7)(x^2-2)\end{aligned}$$

x^2-2 is not a difference of squares because 2 is not a square number. Therefore, this cannot be factored further.

2.

$$\begin{aligned}&2x^4-5x^3+2x-5\\&x^3(2x-5)+1(2x-5)\\&(2x-5)(x^3+1)\qquad\text{Sum of cubes, factor further.}\\&(2x-5)(x+1)(x^2+x+1)\end{aligned}$$

3. Factor by grouping.

$$\begin{aligned}4x^3-8x^2-x+2&=0\\4x^2(x-2)-1(x-2)&=0\\(x-2)(4x^2-1)&=0\\(x-2)(2x-1)(2x+1)&=0\\x&=2,\frac{1}{2},-\frac{1}{2}\end{aligned}$$

Problem Set

Factor the following polynomials using factoring by grouping. Factor each polynomial completely.

1. $x^3-4x^2+3x-12$
2. $x^3+6x^2-9x-54$
3. $3x^3-4x^2+15x-20$
4. $2x^4-3x^3-16x+24$
5. $4x^3+4x^2-25x-25$
6. $4x^3+18x^2-10x-45$
7. $24x^4-40x^3+81x-135$
8. $15x^3+6x^2-10x-4$
9. $4x^3+5x^2-100x-125$

Find all the real-number solutions of the polynomials below.

10. $9x^3-54x^2-4x+24=0$
11. $x^4+3x^3-27x-81=0$
12. $x^3-2x^2-4x+8=0$
13. **Challenge** Find ALL the solutions of $x^6-9x^4-x^2+9=0$.
14. **Challenge** Find ALL the solutions of $x^3+3x^2+16x+48=0$.

Factoring Polynomials in Quadratic Form

Objective

To factor and solve polynomials that are in "quadratic form."

Guidance

The last type of factorable polynomial are those that are in quadratic form. **Quadratic form** is when a polynomial looks like a trinomial or binomial and can be factored like a quadratic. One example is when a polynomial is in the form ax^4+bx^2+c. Another possibility is something similar to the difference of squares, a^4-b^4. This can be factored to $(a^2-b^2)(a^2+b^2)$ or $(a-b)(a+b)(a^2+b^2)$. Always keep in mind that the greatest common factors should be factored out first.

Example A

Factor $2x^4-x^2-15$.

Solution: This particular polynomial is factorable. Let's use the method we learned in the *Factoring when* $a \neq 1$ concept. First, $ac=-30$. The factors of -30 that add up to -1 are -6 and 5. Expand the middle term and then use factoring by grouping.

$$\begin{aligned}
&2x^4-x^2-15\\
&2x^4-6x^2+5x^2-15\\
&2x^2(x^2-3)+5(x^2-3)\\
&(x^2-3)(2x^2+5)
\end{aligned}$$

Both of the factors are not factorable, so we are done.

Example B

Factor $81x^4-16$.

Solution: Treat this polynomial equation like a difference of squares.

$$\begin{aligned}
&81x^4-16\\
&(9x^2-4)(9x^2+4)
\end{aligned}$$

Now, we can factor $9x^2-4$ using the difference of squares a second time.

$$(3x-2)(3x+2)(9x^2+4)$$

$9x^2+4$ cannot be factored because it is a sum of squares. This will have imaginary solutions.

Example C

Find all the real-number solutions of $6x^5-51x^3-27x=0$.

Solution: First, pull out the GCF among the three terms.

$$\begin{aligned}
6x^5-51x^3-27x&=0\\
3x(2x^4-17x^2-9)&=0
\end{aligned}$$

Factor what is inside the parenthesis like a quadratic equation. $ac=-18$ and the factors of -18 that add up to -17 are -18 and 1. Expand the middle term and then use factoring by grouping.

$$\begin{aligned}
6x^5 - 51x^3 - 27x &= 0 \\
3x(2x^4 - 17x^2 - 9) &= 0 \\
3x(2x^4 - 18x^2 + x^2 - 9) &= 0 \\
3x[2x^2(x^2 - 9) + 1(x^2 - 9)] &= 0 \\
3x(x^2 - 9)(2x^2 + 1) &= 0
\end{aligned}$$

Factor $x^2 - 9$ further and solve for x where possible. $2x^2 + 1$ is not factorable.

$$\begin{aligned}
3x(x^2 - 9)(2x^2 + 1) &= 0 \\
3x(x-3)(x+3)(2x^2 + 1) &= 0 \\
x &= -3, 0, 3
\end{aligned}$$

Guided Practice

Factor the following polynomials.

1. $3x^4 + 14x^2 + 8$

2. $36x^4 - 25$

3. Find all the real-number solutions of $8x^5 + 26x^3 - 24x = 0$.

Answers

1. $ac = 24$ and the factors of 24 that add up to 14 are 12 and 2.

$$\begin{aligned}
&3x^4 + 14x^2 + 8 \\
&3x^4 + 12x^2 + 2x^2 + 8 \\
&3x^2(x^2 + 4) + 2(x^4 + 4) \\
&(x^2 + 4)(3x^2 + 2)
\end{aligned}$$

2. Factor this polynomial like a difference of squares.

$$\begin{aligned}
&36x^4 - 25 \\
&(6x^2 - 5)(6x^2 + 5)
\end{aligned}$$

6 and 5 are not square numbers, so this cannot be factored further.

3. Pull out a $2x$ from each term.

$$\begin{aligned}
8x^5 + 26x^3 - 24x &= 0 \\
2x(4x^4 + 13x - 12) &= 0 \\
2x(4x^4 + 16x^2 - 3x^2 - 12) &= 0 \\
2x[4x^2(x^2 + 4) - 3(x^2 + 4)] &= 0 \\
2x(x^2 + 4)(4x^2 - 3) &= 0
\end{aligned}$$

Set each factor equal to zero.

$$\begin{array}{ccccc} & & & & 4x^2-3=0 \\ 2x=0 & x^2+4=0 & & & \\ & & and & & x^2=\frac{3}{4} \\ x=0 & x^2=-4 & & & \\ & & & & x=\pm\frac{\sqrt{3}}{2} \end{array}$$

Notice the second factor will give imaginary solutions.

Vocabulary

Quadratic form
When a polynomial looks a trinomial or binomial and can be factored like a quadratic equation.

Problem Set

Factor the following quadratics completely.

1. x^4-6x^2+8
2. x^4-4x^2-45
3. $4x^4-11x^2-3$
4. $6x^4+19x^2+8$
5. x^4-81
6. $16x^4-1$
7. $6x^5+26x^3-20x$
8. $4x^6-36x^2$
9. $625-81x^4$

Find all the real-number solutions to the polynomials below.

10. $2x^4-5x^2-12=0$
11. $16x^4-49=0$
12. $12x^6+69x^4+45x^2=0$

14.4 Dividing Polynomials

Objective

To divide one polynomial by another using long or synthetic division.

Review Queue

Divide the following numbers by hand.

1. $60 \div 4$
2. $\sqrt[18]{1512}$
3. $825 \div 5$
4. $\sqrt[7.6]{3214.8}$

Long Division

Objective

To use long division to divide polynomials.

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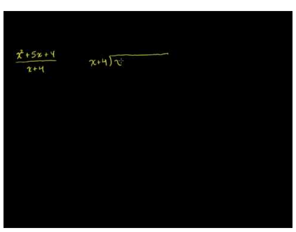

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Khan Academy: Polynomial Division

Guidance

Even though it does not seem like it, factoring is a form of division. Each factor goes into the larger polynomial evenly, without a remainder. For example, take the polynomial $2x^3-3x^2-8x+12$. If we use factoring by grouping, we find that the factors are $(2x-3)(x-2)(x+2)$. If we multiply these three factors together, we will get the original polynomial. So, if we divide by $2x-3$, we should get x^2-4.

$$2x-3\ \overline{)\ 2x^3-3x^2-8x+12}$$

How many times does $2x$ go into $2x^3$? x^2 times.

$$\begin{array}{rl}
 & x^2 \\
2x-3 & \overline{)\,2x^3 -3x^2 -8x+12} \\
 & \underline{\;2x^3 -3x^2} \\
 & \qquad\quad 0
\end{array}$$

Place x^2 above the x^2 term in the polynomial.

Multiply x^2 by both terms in the **divisor** ($2x$ and -3) and place them until their like terms. *Subtract* from the **dividend** ($2x^3-3x^2-8x+12$). Pull down the next two terms and repeat.

$$\begin{array}{rl}
 & x^2 \qquad\quad -4 \\
2x-3 & \overline{)\,2x^3 -3x^2 -8x+12} \\
 & \underline{\;2x^3 -3x^2} \\
 & \qquad\qquad\; -8x+12 \\
 & \qquad\qquad\; \underline{-8x+12}
\end{array}$$

$-8x$ divided by $2x = -4$

After multiplying both terms in the divisor by -4, place that under the terms you brought down. When subtracting we notice that everything cancels out. Therefore, just like we thought, x^2-4 is a factor.

When dividing polynomials, not every divisor will go in evenly to the dividend. If there is a remainder, write it as a fraction over the divisor.

Example A

$(2x^3-6x^2+5x-20) \div (x^2-5)$

Solution: Set up the problem using a long division bar.

$$x^2-5 \overline{)\,2x^3 -6x^2 +5x -20}$$

How many times does x^2 go into $2x^3$? $2x$ times.

$$\begin{array}{rl}
 & 2x \\
x^2-5 & \overline{)\,2x^3 \;\; -6x^2 +5x -20} \\
 & \underline{\;2x^3 -10x^2} \\
 & \qquad\quad 4x^2 +5x -20
\end{array}$$

Multiply $2x$ by the divisor. *Subtract* that from the dividend.

Repeat the previous steps. Now, how many times does x^2 go into $4x^2$? 4 times.

$$\begin{array}{rl}
 & 2x \quad +4 \\
x^2-5 & \overline{)\,2x^3 \;\; -6x^2 +5x -20} \\
 & \underline{\;2x^3 -10x^2} \\
 & \qquad\quad 4x^2 +5x -20 \\
 & \qquad\quad \underline{4x^2 \qquad\; -20} \\
 & \qquad\qquad\quad 5x
\end{array}$$

At this point, we are done. x^2 cannot go into $5x$ because it has a higher degree. Therefore, $5x$ is a remainder. The complete answer would be $2x+4+\frac{5x}{x^2-5}$.

Example B

$(3x^4+x^3-17x^2+19x-6)\div(x^2-2x+1)$. Determine if x^2-2x+1 goes evenly into $3x^4+x^3-17x^2+19x-6$. If so, try to factor the divisor and quotient further.

Solution: First, do the long division. If x^2-2x+1 goes in evenly, then the remainder will be zero.

$$
\begin{array}{rrrrrr}
 & 3x^2 & +7x & -6 & & \\ \cline{2-6}
x^2-2x+1 \,) & 3x^4 & +x^3 & -17x^2 & +19x & -6 \\
 & 3x^4 & -6x^3 & 3x^2 & & \\ \cline{2-6}
 & & 7x^3 & -20x^2 & +19x & \\
 & & 7x^3 & -14x^2 & +7x & \\ \cline{3-6}
 & & & -6x^2 & +12x & -6 \\
 & & & -6x^2 & +12x & -6 \\ \cline{4-6}
 & & & & & 0
\end{array}
$$

This means that x^2-2x+1 and $3x^2+7x-6$ both go evenly into $3x^4+x^3-17x^2+19x-6$. Let's see if we can factor either x^2-2x+1 or $3x^2+7x-6$ further.

$x^2-2x+1=(x-1)(x-1)$ and $3x^2+7x-6=(3x-2)(x+3)$.

Therefore, $3x^4+x^3-17x^2+19x-6=(x-1)(x-1)(x+3)(3x-2)$. You can multiply these to check the work. A binomial with a degree of one is a **factor** of a larger polynomial, $f(x)$, if it goes evenly into it. In this example, $(x-1)(x-1)(x+3)$ and $(3x-2)$ are all factors of $3x^4+x^3-17x^2+19x-6$. We can also say that 1, 1, -3, and $\frac{2}{3}$ are all solutions of $3x^4+x^3-17x^2+19x-6$.

Factor Theorem: A polynomial, $f(x)$, has a factor, $(x-k)$, if and only if $f(k)=0$.

In other words, if k is a **solution** or a **zero**, then the factor, $(x-k)$ divides evenly into $f(x)$.

Example C

Determine if 5 is a solution of $x^3+6x^2-8x+15$.

Solution: To see if 5 is a solution, we need to divide the factor into $x^3+6x^2-8x+15$. The factor that corresponds with 5 is $(x-5)$.

$$
\begin{array}{rrrrr}
 & x^2 & +11x & +5 & \\ \cline{2-5}
x-5 \,) & x^3 & +6x^2 & -50x & +15 \\
 & x^3 & -5x^2 & & \\ \cline{2-5}
 & & 11x^2 & -50x & \\
 & & 11x^2 & -55x & \\ \cline{3-5}
 & & & 5x & +15 \\
 & & & 5x & -25 \\ \cline{4-5}
 & & & & 40
\end{array}
$$

Since there is a remainder, 5 is not a solution.

Guided Practice

1. $(5x^4+6x^3-12x^2-3)\div(x^2+3)$

2. Is $(x+4)$ a factor of $x^3-2x^2-51x-108$? If so, find any other factors.

3. What are the real-number solutions to #2?

4. Determine if 6 is a solution to $2x^3-9x^2-12x-24$.

Answers

1. Make sure to put a placeholder in for the $x-$term.

$$
\begin{array}{rrrrrr}
 & 5x^2 & +6x & -27 & & \\ \cline{2-6}
x^2+3\,) & 5x^4 & +6x^3 & -12x^2 & \color{red}{+0x} & -3 \\
 & 5x^4 & & +15x^2 & & \\ \cline{2-6}
 & & 6x^3 & -27x^2 & \color{red}{+0x} & \\
 & & 6x^3 & & +18x & \\ \cline{3-6}
 & & & -27x^2 & -18x & -3 \\
 & & & -27x^2 & & -81 \\ \cline{4-6}
 & & & & -18x & +78
\end{array}
$$

The final answer is $5x^2+6x-27-\frac{18x-78}{x^2+3}$.

2. Divide $(x+4)$ into $x^3-2x^2-51x-108$ and if the remainder is zero, it is a factor.

$$
\begin{array}{rrrrr}
 & x^2 & -6x & -27 & \\ \cline{2-5}
x+4\,) & x^3 & -2x^2 & -51x & -108 \\
 & x^3 & +4x^2 & & \\ \cline{2-5}
 & & -6x^2 & -51x & \\
 & & -6x^2 & -24x & \\ \cline{3-5}
 & & & -27x & -108 \\
 & & & -27x & -108 \\ \cline{4-5}
 & & & & 0
\end{array}
$$

$x+4$ is a factor. Let's see if $x^2-6x-27$ factors further. Yes, the factors of -27 that add up to -6 are -9 and 3. Therefore, the factors of $x^3-2x^2-51x-108$ are $(x+4),(x-9)$, and $(x+3)$.

3. The solutions would be -4, 9, and 3; the opposite sign of each factor.

4. To see if 6 is a solution, we need to divide $(x-6)$ into $2x^3-9x^2-12x-24$.

$$
\begin{array}{rrrrr}
 & 2x^2 & +3x & +6 & \\ \cline{2-5}
x-6\,) & 2x^3 & -9x^2 & -12x & -24 \\
 & 2x^3 & -12x^2 & & \\ \cline{2-5}
 & & 3x^2 & -12x & \\
 & & 3x^2 & -18x & \\ \cline{3-5}
 & & & 6x & -24 \\
 & & & 6x & -36 \\ \cline{4-5}
 & & & & 12
\end{array}
$$

Because the remainder is not zero, 6 is not a solution.

Vocabulary

Long division (of polynomial)
The process of dividing polynomials where the divisor has two or more terms.

Divisor
The polynomial that divides into another polynomial.

Dividend
The polynomial that the divisor goes into. The polynomial under the division bar.

Quotient
The answer to a division problem.

Problem Set

Divide the following polynomials using long division.

1. $(2x^3+5x^2-7x-6)\div(x+1)$
2. $(x^4-10x^3+15x-30)\div(x-5)$
3. $(2x^4-8x^3+4x^2-11x-1)\div(x^2-1)$
4. $(3x^3-4x^2+5x-2)\div(3x+2)$
5. $(3x^4-5x^3-21x^2-30x+8)\div(x-4)$
6. $(2x^5-5x^3+6x^2-15x+20)\div(2x^2+3)$

Determine all the real-number solutions to the following polynomials, given one zero.

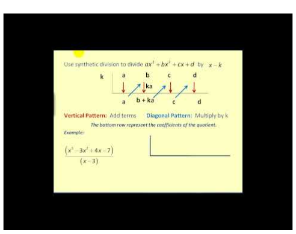
7. $x^3-9x^2+27x-15; -5$
8. $6x^3-37x^2+5x+6; 6$
9. Find a polynomial with the zeros 4, -2, and $\frac{3}{2}$.
10. **Challenge** Find *two* polynomials with the zeros 8, 5, 1, and -1.

Synthetic Division

Objective

Use synthetic division as a short-cut and alternative to long division (in certain cases) and to find zeros.

Watch This



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James Sousa: Polynomial Division: Synthetic Division

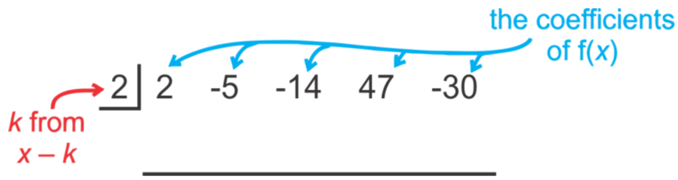
Guidance

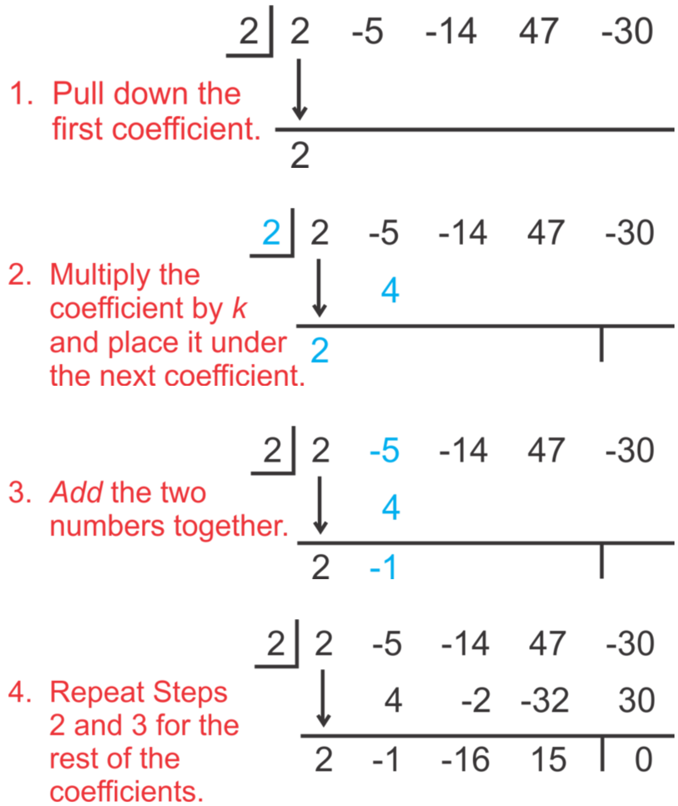
Synthetic division is an alternative to long division from the previous concept. It can also be used to divide a polynomial by a possible factor, $x-k$. However, synthetic division cannot be used to divide larger polynomials, like quadratics, into another polynomial.

Example A

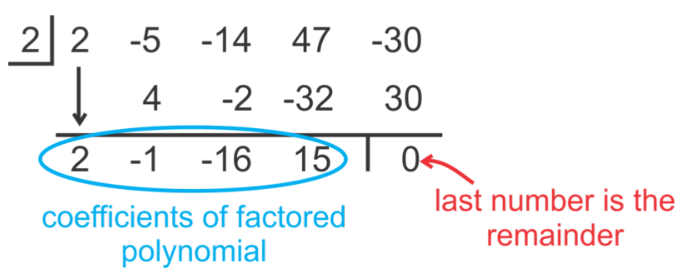
Divide $2x^4 - 5x^3 - 14x^2 - 37x - 30$ by $x - 2$.

Solution: Using synthetic division, the setup is as follows:





To "read" the answer, use the numbers as follows:



Therefore, 2 is a solution, because the remainder is zero. The factored polynomial is $2x^3 - x^2 - 16x + 15$. Notice that when we synthetically divide by k, the "leftover" polynomial is one degree less than the original. We could also write $(x-2)(2x^3 - x^2 - 16x + 15) = 2x^4 - 5x^3 - 14x^2 + 47x - 30$.

Example B

Determine if 4 is a solution to $f(x) = 5x^3 + 6x^2 - 24x - 16$.

Using synthetic division, we have:

$$\begin{array}{r|rrrr}
4 & 5 & 6 & -24 & -16 \\
 & \downarrow & 20 & 104 & 320 \\
\hline
 & 5 & 26 & 80 & \big| \ 304
\end{array}$$

The remainder is 304, so 4 is not a solution. Notice if we substitute in $x = 4$, also written $f(4)$, we would have $f(4) = 5(4)^3 + 6(4)^2 - 24(4) - 16 = 304$. This leads us to the Remainder Theorem.

Remainder Theorem: If $f(k) = r$, then r is also the remainder when dividing by $(x-k)$.

This means that if you substitute in $x = k$ or divide by k, what comes out of $f(x)$ is the same. r is the remainder, but also is the corresponding $y-$value. Therefore, the point (k,r) would be on the graph of $f(x)$.

Example C

Determine if $(2x-5)$ is a factor of $4x^4 - 9x^2 - 100$.

Solution: If you use synthetic division, the factor is not in the form $(x-k)$. We need to solve the possible factor for zero to see what the possible solution would be. Therefore, we need to put $\frac{5}{2}$ up in the left-hand corner box. Also, not every term is represented in this polynomial. When this happens, you must put in zero placeholders. In this example, we need zeros for the x^3-term and the $x-$term.

$$\begin{array}{r|rrrrr}
5/2 & 4 & 0 & -9 & 0 & -100 \\
 & \downarrow & 10 & 25 & 40 & 100 \\
\hline
 & 4 & 10 & 16 & 40 & \big| \ 0
\end{array}$$

This means that $\frac{5}{2}$ is a zero and its corresponding binomial, $(2x-5)$, is a factor.

Guided Practice

1. Divide $x^3 - 9x^2 + 12x - 27$ by $(x+3)$. Write the resulting polynomial with the remainder (if there is one).

2. Divide $2x^4 - 11x^3 + 12x^2 + 9x - 2$ by $(2x+1)$. Write the resulting polynomial with the remainder (if there is one).

3. Is 6 a solution for $f(x) = x^3 - 8x^2 + 72$? If so, find the real-number zeros (solutions) of the resulting polynomial.

Answers

1. Using synthetic division, divide by -3.

$$\begin{array}{r|rrrr}
-3 & 1 & 9 & 12 & -27 \\
 & \downarrow & -3 & -18 & 18 \\
\hline
 & 1 & 6 & -6 & \big| \ -9
\end{array}$$

The answer is $x^2 + 6x - 6 - \frac{9}{x+3}$.

2. Using synthetic division, divide by $-\frac{1}{2}$.

$$\begin{array}{r|rrrrr}
-1/2 & 2 & -11 & 12 & 9 & -2 \\
 & \downarrow & -1 & 6 & -9 & 0 \\
\hline
 & 2 & -12 & 18 & 0 & \big| \ -2
\end{array}$$

The answer is $2x^3 - 12x^2 + 18x - \frac{2}{2x+1}$.

3. Put a zero placeholder for the $x-$term. Divide by 6.

6	1	-8	0	72
	↓	6	-12	-72
	1	-2	-12	0

The resulting polynomial is $x^2 - 2x - 12$. While this quadratic does not factor, we can use the Quadratic Formula to find the other roots.

$$x = \frac{2 \pm \sqrt{2^2 - 4(1)(-12)}}{2} = \frac{2 \pm \sqrt{4+48}}{2} = \frac{2 \pm 2\sqrt{13}}{2} = 1 \pm \sqrt{13}$$

The solutions to this polynomial are 6, $1 + \sqrt{13} \approx 4.61$ and $1 - \sqrt{13} \approx -2.61$.

Vocabulary

Synthetic Division
An alternative to long division for dividing $f(x)$ by k where only the coefficients of $f(x)$ are used.

Remainder Theorem
If $f(k) = r$, then r is also the remainder when dividing by $(x - k)$.

Problem Set

Use synthetic division to divide the following polynomials. Write out the remaining polynomial.

1. $(x^3 + 6x^2 + 7x + 10) \div (x + 2)$
2. $(4x^3 - 15x^2 - 120x - 128) \div (x - 8)$
3. $(4x^2 - 5) \div (2x + 1)$
4. $(2x^4 - 15x^3 - 30x^2 - 20x + 42) \div (x + 9)$
5. $(x^3 - 3x^2 - 11x + 5) \div (x - 5)$
6. $(3x^5 + 4x^3 - x - 2) \div (x - 1)$
7. Which of the division problems above generate no remainder? What does that mean?
8. What is the difference between a zero and a factor?
9. Find $f(-2)$ if $f(x) = 2x^4 - 5x^3 - 10x^2 + 21x - 4$.
10. Now, divide $2x^4 - 5x^3 - 10x^2 + 21x - 4$ by $(x + 2)$ synthetically. What do you notice?

Find all real zeros of the following polynomials, given one zero.

11. $12x^3 + 76x^2 + 107x - 20; -4$
12. $x^3 - 5x^2 - 2x + 10$

CHAPTER 15 Algebra Review - Quadratic Functions

Chapter Outline

Quadratic Functions

In this chapter we explore functions that are no longer linear. We will introduce a quadratic equation or an equation where the largest $x-$value is squared. In addition, we will learn how to solve a quadratic equation: by factoring, using square roots, completing the square, and the Quadratic Formula. We will also introduce complex, or imaginary, numbers. Lastly, we will analyze the graph of a quadratic and use it to approximate data.

15.1 Solving Quadratics by Factoring

Objective

To factor and solve any quadratic equation that is considered "factorable."

Review Queue

Solve the following equations.

1. $5x-12=2x+9$
2. $\frac{1}{3}x+\frac{5}{2}=-\frac{1}{3}x-\frac{7}{2}$
3. Solve the system of equations using any method:

$$\begin{aligned} 2x-y&=12\\ -3x+2y&=-19 \end{aligned}$$

4. Find two numbers whose sum is 10 and product is 16.

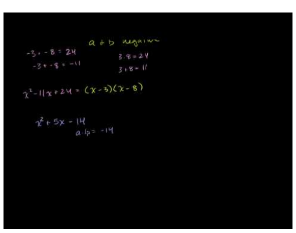
Factoring When the First Coefficient Equals 1

Objective

To factor a quadratic equation in the form x^2+bx+c.

Watch This

Watch the first few examples in this video, until about 12:40.



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Khan Academy: Factoring Quadratic Expressions

Guidance

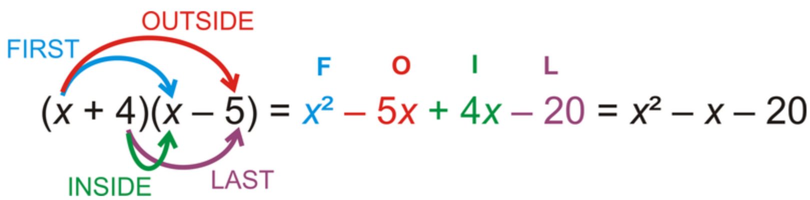
In this chapter we will be discussing quadratic equations. A **quadratic equation** has the form ax^2+bx+c, where $a\neq 0$ (If $a=0$, then the equation would be linear). For all quadratic equations, the 2 is the largest and only exponent. A quadratic equation can also be called a **trinomial** when all three terms are present.

There are four ways to solve a quadratic equation. The easiest is **factoring**. In this concept, we are going to focus on factoring when $a=1$ or when there is no number in front of x^2. First, let's start with a review of multiplying two factors together.

Example A

Multiply $(x+4)(x-5)$.

Solution: Even though this is not a quadratic, the product of the two **factors** will be. Remember from previous math classes that a factor is a number that goes evenly into a larger number. For example, 4 and 5 are factors of 20. So, to determine the larger number that $(x+4)$ and $(x-5)$ go into, we need to multiply them together. One method for multiplying two polynomial factors together is called FOIL. To do this, you need to multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and the LAST terms together and then combine like terms.



Therefore $(x+4)(x-5) = x^2 - x - 20$. We can also say that $(x+4)$ and $(x-5)$ are factors of $x^2 - x - 20$.

More Guidance

Now, we will "undo" the multiplication of two factors by factoring. In this concept, we will only address quadratic equations in the form $x^2 + bx + c$, or when $a = 1$.

Investigation: Factoring $x^2 + bx + c$

1. From the previous example, we know that $(x+m)(x+n) = x^2 + bx + c$. FOIL $(x+m)(x+n)$.

$$(x+m)(x+n) \Rightarrow x^2 + \underbrace{nx + mx}_{bx} + \underbrace{mn}_{c}$$

2. This shows us that the **constant** term, or c, is equal to the <u>product</u> of the constant numbers inside each factor. It also shows us that the **coefficient** in front of x, or b, is equal to the <u>sum</u> of these numbers.

3. Group together the first two terms and the last two terms. Find the Greatest Common Factor, or GCF, for each pair.

$$\begin{aligned}&(x^2 + nx) + (mx + mn)\\ &x(x+n) + m(x+n)\end{aligned}$$

4. Notice that what is inside both sets of parenthesis in Step 3 is the same. This number, $(x+n)$, is the GCF of $x(x+n)$ and $m(x+n)$. You can pull it out in front of the two terms and leave the $x+m$.

$$\begin{aligned}&x(x+n) + m(x+n)\\ &(x+n)(x+m)\end{aligned}$$

We have now shown how to go from FOIL-ing to factoring and back. Let's apply this idea to an example.

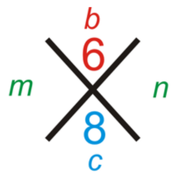
Example B

Factor $x^2 + 6x + 8$.

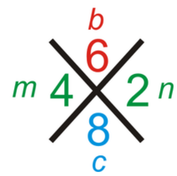
Solution: Let's use the investigation to help us.

$$x^2 + 6x + 8 = (x+m)(x+n)$$

So, from Step 2, b will be equal to the sum of m and n and c will be equal to their product. Applying this to our problem, $6 = m+n$ and $8 = mn$. To organize this, use an "X". Place the sum in the top and the product in the bottom.



Factors of 8	Sum
4 · 2	6
-4 · -2	-6
1 · 8	9
-1 · -8	-9



The green pair above is the only one that also adds up to 6. Now, move on to Step 3 from our investigation. We need to rewrite the $x-$term, or b, as a sum of m and n.

$$\begin{aligned}&x^2+6x+8\\&x^2+4x+2x+8\\&(x^2+4x)+(2x+8)\\&x(x+4)+2(x+4)\end{aligned}$$

Moving on to Step 4, we notice that the $(x+4)$ term is the same. Pull this out and we are done.

$$x(x+4)+2(x+4)$$
$$(x+4)(x+2)$$

Therefore, the factors of x^2+6x+8 are $(x+4)(x+2)$. You can FOIL this to check your answer.

Example C

Factor $x^2+12x-28$.

Solution: We can approach this problem in exactly the same way we did Example B. This time, we will not use the "X." What are the factors of -28 that also add up to 12? Let's list them out to see:

$$-4\cdot 7, 4\cdot -7, 2\cdot -14, -2\cdot 14, 1\cdot -28, -1\cdot 28$$

The **red** pair above is the one that works. Notice that we only listed the factors of *negative* 28.

$$\begin{aligned}&x^2+12x-28\\&x^2-2x+14x-28\\&(x^2-2x)+(14x-28)\\&x(x-2)+14(x-2)\\&(x-2)(x+14)\end{aligned}$$

By now, you might have a couple questions:

1. Does it matter which $x-$term you put first? NO, order does not matter. In the previous example, we could have put $14x$ followed by $-2x$. We would still end up with the same answer.
2. Can I skip the "expanded" part (Steps 3 and 4 in the investigation)? YES and NO. Yes, if $a = 1$ No, if $a \neq 1$ (the next concept). If $a = 1$, then $x^2 + bx + c = (x+m)(x+n)$ such that $m+n = b$ and $mn = c$. Consider this a shortcut.

Example D

Factor $x^2 - 4x$.

Solution: This is an example of a quadratic that is not a trinomial because it only has two terms, also called a **binomial**. There is no c, or constant term. To factor this, we need to look for the GCF. In this case, the largest number that can be taken out of both terms is an x.

$$x^2 - 4x = x(x-4)$$

Therefore, the factors are x and $x-4$.

Guided Practice

1. Multiply $(x-3)(x+8)$.

Factor the following quadratics, if possible.

2. $x^2 - 9x + 20$

3. $x^2 + 7x - 30$

4. $x^2 + x + 6$

5. $x^2 + 10x$

Answers

1. FOIL-ing our factors together, we get:

$$(x-3)(x+8) = x^2 + 8x - 3x - 24 = x^2 + 5x - 24$$

2. Using the "X," we have:

From the shortcut above, $-4 + -5 = -9$ and $-4 \cdot -5 = 20$.

$$x^2 - 9x + 20 = (x-4)(x-5)$$

3. Let's list out all the factors of -30 and their sums. The sums are in red.

$$-10 \cdot 3\ (-7), -3 \cdot 10\ (7), -2 \cdot 15\ (13), -15 \cdot 2\ (-13), -1 \cdot 30\ (29), -30 \cdot 1\ (-29)$$

From this, the factors of -30 that add up to 7 are -3 and 10. $x^2+7x-30=(x-3)(x+10)$

4. There are no factors of 6 that add up to 1. If we had -6, then the trinomial would be factorable. But, as is, this is not a factorable trinomial.

5. The only thing we can do here is to take out the GCF. $x^2+10x=x(x+10)$

Vocabulary

Quadratic Equation
: An equation where the largest exponent is a 2 and has the form ax^2+bx+c, $a \neq 0$.

Trinomial
: A quadratic equation with three terms.

Binomial
: A quadratic equation with two terms.

Factoring
: A way to break down a quadratic equation into smaller factors.

Factor
: A number that goes evenly into a larger number.

FOIL
: A method used to multiply together two factors. You multiply the FIRST terms, OUTSIDE terms, INSIDE terms, and LAST terms and then combine any like terms.

Coefficient
: The number in front of a variable.

Constant
: A number that is added or subtracted within an equation.

Problem Set

Multiply the following factors together.

1. $(x+2)(x-8)$
2. $(x-9)(x-1)$
3. $(x+7)(x+3)$

Factor the following quadratic equations. If it cannot be factored, write *not factorable*. You can use either method presented in the examples.

4. x^2-x-2
5. $x^2+2x-24$
6. x^2-6x
7. x^2+6x+9
8. $x^2+8x-10$
9. $x^2-11x+30$
10. $x^2+13x-30$

11. $x^2+11x+28$
12. $x^2-8x+12$
13. $x^2-7x-44$
14. $x^2-8x-20$
15. x^2+4x+3
16. $x^2-5x+36$
17. $x^2-5x-36$
18. x^2+x

Challenge Fill in the X's below with the correct numbers.

19.


20.


Factoring When the First Coefficient Doesn't Equal 1

Objective

To multiply factors and factor quadratic equations in the form ax^2+bx+c by expanding the $x-$term.

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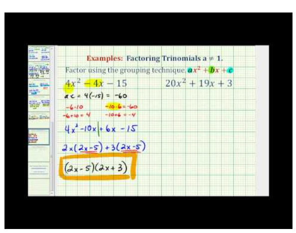

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James Sousa: Ex: Factor Trinomials When A is NOT Equal to 1 - Grouping Method

Guidance

When we add a number in front of the x^2 term, it makes factoring a little trickier. We still follow the investigation from the previous section, but *cannot* use the shortcut. First, let's try FOIL-ing when the coefficients in front of the $x-$terms are not 1.

Example A

Multiply $(3x-5)(2x+1)$

Solution: We can still use FOIL.

FIRST $3x \cdot 2x = 6x^2$

OUTSIDE $3x \cdot 1 = 3x$

INSIDE $-5 \cdot 2x = -10x$

LAST $-5 \cdot 1 = -5$

Combining all the terms together, we get: $6x^2 + 3x - 10x - 5 = 6x^2 - 7x - 5$.

Now, let's work backwards and factor a trinomial to get two factors. Remember, you can always check your work by multiplying the final factors together.

Example B

Factor $6x^2 - x - 2$.

Solution: This is a factorable trinomial. When there is a coefficient, or number in front of x^2, you must follow all the steps from the investigation in the previous concept; no shortcuts. Also, m and n no longer have a product of c and a sum of b. This would not take the coefficient of x^2 into account. What we need to do is multiply together a and c (from $ax^2 + bx + c$) and then find the two numbers whose product is ac and sum is b. Let's use the X to help us organize this.

Now, we can see, we need the two factors of -12 that also add up to -1.

TABLE 15.1:

Factors	Sum
$-1, 12$	11
1, -12	-11
2, -6	-4
-2, 6	4
$3, -4$	-1
-3, 4	1

The factors that work are 3 and -4. Now, take these factors and rewrite the $x-$term expanded using 3 and -4 (Step 3 from the investigation in the previous concept).

$$6x^2 - x - 2$$

$$6x^2 - 4x + 3x - 2$$

Next, group the first two terms together and the last two terms together and pull out any common factors.

$$(6x^2 - 4x) + (3x - 2)$$
$$2x(3x - 2) + 1(3x - 2)$$

Just like in the investigation, what is in the parenthesis is *the same*. We now have two terms that both have $(3x-2)$ as factor. Pull this factor out.

$$2x(3x-2)+1(3x-2)$$
$$(3x-2)(2x+1)$$

The factors of $6x^2-x-2$ are $(3x-2)(2x+1)$. You can FOIL these to check your answer.

Example C

Factor $4x^2+8x-5$.

Solution: Let's make the steps from Example B a little more concise.

1. Find ac and the factors of this number that add up to b.

$4\cdot -5=-20$ The factors of -20 that add up to 8 are 10 and -2.

2. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 1.

$$4x^2+8x-5$$
$$\swarrow\searrow$$
$$4x^2+10x-2x-5$$

3. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&(4x^2+10x)+(-2x-5)\\&2x(2x+5)-1(2x+5)\\&(2x+5)(2x-1)\end{aligned}$$

<u>Alternate Method</u>: What happens if we list $-2x$ before $10x$ in Step 2?

$$\begin{aligned}&4x^2-2x+10x-5\\&(4x^2-2x)(10x-5)\\&2x(2x-1)+5(2x-1)\\&(2x-1)(2x+5)\end{aligned}$$

This tells us it does not matter which $x-$term we list first in Step 2 above.

Example D

Factor $12x^2-22x-20$.

Solution: Let's use the steps from Example C, but we are going to add an additional step at the beginning.

1. Look for any common factors. Pull out the GCF of all three terms, if there is one.

$$12x^2-22x-20=2(6x^2-11x-10)$$

This will make it much easier for you to factor what is inside the parenthesis.

2. Using what is inside the parenthesis, find ac and determine the factors that add up to b.

$$6 \cdot -10 = -60 \rightarrow -15 \cdot 4 = -60, \ -15+4 = -11$$

The factors of -60 that add up to -11 are -15 and 4.

3. Rewrite the trinomial with the $x-$term expanded, using the two factors from Step 2.

$$\begin{aligned}&2(6x^2-11x-10)\\&2(6x^2-15x+4x-10)\end{aligned}$$

4. Group the first two and second two terms together, find the GCF and factor again.

$$\begin{aligned}&2(6x^2-15x+4x-10)\\&2\left[(6x^2-15x)+(4x-10)\right]\\&2\left[3x(2x-5)+2(2x-5)\right]\\&2(2x-5)(3x+2)\end{aligned}$$

Guided Practice

1. Multiply $(4x-3)(3x+5)$.

Factor the following quadratics, if possible.

2. $15x^2-4x-3$

3. $3x^2+6x-12$

4. $24x^2-30x-9$

5. $4x^2+4x-48$

Answers

1. FOIL: $(4x-3)(3x+5) = 12x^2+20x-9x-15 = 12x^2+11x-15$

2. Use the steps from the examples above. There is no GCF, so we can find the factors of ac that add up to b.

$15 \cdot -3 = -45$ The factors of -45 that add up to -4 are -9 and 5.

$$\begin{aligned}&15x^2-4x-3\\&(15x^2-9x)+(5x-3)\\&3x(5x-3)+1(5x-3)\\&(5x-3)(3x+1)\end{aligned}$$

3. $3x^2+6x-12$ has a GCF of 3. Pulling this out, we have $3(x^2+2x-6)$. There is no number in front of x^2, so we see if there are any factors of -6 that add up to 2. There are not, so this trinomial is not factorable.

4. $24x^2-30x-9$ also has a GCF of 3. Pulling this out, we have $3(8x^2-10x-3)$. $ac = -24$. The factors of -24 than add up to -10 are -12 and 2.

$$3(8x^2 - 10x - 3)$$
$$3\left[(8x^2 - 12x) + (2x - 3)\right]$$
$$3\left[4x(2x-3) + 1(2x-3)\right]$$
$$3(2x-3)(4x+1)$$

5. $4x^2 + 4x - 48$ has a GCF of 4. Pulling this out, we have $4(x^2 + x - 12)$. This trinomial does not have a number in front of x^2, so we can use the shortcut from the previous concept. What are the factors of -12 that add up to 1?

$$4(x^2 + x - 12)$$
$$4(x+4)(x-3)$$

Problem Set

Multiply the following expressions.

1. $(2x-1)(x+5)$
2. $(3x+2)(2x-3)$
3. $(4x+1)(4x-1)$

Factor the following quadratic equations, if possible. If they cannot be factored, write *not factorable*. Don't forget to look for any GCFs first.

4. $5x^2 + 18x + 9$
5. $6x^2 - 21x$
6. $10x^2 - x - 3$
7. $3x^2 + 2x - 8$
8. $4x^2 + 8x + 3$
9. $12x^2 - 12x - 18$
10. $16x^2 - 6x - 1$
11. $5x^2 - 35x + 60$
12. $2x^2 + 7x + 3$
13. $3x^2 + 3x + 27$
14. $8x^2 - 14x - 4$
15. $10x^2 + 27x - 9$
16. $4x^2 + 12x + 9$
17. $15x^2 + 35x$
18. $6x^2 - 19x + 15$
19. Factor $x^2 - 25$. What is b?
20. Factor $9x^2 - 16$. What is b? What types of numbers are a and c?

Factoring Special Quadratics

Objective

To factor perfect square trinomials and the difference of squares.

Watch This

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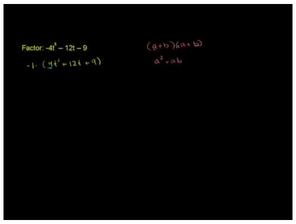

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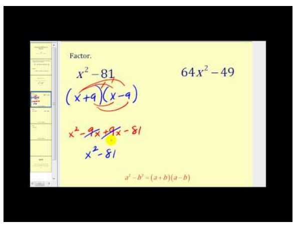
Khan Academy: U09_L2_TI_we1 Factoring Special Products 1

Then, watch the first part of this video, until about 3:10



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James Sousa: Factoring a Difference of Squares

Guidance

There are a couple of special quadratics that, when factored, have a pattern.

Investigation: Multiplying $(a+b)^2$

1. Rewrite $(a+b)^2$ as the product of two factors. Expand $(a+b)^2$. $(a+b)^2 = (a+b)(a+b)$

2. FOIL your answer from Step 1. This is a **perfect square trinomial.** $a^2+2ab+b^2$

3. $(a-b)^2$ also produces a perfect square trinomial. $(a-b)^2 = a^2-2ab+b^2$

4. Apply the formula above to factoring $9x^2-12x+4$. First, find a and b.

$$\begin{aligned} a^2 = 9x^2&, \ b^2 = 4 \\ a = 3x&, \quad b = 2 \end{aligned}$$

5. Now, plug a and b into the appropriate formula.

$$\begin{aligned} (3x-2)^2 &= (3x)^2 - 2(3x)(2) + 2^2 \\ &= 9x^2 - 12x + 4 \end{aligned}$$

Investigation: Multiplying $(a-b)(a+b)$

1. FOIL $(a-b)(a+b)$.

$$\begin{aligned} (a-b)(a+b) &= a^2+ab-ab-b^2 \\ &= a^2-b^2 \end{aligned}$$

2. This is a **difference of squares.** The difference of squares will always factor to be $(a+b)(a-b)$.

3. Apply the formula above to factoring $25x^2-16$. First, find a and b.

$$\begin{aligned} a^2 &= 25x^2, \ b^2 = 16 \\ a &= 5x, \qquad b = 4 \end{aligned}$$

4. Now, plug a and b into the appropriate formula. $(5x-4)(5x+4) = (5x)^2 - 4^2$

**It is important to note that if you forget these formulas or do not want to use them, you can still factor all of these quadratics the same way you did in the previous two concepts.

Example A

Factor x^2-81.

Solution: Using the formula from the investigation above, we need to first find the values of a and b.

$$\begin{aligned} x^2 - 81 &= a^2 - b^2 \\ a^2 &= x^2, \ b^2 = 81 \\ a &= x, \qquad b = 9 \end{aligned}$$

Now, plugging x and 9 into the formula, we have $x^2-81=(x-9)(x+9)$. To solve for a and b, we found the **square root** of each number. Recall that the square root is a number that, when multiplied by itself, produces another number. This other number is called a **perfect square**.

Alternate Method

Rewrite x^2-81 so that the middle term is present. $x^2+0x-81$

Using the method from the previous two concepts, what are the two factors of -81 that add up to 0? 9 and -9

Therefore, the factors are $(x-9)(x+9)$.

Example B

Factor $36x^2+120x+100$.

Solution: First, check for a GCF.

$$4(9x^2+30x+25)$$

Now, double-check that the quadratic equation above fits into the perfect square trinomial formula.

$$\begin{array}{ccc} a^2 = 9x^2 & b^2 = 25 & \\ \sqrt{a^2} = \sqrt{9x^2} & \sqrt{b^2} = \sqrt{25} & 2ab = 30x \\ a = 3x & b = 5 & 2(3x)(5) = 30x \end{array}$$

Using a and b above, the equation factors to be $4(3x+5)^2$. If you did not factor out the 4 in the beginning, the formula will still work. a would equal $6x$ and b would equal 10, so the factors would be $(6x+10)^2$. If you expand and find the GCF, you would have $(6x+10)^2=(6x+10)(6x+10)=2(3x+5)2(3x+5)=4(3x+5)^2$.

Alternate Method

First, find the GCF. $4(9x^2+30x+25)$

Then, find ac and expand b accordingly. $9 \cdot 25 = 225$, the factors of 225 that add up to 30 are 15 and 15.

$$\begin{aligned}
&4(9x^2+30x+25)\\
&4(9x^2+15x+15x+25)\\
&4\left[(9x^2+15x)+(15x+25)\right]\\
&4\left[3x(3x+5)+5(3x+5)\right]\\
&4(3x+5)(3x+5)\ or\ 4(3x^2+5)
\end{aligned}$$

Again, notice that if you do not use the formula discovered in this concept, you can still factor and get the correct answer.

Example C

Factor $48x^2-147$.

Solution: At first glance, this does not look like a difference of squares. 48 nor 147 are square numbers. But, if we take a 3 out of both, we have $3(16x^2-49)$. 16 and 49 are both square numbers, so now we can use the formula.

$$\begin{aligned}
16x^2 &= a^2 & 49 &= b^2\\
4x &= a & 7 &= b
\end{aligned}$$

The factors are $(4x-7)(4x+7)$.

Guided Practice

Factor the following quadratic equations.

1. x^2-4

2. $2x^2-20x+50$

3. $81x^2+144+64$

Answers

1. $a=x$ and $b=2$. Therefore, $x^2-4=(x-2)(x+2)$.

2. Factor out the GCF, 2. $2(x^2-10x+25)$. This is now a perfect square trinomial with $a=x$ and $b=5$.

$$2(x^2-10x+25)=2(x-5)^2.$$

3. This is a perfect square trinomial and no common factors. Solve for a and b.

$$\begin{aligned}
81x^2 &= a^2 & 64 &= b^2\\
9x &= a & 8 &= b
\end{aligned}$$

The factors are $(9x+8)^2$.

Vocabulary

Perfect Square Trinomial
A quadratic equation in the form $a^2+2ab+b^2$ or $a^2-2ab+b^2$.

Difference of Squares
A quadratic equation in the form a^2-b^2.

Square Root
A number, that when multiplied by itself produces another number. 3 is the square root of 9.

Perfect Square
A number that has a square root that is an integer. 25 is a perfect square.

Problem Set

1. List the perfect squares that are less than 200.
2. Why do you think there is no *sum of squares* formula?

Factor the following quadratics, if possible.

3. x^2-1
4. x^2+4x+4
5. $16x^2-24x+9$
6. $-3x^2+36x-108$
7. $144x^2-49$
8. $196x^2+140x+25$
9. $100x^2+1$
10. $162x^2+72x+8$
11. $225-x^2$
12. $121-132x+36x^2$
13. $5x^2+100x-500$
14. $256x^2-676$
15. **Error Analysis** Spencer is given the following problem: Multiply $(2x-5)^2$. Here is his work:

$$(2x-5)^2=(2x)^2-5^2=4x^2-25$$

His teacher tells him the answer is $4x^2-20x+25$. What did Spencer do wrong? Describe his error and correct the problem.

Solving Quadratics by Factoring

Objective

To solve factorable quadratic equations for x.

Watch This

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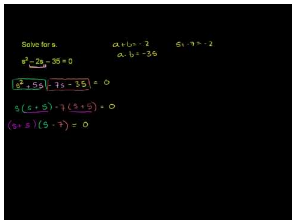

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Khan Academy: Solving Quadratic Equations by Factoring.avi

Guidance

In this lesson we have not actually solved for x. Now, we will apply factoring to solving a quadratic equation. It adds one additional step to the end of what you have already been doing. Let's go through an example.

Example A

Solve $x^2-9x+18=0$ by factoring.

Solution: The only difference between this problem and previous ones from the concepts before is the addition of the $=$ sign. Now that this is present, we need to solve for x. We can still factor the way we always have. Because $a=1$, determine the two factors of 18 that add up to -9.

$$\begin{aligned} x^2-9x+18&=0\\ (x-6)(x-3)&=0 \end{aligned}$$

Now, we have two factors that, when multiplied, equal zero. Recall that when two numbers are multiplied together and one of them is zero, the product is always zero.

Zero-Product Property: If $ab=0$, then $a=0$ or $b=0$.

This means that $x-6=0$ OR $x-3=0$. Therefore, $x=6$ or $x=3$. There will always be the same number of solutions as factors.

Check your answer:

$$\begin{aligned} 6^2-9(6)+18&=0 \quad or \quad & 3^2-9(3)+18&=0\\ 36-54+18&=0 & 9-27+18&=0\ \boxed{\checkmark} \end{aligned}$$

Example B

Solve $6x^2+x-4=11$ by factoring.

Solution: At first glance, this might not look factorable to you. However, before we factor, we must combine like terms. Also, the Zero-Product Property tells us that in order to solve for the factors, one side of the equation must be zero.

$$\begin{aligned} 6x^2+x-4&=\cancel{11}\\ \underline{\qquad\qquad -11}&\underline{=-\cancel{11}}\\ 6x^2+x-15&=0 \end{aligned}$$

Now, factor. The product of ac is -90. What are the two factors of -90 that add up to 1? 10 and -9. Expand the $x-$term and factor.

$$\begin{aligned}6x^2+x-15&=0\\6x^2-9x+10x-15&=0\\3x(2x-3)+5(2x-3)&=0\\(2x-3)(3x+5)&=0\end{aligned}$$

Lastly, set each factor equal to zero and solve.

$$\begin{aligned}2x-3&=0 & & & 3x+5&=0\\2x&=3 & &or & 3x&=-5\\x&=\frac{3}{2} & & & x&=-\frac{5}{3}\end{aligned}$$

Check your work:

$$\begin{aligned}6\left(\frac{3}{2}\right)^2+\frac{3}{2}-4&=11 & & & 6\left(-\frac{5}{3}\right)^2-\frac{5}{3}-4&=11\\6\cdot\frac{9}{4}+\frac{3}{2}-4&=11 & &or & 6\cdot\frac{25}{9}-\frac{5}{3}-4&=11\ \boxed{\checkmark}\\\frac{27}{2}+\frac{3}{2}-4&=11 & & & \frac{50}{3}-\frac{5}{3}-4&=11\\15-4&=11 & & & 15-4&=11\end{aligned}$$

Example C

Solve $10x^2-25x=0$ by factoring.

Solution: Here is an example of a quadratic equation without a constant term. The only thing we can do is take out the GCF.

$$\begin{aligned}10x^2-25x&=0\\5x(2x-5)&=0\end{aligned}$$

Set the two factors equal to zero and solve.

$$\begin{aligned}5x&=0 & & & 2x-5&=0\\x&=0 & &or & 2x&=5\\& & & & x&=\frac{5}{2}\end{aligned}$$

Check:

$$\begin{aligned}10(0)^2-25(0)&=0 & & & 10\left(\frac{5}{2}\right)^2-25\left(\frac{5}{2}\right)&=0\\0&=0 & &or & 10\cdot\frac{25}{4}-\frac{125}{2}&=0\ \boxed{\checkmark}\\& & & & \frac{125}{2}-\frac{125}{2}&=0\end{aligned}$$

Guided Practice

Solve the following equations by factoring.

1. $4x^2 - 12x + 9 = 0$

2. $x^2 - 5x = 6$

3. $8x - 20x^2 = 0$

4. $12x^2 + 13x + 7 = 12 - 4x$

Answers

1. $ac = 36$. The factors of 36 that also add up to -12 are -6 and -6. Expand the $x-$term and factor.

$$\begin{aligned} 4x^2 - 12x + 9 &= 0 \\ 4x^2 - 6x - 6x + 9 &= 0 \\ 2x(2x-3) - 3(2x-3) &= 0 \\ (2x-3)(2x-3) &= 0 \end{aligned}$$

The factors are the same. When factoring a perfect square trinomial, the factors will always be the same. In this instance, the solutions for x will also be the same. Solve for x.

$$\begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

When the two factors are the same, we call the solution for x a **double root** because it is the solution twice.

2. Here, we need to get everything on the same side of the equals sign in order to factor.

$$\begin{aligned} x^2 - 5x &= 6 \\ x^2 - 5x - 6 &= 0 \end{aligned}$$

Because there is no number in front of x^2, we need to find the factors of -6 that add up to -5.

$$(x-6)(x+1) = 0$$

Solving each factor for x, we get that $x = 6$ or $x = -1$.

3. Here there is no constant term. Find the GCF to factor.

$$\begin{aligned} 8x - 20x^2 &= 0 \\ 4x(2-5x) &= 0 \end{aligned}$$

Solve each factor for x.

$$\begin{aligned} 4x &= 0 & 2-5x &= 0 \\ x &= 0 \quad or & 2 &= 5x \\ & & \frac{2}{5} &= x \end{aligned}$$

4. This problem is slightly more complicated than #2. Combine all like terms onto the same side of the equals sign so that one side is zero.

$$\begin{aligned} 12x^2+13x+7 &= 12-4x \\ 12x^2+17x-5 &= 0 \end{aligned}$$

$ac = -60$. The factors of -60 that add up to 17 are 20 and -3. Expand the $x-$term and factor.

$$\begin{aligned} 12x^2+17x-5 &= 0 \\ 12x^2+20x-3x-5 &= 0 \\ 4x(3x+5)-1(3x+5) &= 0 \\ (3x+5)(4x-1) &= 0 \end{aligned}$$

Solve each factor for x.

$$\begin{aligned} 3x+5 &= 0 & 4x-1 &= 0 \\ 3x &= -5 \quad or & 4x &= 1 \\ x &= -\frac{5}{3} & x &= \frac{1}{4} \end{aligned}$$

Vocabulary

Solution
The answer to an equation. With quadratic equations, solutions can also be called *zeros* or *roots*.

Double Root
A solution that is repeated twice.

Problem Set

Solve the following quadratic equations by factoring, if possible.

1. $x^2+8x-9=0$
2. $x^2+6x=0$
3. $2x^2-5x=12$
4. $12x^2+7x-10=0$
5. $x^2=9$
6. $30x+25=-9x^2$
7. $2x^2+x-5=0$

8. $16x = 32x^2$
9. $3x^2 + 28x = -32$
10. $36x^2 - 48 = 1$
11. $6x^2 + x = 4$
12. $5x^2 + 12x + 4 = 0$

Challenge Solve these quadratic equations by factoring. They are all factorable.

13. $8x^2 + 8x - 5 = 10 - 6x$
14. $-18x^2 = 48x + 14$
15. $36x^2 - 24 = 96x - 39$
16. **Real Life Application** George is helping his dad build a fence for the backyard. The total area of their backyard is 1600 square feet. The width of the house is half the length of the yard, plus 7 feet. How much fencing does George's dad need to buy?

15.2 Solving Quadratics by Using Square Roots

Objective

Reviewing simplifying square roots and to solve a quadratic equation by using square roots.

Review Queue

1. What is $\sqrt{64}$? Can there be more than one answer?

2. What two numbers should $\sqrt{18}$ be between? How do you know?

3. Find $\sqrt{18}$ on your calculator.

4. Solve $x^2 - 25 = 0$ by factoring.

Simplifying Square Roots

Objective

Simplifying, adding, subtracting and multiplying square roots.

Guidance

Before we can solve a quadratic equation using square roots, we need to review how to simplify, add, subtract, and multiply them. Recall that the **square root** is a number that, when multiplied by itself, produces another number. 4 is the square root of 16, for example. -4 is also the square root of 16 because $(-4)^2 = 16$. The symbol for square root is the **radical** sign, or $\sqrt{}$. The number under the radical is called the **radicand**. If the square root of a number is not an integer, it is an irrational number.

Example A

Find $\sqrt{50}$ using:

a) A calculator.

b) By simplifying the square root.

Solution:

a) To plug the square root into your graphing calculator, typically there is a $\sqrt{}$ or SQRT button. Depending on your model, you may have to enter 50 before or after the square root button. Either way, your answer should be $\sqrt{50} = 7.071067811865\ldots$ In general, we will round to the hundredths place, so 7.07 is sufficient.

b) To simplify the square root, the square numbers must be "pulled out." Look for factors of 50 that are square numbers: 4, 9, 16, 25... 25 is a factor of 50, so break the factors apart.

$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$. *This is the most accurate answer.*

<u>**Radical Rules**</u>

1. $\sqrt{ab} = \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ Any two radicals can be multiplied together.

2. $x\sqrt{a} \pm y\sqrt{a} = x \pm y\sqrt{a}$ The radicands must be the same in order to add or subtract.

3. $\left(\sqrt{a}\right)^2 = \sqrt{a^2} = a$ The square and square root cancel each other out.

Example B

Simplify $\sqrt{45}+\sqrt{80}-2\sqrt{5}$.

Solution: At first glance, it does not look like we can simplify this. But, we can simplify each radical by pulling out the perfect squares.

$$\sqrt{45}=\sqrt{9\cdot 5}=3\sqrt{5}$$
$$\sqrt{80}=\sqrt{16\cdot 5}=4\sqrt{5}$$

Rewriting our expression, we have: $3\sqrt{5}+4\sqrt{5}-2\sqrt{5}$ and all the radicands are the same. Using the Order of Operations, our answer is $5\sqrt{5}$.

Example C

Simplify $2\sqrt{35}\cdot 4\sqrt{7}$.

Solution: Multiply across.

$$2\sqrt{35}\cdot 4\sqrt{7}=2\cdot 4\sqrt{35\cdot 7}=8\sqrt{245}$$

Now, simplify the radical. $8\sqrt{245}=8\sqrt{49\cdot 5}=8\cdot 7\sqrt{5}=56\sqrt{5}$

Guided Practice

Simplify the following radicals.

1. $\sqrt{150}$
2. $2\sqrt{3}-\sqrt{6}+\sqrt{96}$
3. $\sqrt{8}\cdot\sqrt{20}$

Answers

1. Pull out all the square numbers.

$$\sqrt{150}=\sqrt{25\cdot 6}=5\sqrt{6}$$

Alternate Method: Write out the prime factorization of 150.

$$\sqrt{150}=\sqrt{2\cdot 3\cdot 5\cdot 5}$$

Now, pull out any number that has a pair. Write it *once* in front of the radical and multiply together what is left over under the radical.

$$\sqrt{150}=\sqrt{2\cdot 3\cdot 5\cdot 5}=5\sqrt{6}$$

2. Simplify $\sqrt{96}$ to see if anything can be combined. We will use the alternate method above.

$$\sqrt{96}=\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3}=2\cdot 2\sqrt{6}=4\sqrt{6}$$

Rewrite the expression: $2\sqrt{3} - \sqrt{6} + 4\sqrt{6} = 2\sqrt{3} + 3\sqrt{6}$. This is fully simplified. $\sqrt{3}$ and $\sqrt{6}$ cannot be combined because they do not have the same value under the radical.

3. This problem can be done two different ways.

1stMethod: Multiply radicals, then simplify the answer.

$$\sqrt{8} \cdot \sqrt{20} = \sqrt{160} = \sqrt{16 \cdot 10} = 4\sqrt{10}$$

2ndMethod: Simplify radicals, then multiply.

$$\sqrt{8} \cdot \sqrt{20} = \left(\sqrt{4 \cdot 2}\right) \cdot \left(\sqrt{4 \cdot 5}\right) = 2\sqrt{2} \cdot 2\sqrt{5} = 2 \cdot 2\sqrt{2 \cdot 5} = 4\sqrt{10}$$

Depending on the complexity of the problem, either method will work. Pick whichever method you prefer.

Vocabulary

Square Root
A number, that when multiplied by itself, produces another number.

Perfect Square
A number that has an integer for a square root.

Radical
The $\sqrt{}$, or square root, sign.

Radicand
The number under the radical.

Problem Set

Find the square root of each number by using the calculator. Round your answer to the nearest hundredth.

1. 56
2. 12
3. 92

Simplify the following radicals. If it cannot be simplified further, write *cannot be simplified*.

4. $\sqrt{18}$
5. $\sqrt{75}$
6. $\sqrt{605}$
7. $\sqrt{48}$
8. $\sqrt{50} \cdot \sqrt{2}$
9. $4\sqrt{3} \cdot \sqrt{21}$
10. $\sqrt{6} \cdot \sqrt{20}$
11. $\left(4\sqrt{5}\right)^2$
12. $\sqrt{24} \cdot \sqrt{27}$
13. $\sqrt{16} + 2\sqrt{8}$

14. $\sqrt{28}+\sqrt{7}$
15. $-8\sqrt{3}-\sqrt{12}$
16. $\sqrt{72}-\sqrt{50}$
17. $\sqrt{6}+7\sqrt{6}-\sqrt{54}$
18. $8\sqrt{10}-\sqrt{90}+7\sqrt{5}$

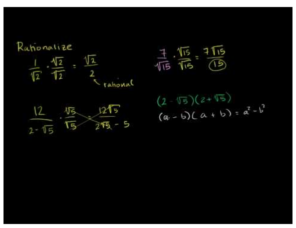
Dividing Square Roots

Objective

To divide radicals and rationalize the denominator.

Watch This

Watch the first part of this video, until about 3:15.



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Khan Academy: How to Rationalize a Denominator

Guidance

Dividing radicals can be a bit more difficult that the other operations. The main complication is that you cannot leave any radicals in the denominator of a fraction. For this reason we have to do something called **rationalizing the denominator**, where you multiply the top and bottom of a fraction by the same radical that is in the denominator. This will cancel out the radicals and leave a whole number.

Radical Rules

4. $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$

5. $\frac{\sqrt{a}}{\sqrt{b}}\cdot\frac{\sqrt{b}}{\sqrt{b}}=\frac{\sqrt{ab}}{b}$

Example A

Simplify $\sqrt{\frac{1}{4}}$.

Solution: Break apart the radical by using Rule #4.

$$\sqrt{\frac{1}{4}}=\frac{\sqrt{1}}{\sqrt{4}}=\frac{1}{2}$$

Example B

Simplify $\frac{2}{\sqrt{3}}$.

Solution: This might look simplified, but radicals cannot be in the denominator of a fraction. This means we need to apply Rule #5 to get rid of the radical in the denominator, or rationalize the denominator. Multiply the top and bottom of the fraction by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Example C

Simplify $\sqrt{\frac{32}{40}}$.

Solution: Reduce the fraction, and then apply the rules above.

$$\sqrt{\frac{32}{40}} = \sqrt{\frac{4}{5}} = \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Guided Practice

Simplify the following expressions using the Radical Rules learned in this concept and the previous concept.

1. $\sqrt{\frac{1}{2}}$

2. $\sqrt{\frac{64}{50}}$

3. $\frac{4\sqrt{3}}{\sqrt{6}}$

Answers

1. $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

2. $\sqrt{\frac{64}{50}} = \sqrt{\frac{32}{25}} = \frac{\sqrt{16 \cdot 2}}{5} = \frac{4\sqrt{2}}{5}$

3. The only thing we can do is rationalize the denominator by multiplying the numerator and denominator by $\sqrt{6}$ and then simplify the fraction.

$$\frac{4\sqrt{3}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{4\sqrt{18}}{6} = \frac{4\sqrt{9 \cdot 2}}{6} = \frac{12\sqrt{2}}{6} = 2\sqrt{2}$$

Vocabulary

Rationalize the denominator
: The process used to get a radical out of the denominator of a fraction.

Problem Set

Simplify the following fractions.

1. $\sqrt{\frac{4}{25}}$

2. $\sqrt{\frac{96}{121}}$
3. $\frac{5\sqrt{2}}{\sqrt{10}}$
4. $\frac{6}{\sqrt{15}}$
5. $\sqrt{\frac{60}{35}}$
6. $8\frac{\sqrt{18}}{\sqrt{30}}$
7. $\frac{12}{\sqrt{6}}$
8. $\sqrt{\frac{208}{143}}$
9. $\frac{21\sqrt{3}}{2\sqrt{14}}$

Challenge Use all the Radical Rules you have learned in the last two oncepts to simplify the expressions.

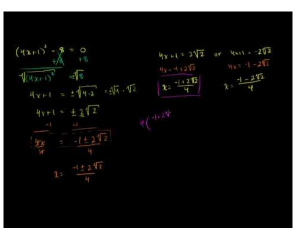
10. $\sqrt{\frac{8}{12}} \cdot \sqrt{15}$
11. $\sqrt{\frac{32}{45}} \cdot \frac{6\sqrt{20}}{\sqrt{5}}$
12. $\frac{\sqrt{24}}{\sqrt{2}} + \frac{8\sqrt{26}}{\sqrt{8}}$

Solving Quadratics Using Square Roots

Objective

To use the properties of square roots to solve certain types of quadratic equations.

Watch This



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Khan Academy: Solving Quadratics by Square Roots

Guidance

Now that you are familiar with square roots, we will use them to solve quadratic equations. Keep in mind, that square roots cannot be used to solve every type of quadratic. In order to solve a quadratic equation by using square roots, an $x-$term *cannot* be present. Solving a quadratic equation by using square roots is very similar to solving a linear equation. In the end, you must isolate the x^2 or whatever is being squared.

Example A

Solve $2x^2 - 3 = 15$.

Solution: Start by isolating the x^2.

$$\begin{aligned} 2x^2-3&=15\\ 2x^2&=18\\ x^2&=9 \end{aligned}$$

At this point, you can take the square root of both sides.

$$\begin{aligned} \sqrt{x^2}&=\pm\sqrt{9}\\ x&=\pm 3 \end{aligned}$$

Notice that x has two solutions; 3 or -3. When taking the square root, always put the $\pm$ (plus or minus sign) in front of the square root. This indicates that the positive or negative answer will be the solution.

Check:

$$\begin{aligned} 2(3)^2-3&=15 & & 2(-3)^2-3=15\\ 2\cdot 9-3&=15 & or & \quad 2\cdot 9-3=15 \boxed{\checkmark}\\ 18-3&=15 & & 18-3=15 \end{aligned}$$

Example B

Solve $\frac{x^2}{16}+3=27$.

Solution: Isolate x^2 and then take the square root.

$$\begin{aligned} \frac{x^2}{16}+3&=27\\ \frac{x^2}{16}&=24\\ x^2&=384\\ x&=\pm\sqrt{384}=\pm 8\sqrt{6} \end{aligned}$$

Example C

Solve $3(x-5)^2+7=43$.

Solution: In this example, x is not the only thing that is squared. Isolate the $(x-5)^2$ and then take the square root.

$$\begin{aligned} 3(x-5)^2+7&=43\\ 3(x-5)^2&=36\\ (x-5)^2&=12\\ x-5&=\pm\sqrt{12}\ or\ \pm 2\sqrt{3} \end{aligned}$$

Now that the square root is gone, add 5 to both sides.

$$x-5=\pm 2\sqrt{3}$$
$$x=5\pm 2\sqrt{3}$$

$x=5+2\sqrt{3}$ or $5-2\sqrt{3}$. We can estimate these solutions as decimals; 8.46 or 1.54. Remember, that the most accurate answer includes the radical numbers.

Guided Practice

Solve the following quadratic equations.

1. $\frac{2}{3}x^2-14=38$
2. $11+x^2=4x^2+5$
3. $(2x+1)^2-6=19$

Answers

1. Isolate x^2 and take the square root.

$$\frac{2}{3}x^2-14=38$$
$$\frac{2}{3}x^2=52$$
$$x^2=78$$
$$x=\pm\sqrt{78}$$

2. Combine all like terms, then isolate x^2.

$$11+x^2=4x^2+5$$
$$-3x^2=-6$$
$$x^2=2$$
$$x=\pm\sqrt{2}$$

3. Isolate what is being squared, take the square root, and then isolate x.

$$(2x+1)^2-6=19$$
$$(2x+1)^2=25$$
$$2x+1=\pm 5$$
$$2x=-1\pm 5$$
$$x=\frac{-1\pm 5}{2}\rightarrow x=\frac{-1+5}{2}=2\ or\ x=\frac{-1-5}{2}=-3$$

Problem Set

Solve the following quadratic equations. Reduce answers as much as possible. No decimals.

1. $x^2=144$

2. $5x^2 - 4 = 16$
3. $8 - 10x^2 = -22$
4. $(x+2)^2 = 49$
5. $6(x-5)^2 + 1 = 19$
6. $\frac{3}{4}x^2 - 19 = 26$
7. $x^2 - 12 = 36 - 2x^2$
8. $9 - \frac{x^2}{3} = -33$
9. $-4(x+7)^2 = -52$
10. $2(3x+4)^2 - 5 = 45$
11. $\frac{1}{3}(x-10)^2 - 8 = 16$
12. $\frac{(x-1)^2}{6} - \frac{8}{3} = \frac{7}{2}$

Use either factoring or solving by square roots to solve the following quadratic equations.

13. $x^2 - 16x + 55 = 0$
14. $2x^2 - 9 = 27$
15. $6x^2 + 23x = -20$
16. **Writing** Write a set of hints that will help you remember when you should solve an equation by factoring and by square roots. Are there any quadratics that can be solved using either method?
17. Solve $x^2 - 9 = 0$ by factoring and by using square roots. Which do you think is easier? Why?
18. Solve $(3x-2)^2 + 1 = 17$ by using square roots. Then, solve $3x^2 - 4x - 4 = 0$ by factoring. What do you notice? What can you conclude?
19. **Real Life Application** The *aspect ratio* of a TV screen is the ratio of the screen's width to its height. For HDTVs, the aspect ratio is 16:9. What is the width and height of a 42 inch screen TV? (42 inches refers to the length of the screen's diagonal.) HINT: Use the Pythagorean Theorem. Round your answers to the nearest hundredth.
20. **Real Life Application** When an object is dropped, its speed continually increases until it reaches the ground. This scenario can be modeled by the equation $h = -16t^2 + h_0$, where h is the height, t is the time (in seconds), and h_0 is the initial height of the object. Round your answers to the nearest hundredth.
    a. If you drop a ball from 200 feet, what is the height after 2 seconds?
    b. After how many seconds will the ball hit the ground?

15.3 Complex Numbers

Objective

To define and use complex and imaginary numbers. Then, solve quadratic equations with imaginary solutions.

Review Queue

1. Can you find $\sqrt{-25}$? Why or why not?
2. Simplify $\sqrt{192} \cdot \sqrt{27}$
3. Simplify $\sqrt{\frac{12}{15}}$
4. Solve $4(x-6)^2 - 7 = 61$

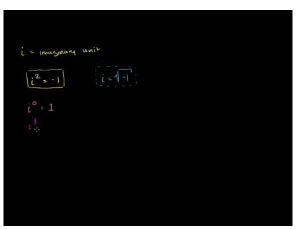
Defining Complex Numbers

Objective

To define, discover the "powers of i," and add and subtract complex and imaginary numbers.

Watch This

First, watch this video.



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Khan Academy: Introduction to i and Imaginary Numbers

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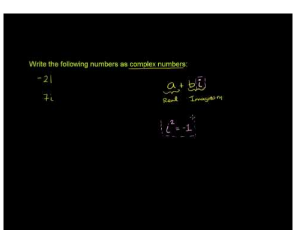

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Khan Academy: Complex Numbers

Guidance

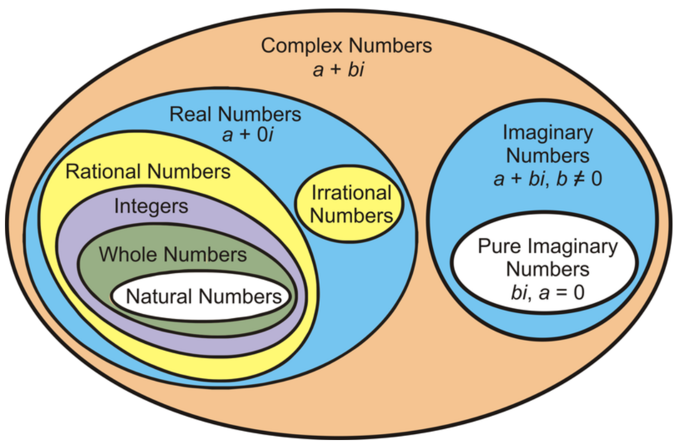
Before this concept, all numbers have been real numbers. 2, -5, $\sqrt{11}$, and $\frac{1}{3}$ are all examples of real numbers. Look at #1 from the Review Queue. With what we have previously learned, we cannot find $\sqrt{-25}$ because you cannot take the square root of a negative number. There is no real number that, when multiplied by itself, equals -25. Let's simplify $\sqrt{-25}$.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1}$$

In order to take the square root of a negative number we are going to assign $\sqrt{-1}$ a variable, i. i represents an **imaginary number**. Now, we can use i to take the square root of a negative number.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1} = 5i$$

All **complex numbers** have the form $a+bi$, where a and b are real numbers. a is the **real part** of the complex number and b is the **imaginary part**. If $b=0$, then a is left and the number is a **real number.** If $a=0$, then the number is only bi and called a **pure imaginary number**. If $b \neq 0$ and $a \neq 0$, the number will be an imaginary number.



Example A

Find $\sqrt{-162}$.

Solution: First pull out the i. Then, simplify $\sqrt{162}$.

$$\sqrt{-162} = \sqrt{-1} \cdot \sqrt{162} = i\sqrt{162} = i\sqrt{81 \cdot 2} = 9i\sqrt{2}$$

Investigation: Powers of i

In addition to now being able to take the square root of a negative number, i also has some interesting properties. Try to find i^2, i^3, and i^4.

1. Write out i^2 and simplify. $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1}^2 = -1$
2. Write out i^3 and simplify. $i^3 = i^2 \cdot i = -1 \cdot i = -i$
3. Write out i^4 and simplify. $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$
4. Write out i^5 and simplify. $i^5 = i^4 \cdot i = 1 \cdot i = i$
5. Write out i^6 and simplify. $i^6 = i^4 \cdot i^2 = 1 \cdot -1 = -1$
6. Do you see a pattern? Describe it and try to find i^{19}.

You should see that the powers of i repeat every 4 powers. So, all the powers that are divisible by 4 will be equal to 1. To find i^{19}, divide 19 by 4 and determine the remainder. That will tell you what power it is the same as.

$$i^{19} = i^{16} \cdot i^3 = 1 \cdot i^3 = -i$$

Example B

Find:

a) i^{32}

b) i^{50}

c) i^7

Solution:

a) 32 is divisible by 4, so $i^{32} = 1$.

b) $50 \div 4 = 12$, with a remainder of 2. Therefore, $i^{50} = i^2 = -1$.

c) $7 \div 4 = 1$, with a remainder of 3. Therefore, $i^7 = i^3 = -i$

Example C

Simplify the complex expressions.

a) $(6-4i)+(5+8i)$

b) $9-(4+i)+(2-7i)$

Solution: To add or subtract complex numbers, you need to combine like terms. Be careful with negatives and properly distributing them. Your answer should always be in **standard form**, which is $a+bi$.

a) $(6-4i)+(5+8i) = 6-4i+5+8i = 11+4i$

b) $9-(4+i)+(2-7i) = 9-4-i+2-7i = 7-8i$

Guided Practice

Simplify.

1. $\sqrt{-49}$

2. $\sqrt{-125}$

3. i^{210}

4. $(8-3i)-(12-i)$

Answers

1. Rewrite $\sqrt{-49}$ in terms of i and simplify the radical.

$$\sqrt{-49} = i\sqrt{49} = 7i$$

2. Rewrite $\sqrt{-125}$ in terms of i and simplify the radical.

$$\sqrt{-125} = i\sqrt{125} = i\sqrt{25 \cdot 5} = 5i\sqrt{5}$$

3. $210 \div 4 = 52$, with a remainder of 2. Therefore, $i^{210} = i^2 = -1$.

4. Distribute the negative and combine like terms.

$$(8-3i)-(12-i)=8-3i-12+i=-4-2i$$

Vocabulary

Imaginary Numbers
Any number with an i associated with it. Imaginary numbers have the form $a+bi$ or bi.

Complex Numbers
All real and imaginary numbers. Complex numbers have the **standard form** $a+bi$, where a or b can be zero. a is the **real part** and bi is the **imaginary part**.

Pure Imaginary Numbers
An imaginary number without a real part, only bi.

Problem Set

Simplify each expression and write in standard form.

1. $\sqrt{-9}$
2. $\sqrt{-242}$
3. $6\sqrt{-45}$
4. $-12i\sqrt{98}$
5. $\sqrt{-32}\cdot\sqrt{-27}$
6. $7i\sqrt{-126}$
7. i^8
8. $16i^{22}$
9. $-9i^{65}$
10. i^{365}
11. $2i^{91}$
12. $\sqrt{-\frac{16}{80}}$
13. $(11-5i)+(6-7i)$
14. $(14+2i)-(20+9i)$
15. $(8-i)-(3+4i)+15i$
16. $-10i-(1-4i)$
17. $(0.2+1.5i)-(-0.6+i)$
18. $6+(18-i)-(2+12i)$
19. $-i+(19+22i)-(8-14i)$
20. $18-(4+6i)+(17-9i)+24i$

Multiplying and Dividing Complex Numbers

Objective

To multiply and divide complex numbers.

Watch This

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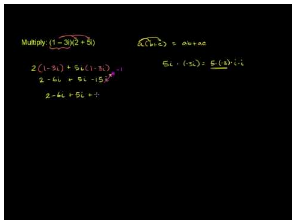

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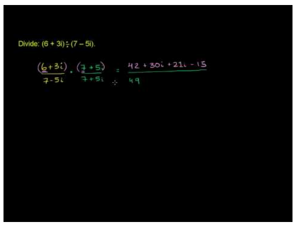
Khan Academy: Multiplying Complex Numbers

Then watch this video.



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Khan Academy: Dividing Complex Numbers

Guidance

When multiplying complex numbers, FOIL the two numbers together (see *Factoring when* $a = 1$ concept) and then combine like terms. At the end, there will be an i^2 term. Recall that $i^2 = -1$ and continue to simplify.

Example A

Simplify:

a) $6i(1-4i)$

b) $(5-2i)(3+8i)$

Solution:

a) Distribute the $6i$ to both parts inside the parenthesis.

$$6i(1-4i) = 6i - 24i^2$$

Substitute $i^2 = -1$ and simplify further.

$$\begin{aligned} &= 6i - 24(-1) \\ &= 24 + 6i \end{aligned}$$

Remember to always put the real part first.

b) FOIL the two terms together.

$$\begin{aligned} (5-2i)(3+8i) &= 15 + 40i - 6i - 16i^2 \\ &= 15 + 34i - 16i^2 \end{aligned}$$

Substitute $i^2 = -1$ and simplify further.

$$
\begin{aligned}
&= 15 + 34i - 16(-1) \\
&= 15 + 34i + 16 \\
&= 31 + 34i
\end{aligned}
$$

More Guidance

Dividing complex numbers is a bit more complicated. Similar to irrational numbers, complex numbers cannot be in the denominator of a fraction. To get rid of the complex number in the denominator, we need to multiply by the **complex conjugate**. If a complex number has the form $a+bi$, then its complex conjugate is $a-bi$. For example, the complex conjugate of $-6+5i$ would be $-6-5i$. Therefore, rather than dividing complex numbers, we multiply by the complex conjugate.

Example B

Simplify $\frac{8-3i}{6i}$.

Solution: In the case of dividing by a pure imaginary number, you only need to multiply the top and bottom by that number. Then, use multiplication to simplify.

$$
\begin{aligned}
\frac{8-3i}{6i} \cdot \frac{6i}{6i} &= \frac{48i-18i^2}{36i^2} \\
&= \frac{18+48i}{-36} \\
&= \frac{18}{-36} + \frac{48}{-36}i \\
&= -\frac{1}{2} - \frac{4}{3}i
\end{aligned}
$$

When the complex number contains fractions, write the number in standard form, keeping the real and imaginary parts separate. Reduce both fractions separately.

Example C

Simplify $\frac{3-5i}{2+9i}$.

Solution: Now we are dividing by $2+9i$, so we will need to multiply the top and bottom by the complex conjugate, $2-9i$.

$$
\begin{aligned}
\frac{3-5i}{2+9i} \cdot \frac{2-9i}{2-9i} &= \frac{6-27i-10i+45i^2}{4-18i+18i-81i^2} \\
&= \frac{6-37i-45}{4+81} \\
&= \frac{-39-37i}{85} \\
&= -\frac{39}{85} - \frac{37}{85}i
\end{aligned}
$$

Notice, by multiplying by the complex conjugate, the denominator becomes a real number and you can split the fraction into its real and imaginary parts.

In both Examples B and C, substitute $i^2 = -1$ to simplify the fraction further. ***Your final answer should never have any power of*** i ***greater than 1.***

Guided Practice

1. What is the complex conjugate of $7-5i$?

Simplify the following complex expressions.

2. $(7-4i)(6+2i)$

3. $\frac{10-i}{5i}$

4. $\frac{8+i}{6-4i}$

Answers

1. $7+5i$

2. FOIL the two expressions.

$$\begin{aligned}(7-4i)(6+2i) &= 42+14i-24i-8i^2 \\ &= 42-10i+8 \\ &= 50-10i\end{aligned}$$

3. Multiply the numerator and denominator by $5i$.

$$\begin{aligned}\frac{10-i}{5i}\cdot\frac{5i}{5i} &= \frac{50i-5i^2}{25i^2} \\ &= \frac{5+50i}{-25} \\ &= \frac{5}{-25}+\frac{50}{-25}i \\ &= -\frac{1}{5}-2i\end{aligned}$$

4. Multiply the numerator and denominator by the complex conjugate, $6+4i$.

$$\begin{aligned}\frac{8+i}{6-4i}\cdot\frac{6+4i}{6+4i} &= \frac{48+32i+6i+4i^2}{36+24i-24i-16i^2} \\ &= \frac{48+38i-4}{36+16} \\ &= \frac{44+38i}{52} \\ &= \frac{44}{52}+\frac{38}{52}i \\ &= \frac{11}{13}+\frac{19}{26}i\end{aligned}$$

Vocabulary

Complex Conjugate

The "opposite" of a complex number. If a complex number has the form $a+bi$, its complex conjugate is $a-bi$. When multiplied, these two complex numbers will produce a real number.

Problem Set

Simplify the following expressions. Write your answers in standard form.

1. $i(2-7i)$
2. $8i(6+3i)$
3. $-2i(11-4i)$
4. $(9+i)(8-12i)$
5. $(4+5i)(3+16i)$
6. $(1-i)(2-4i)$
7. $4i(2-3i)(7+3i)$
8. $(8-5i)(8+5i)$
9. $\frac{4+9i}{3i}$
10. $\frac{6-i}{12i}$
11. $\frac{7+12i}{-5i}$
12. $\frac{4-2i}{6-6i}$
13. $\frac{2-i}{2+i}$
14. $\frac{10+8i}{2+4i}$
15. $\frac{14+9i}{7-20i}$

Solving Quadratic Equations with Complex Number Solutions

Objective

To apply what we have learned about complex numbers and solve quadratic equations with complex number solutions.

Guidance

When you solve a quadratic equation, there will always be two answers. Until now, we thought the answers were always real numbers. In actuality, there are quadratic equations that have imaginary solutions as well. The possible solutions for a quadratic are:

2 real solutions

$$\begin{aligned} x^2-4&=0 \\ x&=-2,2 \end{aligned}$$

Double root

$$\begin{aligned} x^2+4x+4&=0 \\ x&=-2,-2 \end{aligned}$$

2 imaginary solutions

$$\begin{aligned} x^2+4&=0 \\ x&=-2i,2i \end{aligned}$$

Example A

Solve $3x^2+27=0$.

Solution: First, factor out the GCF.

$$3(x^2+9)=0$$

Now, try to factor x^2+9. Rewrite the quadratic as x^2+0x+9 to help. There are no factors of 9 that add up to 0. Therefore, this is not a factorable quadratic. Let's solve it using square roots.

$$\begin{aligned}3x^2+27&=0\\3x^2&=-27\\x^2&=-9\\x&=\pm\sqrt{-9}=\pm 3i\end{aligned}$$

Quadratic equations with imaginary solutions are never factorable.

Example B

Solve $(x-8)^2=-25$

Solution: Solve using square roots.

$$\begin{aligned}(x-8)^2&=-25\\x-8&=\pm 5i\\x&=8\pm 5i\end{aligned}$$

Example C

Solve $2(3x-5)+10=-30$.

Solution: Solve using square roots.

$$\begin{aligned}2(3x-5)^2+10&=-30\\2(3x-5)^2&=-40\\(3x-5)^2&=-20\\3x-5&=\pm 2i\sqrt{5}\\3x&=5\pm 2i\sqrt{5}\\x&=\frac{5}{3}\pm\frac{2\sqrt{5}}{3}i\end{aligned}$$

Guided Practice

1. Solve $4(x-5)^2+49=0$.

2. Solve $-\frac{1}{2}(3x+8)^2-16=2$.

Answers

Both of these quadratic equations can be solved by using square roots.

1.

$$\begin{aligned}4(x-5)^2+49&=0\\4(x-5)^2&=-49\\(x-5)^2&=-\frac{49}{4}\\x-5&=\pm\frac{7}{2}i\\x&=5\pm\frac{7}{2}i\end{aligned}$$

2.

$$\begin{aligned}-\frac{1}{2}(3x+8)^2-16&=2\\-\frac{1}{2}(3x+8)^2&=18\\(3x+8)^2&=-36\\3x+8&=\pm6i\\3x&=-8\pm6i\\x&=-\frac{8}{3}\pm2i\end{aligned}$$

Problem Set

Solve the following quadratic equations.

1. $(x+1)^2=-121$
2. $5x^2+16=-29$
3. $14-4x^2=38$
4. $(x-9)^2-2=-82$
5. $-3(x+6)^2+1=37$
6. $4(x-5)^2-3=-59$
7. $(2x-1)^2+5=-23$
8. $-(6x+5)^2=72$
9. $7(4x-3)^2-15=-68$
10. If a quadratic equation has $4-i$ as a solution, what must the other solution be?
11. If a quadratic equation has $6+2i$ as a solution, what must the other solution be?
12. **Challenge** Recall that the factor of a quadratic equation have the form $(x\pm m)$ where m is any number. Find the quadratic equation that has the solution $3+2i$.

15.4 Completing the Square

Objective

We will introduce another technique to solve quadratic equations, called completing the square.

Review Queue

Solve the following equations. Use the appropriate method.

1. $x^2-18x+32=0$
2. $2(x-4)^2=-54$
3. $4x^2-5x-6=0$
4. $x^2-162=0$

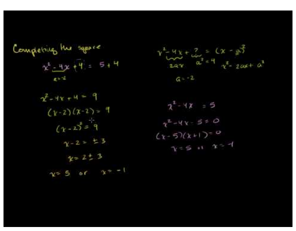
Completing the Square When the First Coefficient Equals 1

Objective

Learning how to complete the square for quadratic equations in the form $x^2+bx+c=0$.

Watch This

Watch the first part of this video, until about 5:25.



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Khan Academy: Solving Quadratic Equations by Completing the Square

Guidance

Completing the square is another technique used to solve quadratic equations. When completing the square, the goal is to make a perfect square trinomial and factor it.

Example A

Solve $x^2-8x-1=10$.

Solution:

1. Write the polynomial so that x^2 and x are on the left side of the equation and the constants on the right. This is only for organizational purposes, but it really helps. Leave a little space after the $x-$term.

$$x^2-8x=11$$

2. Now, "complete the square." Determine what number would make a perfect square trinomial with $x^2 - 8x + c$. To do this, divide the $x-$term by 2 and square that number, or $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{8}{2}\right)^2 = 4^2 = 16$$

3. Add this number to *both sides* in order to keep the equation balanced.

$$x^2 - 8x + 16 = 11 + 16$$

4. Factor the left side to the **square of a binomial** and simplify the right.

$$(x-4)^2 = 27$$

5. Solve by using square roots.

$$\begin{aligned} x - 4 &= \pm 3\sqrt{3} \\ x &= 4 \pm 3\sqrt{3} \end{aligned}$$

Completing the square enables you to solve any quadratic equation using square roots. Through this process, we can make an unfactorable quadratic equation solvable, like the one above. It can also be used with quadratic equations that have imaginary solutions.

Example B

Solve $x^2 + 12x + 37 = 0$

Solution: First, this is not a factorable quadratic equation. Therefore, the only way we know to solve this equation is to complete the square. Follow the steps from Example A.

1. Organize the polynomial, x's on the left, constant on the right.

$$x^2 + 12x = -37$$

2. Find $\left(\frac{b}{2}\right)^2$ and add it to both sides.

$$\begin{aligned} \left(\frac{b}{2}\right)^2 = \left(\frac{12}{2}\right)^2 &= 6^2 = 36 \\ x^2 + 12x + 36 &= -37 + 36 \end{aligned}$$

3. Factor the left side and solve.

$$\begin{aligned} (x+6)^2 &= -1 \\ x + 6 &= \pm i \\ x &= -6 \pm i \end{aligned}$$

Example C

Solve $x^2 - 11x - 15 = 0$.

Solution: This is not a factorable equation. Use completing the square.

1. Organize the polynomial, x's on the left, constant on the right.

$$x^2 - 11x = 15$$

2. Find $\left(\frac{b}{2}\right)^2$ and add it to both sides.

$$\begin{aligned}\left(\frac{b}{2}\right)^2 &= \left(\frac{11}{2}\right)^2 = \frac{121}{4}\\ x^2 - 11x + \frac{121}{4} &= 15 + \frac{121}{4}\end{aligned}$$

3. Factor the left side and solve.

$$\begin{aligned}\left(x - \frac{11}{2}\right)^2 &= \frac{60}{4} + \frac{121}{4}\\ \left(x - \frac{11}{2}\right)^2 &= \frac{181}{4}\\ x - \frac{11}{2} &= \pm\frac{\sqrt{181}}{2}\\ x &= \frac{11}{2} \pm \frac{\sqrt{181}}{2}\end{aligned}$$

Guided Practice

1. Find the value of c that would make $(x^2 - 2x + c)$ a perfect square trinomial. Then, factor the trinomial.

Solve the following quadratic equations by completing the square.

2. $x^2 + 10x + 21 = 0$

3. $x - 5x = 12$

Answers

1. $c = \left(\frac{b}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1^2 = 1$. The factors of $x^2 - 2x + 1$ are $(x-1)(x-1)$ or $(x-1)^2$.

2. Use the steps from the examples above.

$$\begin{aligned}
x^2+10x+21&=0\\
x^2+10x&=-21\\
x^2+10x+\left(\frac{10}{2}\right)^2&=-21+\left(\frac{10}{2}\right)^2\\
x^2+10x+25&=-21+25\\
(x+5)^2&=4\\
x+5&=\pm 2\\
x&=-5\pm 2\\
x&=-7,-3
\end{aligned}$$

3. Use the steps from the examples above.

$$\begin{aligned}
x^2-5x&=12\\
x^2-5x+\left(\frac{5}{2}\right)^2&=12+\left(\frac{5}{2}\right)^2\\
x^2-5x+\frac{25}{4}&=\frac{48}{4}+\frac{25}{4}\\
\left(x-\frac{5}{2}\right)^2&=\frac{73}{4}\\
x-\frac{5}{2}&=\pm\frac{\sqrt{73}}{2}\\
x&=\frac{5}{2}\pm\frac{\sqrt{73}}{2}
\end{aligned}$$

Vocabulary

Binomial
A mathematical expression with two terms.

Square of a Binomial
A binomial that is squared.

Complete the Square
The process used to solve unfactorable quadratic equations.

Problem Set

Determine the value of c that would complete the perfect square trinomial.

1. x^2+4x+c
2. x^2-2x+c
3. $x^2+16x+c$

Rewrite the perfect square trinomial as a square of a binomial.

4. x^2+6x+9
5. $x^2-7x+\frac{49}{4}$
6. $x^2-\frac{1}{2}x+\frac{1}{16}$

Solve the following quadratic equations by completing the square.

7. $x^2+6x-15=0$
8. $x^2+10x+29=0$
9. $x^2-14x+9=-60$
10. $x^2+3x+18=-2$
11. $x^2-9x-5=23$
12. $x^2-20x=60$

Solve the following quadratic equations by factoring, square roots, or completing the square.

13. $x^2+x-30=0$
14. $x^2-18x+90=0$
15. $x^2+15x+56=0$
16. $x^2+3x-24=12$
17. $(x-2)^2-20=-45$
18. $x^2+24x+44=-19$
19. Solve $x^2+7x-44=0$ by factoring and completing the square. Which method do you prefer?
20. **Challenge** Solve $x^2+\frac{17}{8}x-2=-9$.

Completing the Square When the First Coefficient Doesn't Equal 1

Objective

Learning how to complete the square for quadratic equations in the form $ax^2+bx+c=0$.

Guidance

When there is a number in front of x^2, it will make completing the square a little more complicated. See how the steps change in Example A.

Example A

Solve $3x^2-9x+11=0$

Solution:

1. Write the polynomial so that x^2 and x are on the left side of the equation and the constants on the right.

$$3x^2-9x=-11$$

2. Pull out a from everything on the left side. Even if b is not divisible by a, the coefficient of x^2 needs to be 1 in order to complete the square.

$$3(x^2-3x+\underline{\quad})=-11$$

3. Now, complete the square. Determine what number would make a perfect square trinomial.

To do this, divide the $x-$term by 2 and square that number, or $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

4. Add this number to the interior of the parenthesis on the left side. On the right side, you will need to add $a \cdot \left(\frac{b}{2}\right)^2$ to keep the equation balanced.

$$3\left(x^2 - 3x + \frac{9}{4}\right) = -11 + \frac{27}{4}$$

5. Factor the left side and simplify the right.

$$3\left(x - \frac{3}{2}\right)^2 = -\frac{17}{4}$$

6. Solve by using square roots.

$$\left(x - \frac{3}{2}\right)^2 = -\frac{17}{12}$$
$$x - \frac{3}{2} = \pm\frac{i\sqrt{17}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$x = \frac{3}{2} \pm \frac{\sqrt{51}}{6} i$$

Be careful with the addition of step 2 and the changes made to step 4. A very common mistake is to add $\left(\frac{b}{2}\right)^2$ to both sides, without multiplying by a for the right side.

Example B

Solve $4x^2 + 7x - 18 = 0$.

Solution: Let's follow the steps from Example A.

1. Write the polynomial so that x^2 and x are on the left side of the equation and the constants on the right.

$$4x^2 - 7x = 18$$

2. Pull out a from everything on the left side.

$$4\left(x^2 + \frac{7}{4}x + \_\_\right) = 18$$

3. Now, complete the square. Find $\left(\frac{b}{2}\right)^2$.

$$\left(\frac{b}{2}\right)^2 = \left(\frac{7}{8}\right)^2 = \frac{49}{64}$$

4. Add this number to the interior of the parenthesis on the left side. On the right side, you will need to add $a \cdot \left(\frac{b}{2}\right)^2$ to keep the equation balanced.

$$4\left(x^2 + \frac{7}{4}x + \frac{49}{64}\right) = 18 + \frac{49}{16}$$

5. Factor the left side and simplify the right.

$$4\left(x + \frac{7}{8}\right)^2 = \frac{337}{16}$$

6. Solve by using square roots.

$$\begin{aligned}
\left(x + \frac{7}{8}\right)^2 &= \frac{337}{64} \\
x + \frac{7}{8} &= \pm\frac{\sqrt{337}}{8} \\
x &= -\frac{7}{8} \pm \frac{\sqrt{337}}{8}
\end{aligned}$$

Guided Practice

Solve the following quadratic equations by completing the square.

1. $5x^2 + 29x - 6 = 0$

2. $8x^2 - 32x + 4 = 0$

Answers

Use the steps from the examples above to solve for x.

1.

$$\begin{aligned}
5x^2+29x-6&=0\\
5\left(x^2+\frac{29}{5}x\right)&=6\\
5\left(x^2+\frac{29}{5}x+\frac{841}{100}\right)&=6+\frac{841}{20}\\
5\left(x+\frac{29}{10}\right)^2&=\frac{961}{20}\\
\left(x+\frac{29}{10}\right)^2&=\frac{961}{100}\\
x+\frac{29}{10}&=\pm\frac{31}{10}\\
x&=-\frac{29}{10}\pm\frac{31}{10}\\
x&=-6,\frac{1}{5}
\end{aligned}$$

2.

$$\begin{aligned}
8x^2-32x+4&=0\\
8(x^2-4x)&=-4\\
8(x^2-4x+4)&=-4+32\\
8(x-2)^2&=28\\
(x-2)^2&=\frac{7}{2}\\
x-2&=\pm\frac{\sqrt{7}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\
x&=2\pm\frac{\sqrt{14}}{2}
\end{aligned}$$

Problem Set

Solve the quadratic equations by completing the square.

1. $6x^2-12x-7=0$
2. $-4x^2+24x-100=0$
3. $5x^2-30x+55=0$
4. $2x^2-x-6=0$
5. $\frac{1}{2}x^2+7x+8=0$
6. $-3x^2+4x+15=0$

Solve the following equations by factoring, using square roots, or completing the square.

7. $4x^2-4x-8=0$
8. $2x^2+9x+7=0$
9. $-5(x+4)^2-19=26$
10. $3x^2+30x-5=0$

11. $9x^2 - 15x - 6 = 0$
12. $10x^2 + 40x + 88 = 0$

Problems 13-15 build off of each other.

13. **Challenge** Complete the square for $ax^2 + bx + c = 0$. Follow the steps from Examples A and B. Your final answer should be in terms of a, b, and c.
14. For the equation $8x^2 + 6x - 5 = 0$, use the formula you found in #13 to solve for x.
15. Is the equation in #14 factorable? If so, factor and solve it.
16. **Error Analysis** Examine the worked out problem below.

$$\begin{aligned}
4x^2 - 48x + 11 &= 0 \\
4(x^2 - 12x + \underline{\quad}) &= -11 \\
4(x^2 - 12x + 36) &= -11 + 36 \\
4(x-6)^2 &= 25 \\
(x-6)^2 &= \frac{25}{4} \\
x - 6 &= \pm\frac{5}{2} \\
x &= 6 \pm \frac{5}{2} \rightarrow \frac{17}{2}, \frac{7}{2}
\end{aligned}$$

Plug the answers into the original equation to see if they work. If not, find the error and correct it.

15.5 The Quadratic Formula

Objective

To derive and use the Quadratic Formula to solve quadratic equations and determine how many solutions an equation has.

Review Queue

Solve each equation by completing the square.

1. $x^2 - 4x + 20 = 0$
2. $4x^2 - 12x - 33 = 0$

Solve each equation by factoring.

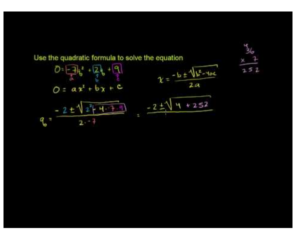
3. $12x^2 + 31x + 20 = 0$
4. $5x^2 - 30x - 23 = x^2 - 77$

Deriving and Using the Quadratic Formula

Objective

Deriving the Quadratic Formula and using it to solve any quadratic equation.

Watch This



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Khan Academy: Quadratic Formula 1

Guidance

The last way to solve a quadratic equation is the **Quadratic Formula.** This formula is derived from completing the square for the equation $ax^2 + bx + c = 0$ (see #13 from the Problem Set in the previous concept). We will derive the formula here.

Investigation: Deriving the Quadratic Formula

Walk through each step of completing the square of $ax^2 + bx + c = 0$.

1. Move the constant to the right side of the equation. $ax^2 + bx = -c$
2. "Take out" a from everything on the left side of the equation. $a\left(x^2 + \frac{b}{a}x\right) = -c$
3. Complete the square using $\frac{b}{a}$. $\left(\frac{b}{2}\right)^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$
4. Add this number to both sides. Don't forget on the right side, you need to multiply it by a (to account for the a

outside the parenthesis). $a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)=-c+\frac{b^2}{4a}$

5. Factor the quadratic equation inside the parenthesis and give the right hand side a common denominator. $a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a}$

6. Divide both sides by a. $\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$

7. Take the square root of both sides. $x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$

8. Subtract $\frac{b}{2a}$ from both sides to get x by itself. $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

This formula will enable you to solve any quadratic equation as long as you know a, b, and c (from $ax^2+bx+c=0$).

Example A

Solve $9x^2-30x+26=0$ using the Quadratic Formula.

Solution: First, make sure one side of the equation is zero. Then, find a, b, and c. $a=9, b=-30, c=26$. Now, plug in the values into the formula and solve for x.

$$\begin{aligned}
x&=\frac{-(-30)\pm\sqrt{(-30)^2-4(9)(26)}}{2(9)}\\
&=\frac{30\pm\sqrt{900-936}}{18}\\
&=\frac{30\pm\sqrt{-36}}{18}\\
&=\frac{30\pm 6i}{18}\\
&=\frac{5}{3}\pm\frac{1}{3}i
\end{aligned}$$

Example B

Solve $2x^2+5x-15=-x^2+7x+2$ using the Quadratic Formula.

Solution: Let's get everything onto the left side of the equation.

$$\begin{aligned}
&2x^2+5x-15=-x^2+7x+2\\
&3x^2-2x-13=0
\end{aligned}$$

Now, use $a=3, b=-2$, and $c=-13$ and plug them into the Quadratic Formula.

$$\begin{aligned}
x&=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-13)}}{2(3)}\\
&=\frac{2\pm\sqrt{4+156}}{6}\\
&=\frac{2\pm\sqrt{160}}{6}\\
&=\frac{2\pm 4\sqrt{10}}{3}
\end{aligned}$$

Example C

Solve $x^2+20x+51=0$ by factoring, completing the square, and the Quadratic Formula.

Solution: While it might not look like it, 51 is not a prime number. Its factors are 17 and 3, which add up to 20.

$$\begin{aligned}x^2+20x+51&=0\\(x+17)(x+13)&=0\\x&=-17,-3\end{aligned}$$

Now, solve by completing the square.

$$\begin{aligned}x^2+20x+51&=0\\x^2+20x&=-51\\x^2+20x+100&=-51+100\\(x+10)^2&=49\\x+10&=\pm 7\\x&=-10\pm 7\rightarrow -17,-3\end{aligned}$$

Lastly, let's use the Quadratic Formula. $a=1, b=20, c=51$.

$$\begin{aligned}x&=\frac{-20\pm\sqrt{20^2-4(1)(51)}}{2(1)}\\&=\frac{-20\pm\sqrt{400-204}}{2}\\&=\frac{-20\pm\sqrt{196}}{2}\\&=\frac{-20\pm 14}{2}\\&=-17,-3\end{aligned}$$

Notice that no matter how you solve this, or any, quadratic equation, the answer will always be the same.

Guided Practice

1. Solve $-6x^2+15x-22=0$ using the Quadratic Formula.

2. Solve $2x^2-x-15=0$ using all three methods.

Answers

1. $a=-6, b=15$, and $c=-22$

$$\begin{aligned}x&=\frac{-15\pm\sqrt{15^2-4(-6)(-22)}}{2(-6)}\\&=\frac{-15\pm\sqrt{225-528}}{-12}\\&=\frac{-15\pm i\sqrt{303}}{-12}\\&=\frac{5}{4}\pm\frac{\sqrt{303}}{12}i\end{aligned}$$

2. Factoring: $ac = -30$. The factors of -30 that add up to -1 are -6 and 5. Expand the $x-$term.

$$\begin{aligned} 2x^2 - 6x + 5x - 15 &= 0 \\ 2x(x-3) + 5(x-3) &= 0 \\ (x-3)(2x+5) &= 0 \\ x &= 3, -\frac{5}{2} \end{aligned}$$

Complete the square

$$\begin{aligned} 2x^2 - x - 15 &= 0 \\ 2x^2 - x &= 15 \\ 2\left(x^2 - \frac{1}{2}x\right) &= 15 \\ 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) &= 15 + \frac{1}{8} \\ 2\left(x - \frac{1}{4}\right)^2 &= \frac{121}{8} \\ \left(x - \frac{1}{4}\right)^2 &= \frac{121}{16} \\ x - \frac{1}{4} &= \pm\frac{11}{4} \\ x &= \frac{1}{4} \pm \frac{11}{4} \rightarrow 3, -\frac{5}{2} \end{aligned}$$

Quadratic Formula

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1^2 - 4(2)(-15)}}{2(2)} \\ &= \frac{1 \pm \sqrt{1+120}}{4} \\ &= \frac{1 \pm \sqrt{121}}{4} \\ &= \frac{1 \pm 11}{4} \\ &= \frac{12}{4}, -\frac{10}{4} \rightarrow 3, -\frac{5}{2} \end{aligned}$$

Vocabulary

Quadratic Formula: For any quadratic equation in the form $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Problem Set

Solve the following equations using the Quadratic Formula.

1. $x^2 + 8x + 9 = 0$

2. $4x^2 - 13x - 12 = 0$
3. $-2x^2 + x + 5 = 0$
4. $7x^2 - 11x + 12 = 0$
5. $3x^2 + 4x + 5 = 0$
6. $x^2 - 14x + 49 = 0$

Choose any method to solve the equations below.

7. $x^2 + 5x - 150 = 0$
8. $8x^2 - 2x - 3 = 0$
9. $-5x^2 + 18x - 24 = 0$
10. $10x^2 + x - 2 = 0$
11. $x^2 - 16x + 4 = 0$
12. $9x^2 - 196 = 0$

Solve the following equations using all three methods.

13. $4x^2 + 20x + 25 = 0$
14. $x^2 - 18x - 63 = 0$
15. **Writing** Explain when you would use the different methods to solve different types of equations. Would the type of answer (real or imaginary) help you decide which method to use? Which method do you think is the easiest?

Using the Discriminant

Objective

Using the discriminant of the Quadratic Formula to determine how many real solutions an equation has.

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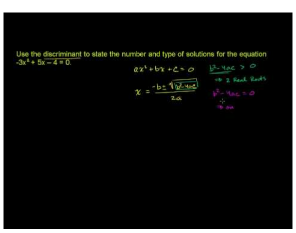

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Khan Academy: Discriminant for Types of Solutions for a Quadratic

Guidance

From the previous concept, the Quadratic Formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The expression under the radical, $b^2 - 4ac$, is called the **discriminant.** You can use the discriminant to determine the number and type of solutions an equation has.

Investigation: Solving Equations with Different Types of Solutions

1. Solve $x^2 - 8x - 20 = 0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{144}}{2} \\ &= \frac{8 \pm 12}{2} \rightarrow 10, -2 \end{aligned}$$

2. Solve $x^2 - 8x + 6 = 0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{0}}{2} \\ &= \frac{8 \pm 0}{2} \rightarrow 4 \end{aligned}$$

3. Solve $x^2 - 8x + 20 = 0$ using the Quadratic Formula. What is the value of the discriminant?

$$\begin{aligned} x &= \frac{8 \pm \sqrt{-16}}{2} \\ &= \frac{8 \pm 4i}{2} \rightarrow 4 \pm 2i \end{aligned}$$

4. Look at the values of the discriminants from Steps 1-3. How do they differ? How does that affect the final answer?

From this investigation, we can conclude:

- If $b^2 - 4ac > 0$, then the equation has two real solutions.
- If $b^2 - 4ac = 0$, then the equation has one real solution; a double root.
- If $b^2 - 4ac < 0$, then the equation has two imaginary solutions.

Example A

Determine the type of solutions $4x^2 - 5x + 17 = 0$ has.

Solution: Find the discriminant.

$$\begin{aligned} b^2 - 4ac &= (-5)^2 - 4(4)(17) \\ &= 25 - 272 \end{aligned}$$

At this point, we know the answer is going to be negative, so there is no need to continue (unless we were solving the problem). This equation has two imaginary solutions.

Example B

Solve the equation from Example A to prove that it does have two imaginary solutions.

Solution: Use the Quadratic Formula.

$$x = \frac{5 \pm \sqrt{25 - 272}}{8} = \frac{5 \pm \sqrt{-247}}{8} = \frac{5}{8} \pm \frac{\sqrt{247}}{8} i$$

Guided Practice

1. Use the discriminant to determine the type of solutions $-3x^2 - 8x + 16 = 0$ has.

2. Use the discriminant to determine the type of solutions $25x^2 - 80x + 64 = 0$ has.

3. Solve the equation from #1.

Answers

1.

$$\begin{aligned} b^2 - 4ac &= (-8)^2 - 4(-3)(16) \\ &= 64 + 192 \\ &= 256 \end{aligned}$$

This equation has two real solutions.

2.

$$\begin{aligned} b^2 - 4ac &= (-80)^2 - 4(25)(64) \\ &= 6400 - 6400 \\ &= 0 \end{aligned}$$

This equation has one real solution.

3. $x = \frac{8 \pm \sqrt{256}}{-6} = \frac{8 \pm 16}{-6} = -4, \frac{4}{3}$

Vocabulary

Discriminant
The value under the radical in the Quadratic Formula, $b^2 - 4ac$. The discriminant tells us number and type of solution(s) a quadratic equation has.

Problem Set

Determine the number and type of solutions each equation has.

1. $x^2 - 12x + 36 = 0$
2. $5x^2 - 9 = 0$
3. $2x^2 + 6x + 15 = 0$
4. $-6x^2 + 8x + 21 = 0$
5. $x^2 + 15x + 26 = 0$
6. $4x^2 + x + 1 = 0$

Solve the following equations using the Quadratic Formula.

7. $x^2 - 17x - 60 = 0$
8. $6x^2 - 20 = 0$
9. $2x^2 + 5x + 11 = 0$

Challenge Determine the values for c that make the equation have a) two real solutions, b) one real solution, and c) two imaginary solutions.

10. $x^2 + 2x + c = 0$
11. $x^2 - 6x + c = 0$
12. $x^2 + 12x + c = 0$
13. What is the discriminant of $x^2 + 2kx + 4 = 0$? Write your answer in terms of k.
14. For what values of k will the equation have a) two real solutions, b) one real solution, and c) two imaginary solutions?

15.6 Parabolas

Objective

To analyze a parabola: find the vertex, focus, directrix, and graph.

Review Queue

1. Graph $y = 2x^2$. Find the vertex, axis of symmetry and any intercepts.
2. Graph $y = x^2 + 10x - 24$. Find the axis of symmetry and the vertex.
3. What are the intercepts of $y = x^2 - 7x - 18$?

Parabolas with Vertex at (0, 0)

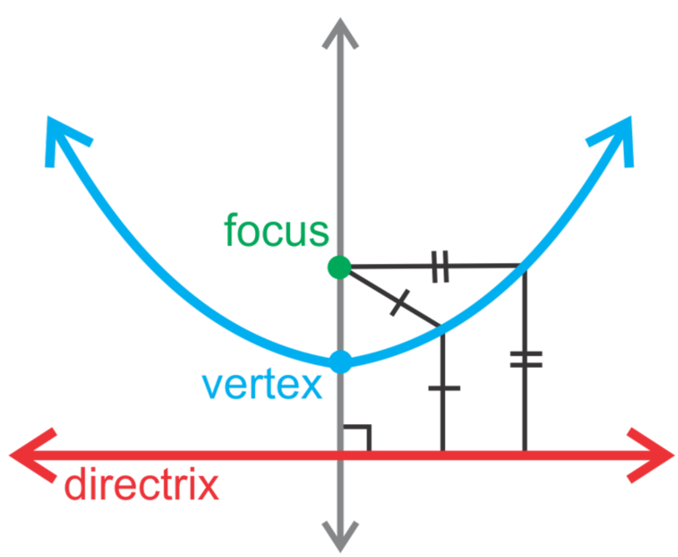
Objective

To write and graph the equation of a parabola, with vertex $(0,0)$, and find the focus, directrix, and vertex.

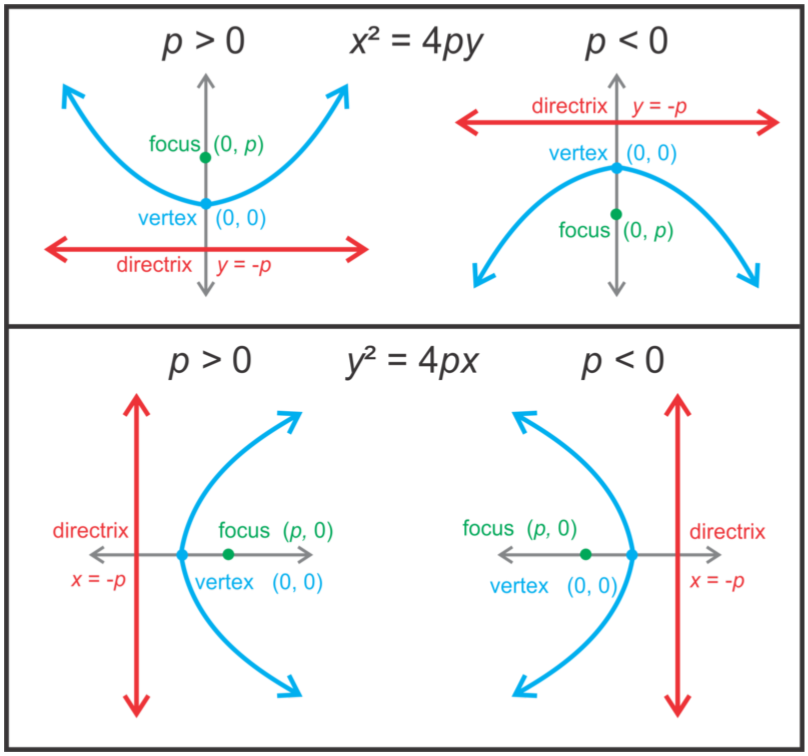
Guidance

You already know that the graph of a parabola has the parent graph $y = x^2$, with a vertex of $(0,0)$ and an axis of symmetry of $x = 0$. A parabola can also be defined in a different way. It has a property such that any point on it is equidistant from another point, called the **focus**, and a line called the **directrix**.

The focus is one the axis of symmetry and the vertex is halfway between it and the directrix. The directrix is perpendicular to the axis of symmetry.



Until now, we have been used to seeing the equation of a parabola like $y = ax^2$. In this concept, we will rewrite the equation to look like $x^2 = 4py$ where p is used to find the focus and directrix. We will also draw the parabola with a horizontal orientation, such that the equation will be $y^2 = 4px$.



Notice, that when the parabola opens to the left or right, the y is squared. In this concept, the vertex will be $(0,0)$.

Example A

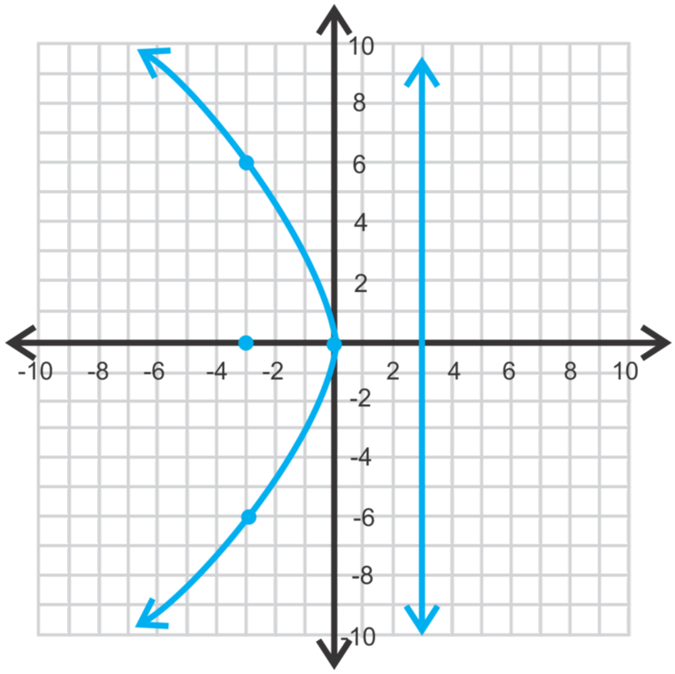
Analyze the equation $y^2 = -12x$. Find the focus, directrix, and determine if the function opens up, down, to the left or right. Then graph the parabola.

Solution: To find the focus and directrix, we need to find p. We can set $-12 = 4p$ and solve for p.

$$\begin{aligned} -12 &= 4p \\ -3 &= p \end{aligned}$$

Because y is squared, we know that the parabola opens to the left or right. Because p is negative, we know it is going to open to the left, towards the negative side of the x-axis. Using the pictures above, this parabola is like the second one under $y^2 = 4px$. Therefore, the focus is $(-3,0)$ and the directrix is $x = 3$. To graph the parabola, plot the vertex, focus, directrix, and sketch the curve. Find at least one or two points on the curve to make sure your sketch is accurate. For example, because $(-3,6)$ is on the parabola, then $(-3,-6)$ is also on the parabola because it is symmetrical.

Notice that the points $(-3,6)$ and $(-3,-6)$ are equidistant from the focus and the directrix. They are both 6 units from each.



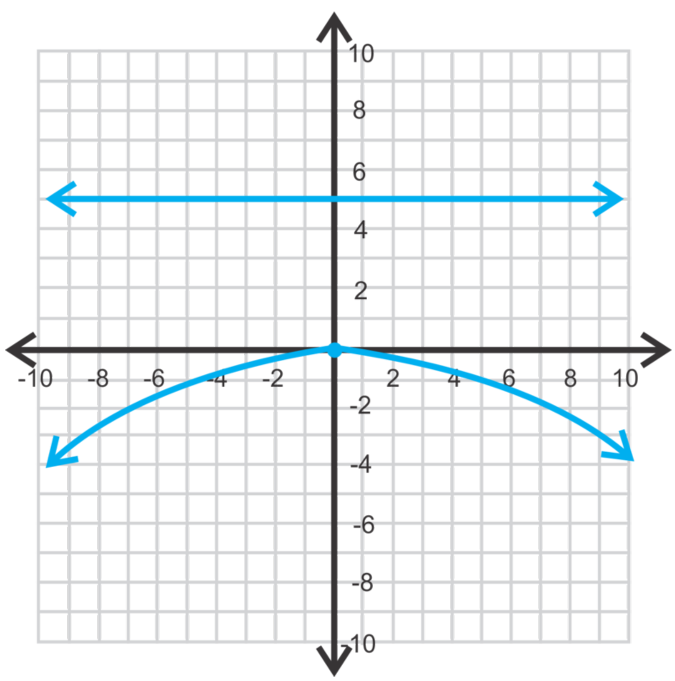
Example B

The focus of a parabola is $\left(0, \frac{1}{2}\right)$. Find the equation of the parabola.

Solution: Because the p value is the y-value and positive, this parabola is going to open up. So, the general equation is $x^2 = 4py$. Plugging in $\frac{1}{2}$ for p, we have $x^2 = 4 \cdot \frac{1}{2}y$ or $x^2 = 2y$.

Example C

Find the equation of the parabola below.



Solution: The equation of the directrix is $y = 5$, which means that $p = -5$ and the general equation will be $x^2 = 4py$. Plugging in -5 for p, we have $x^2 = -20y$.

Guided Practice

1. Determine if the parabola $x^2 = -2y$ opens up, down, left or right.

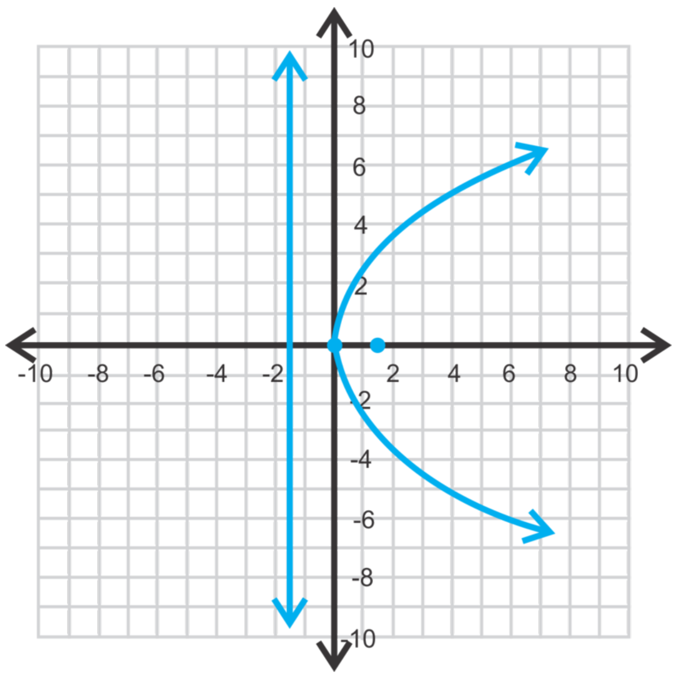
2. Find the focus and directrix of $y^2 = 6x$. Then, graph the parabola.

3. Find the equation of the parabola with directrix $x = -\frac{3}{8}$.

Answers

1. Down; p is negative and x is squared.

2. Solving for p, we have $4p = 6 \rightarrow p = \frac{3}{2}$. Because y is squared and p is positive, the parabola will open to the right. The focus is $\left(\frac{3}{2}, 0\right)$ and the directrix is $x = -\frac{3}{2}$.



3. If the directrix is negative and vertical $(x =)$, we know that the equation is going to be $y^2 = 4px$ and the parabola will open to the right, making p positive; $p = \frac{3}{8}$. Therefore, the equation will be $y^2 = 4 \cdot \frac{3}{8} \cdot x \rightarrow y^2 = \frac{3}{2}x$.

Vocabulary

Parabola
: The set of points that are equidistant from a fixed point on the interior of the curve, called the **focus**, and a line on the exterior, called the **directrix**. The directrix is vertical or horizontal, depending on the orientation of the parabola.

Problem Set

Determine if the parabola opens to the left, right, up or down.

1. $x^2 = 4y$
2. $y^2 = -\frac{1}{2}x$
3. $x^2 = -y$

Find the focus and directrix of the following parabolas.

4. $x^2 = -2y$
5. $y^2 = \frac{1}{4}x$

6. $y^2 = -5x$

Graph the following parabolas. Identify the focus and directrix as well.

7. $x^2 = 8y$
8. $y^2 = \frac{1}{2}x$
9. $x^2 = -3y$

Find the equation of the parabola given that the vertex is $(0,0)$ and the focus or directrix.

10. focus: $(4,0)$
11. directrix: $x = 10$
12. focus: $\left(0, \frac{7}{2}\right)$
13. In the *Quadratics* chapter, the basic parabolic equation was $y = ax^2$. Now, we write $x^2 = 4py$. Rewrite p in terms of a and determine how they affect each other.
14. ***Challenge*** Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$, to prove that the point $(4,2)$ is on the parabola $x^2 = 8y$.
15. ***Real World Application*** A satellite dish is a 3-dimensional parabola used to retrieve sound, TV, or other waves. Assuming the vertex is $(0,0)$, where would the focus have to be on a satellite dish that is 4 feet wide and 9 inches deep? You may assume the parabola has a vertical orientation (opens up).

Parabolas with Vertex at (h, k)

Objective

To write and graph the equation of a parabola with vertex (h,k) and find the focus, directrix, and vertex.

Guidance

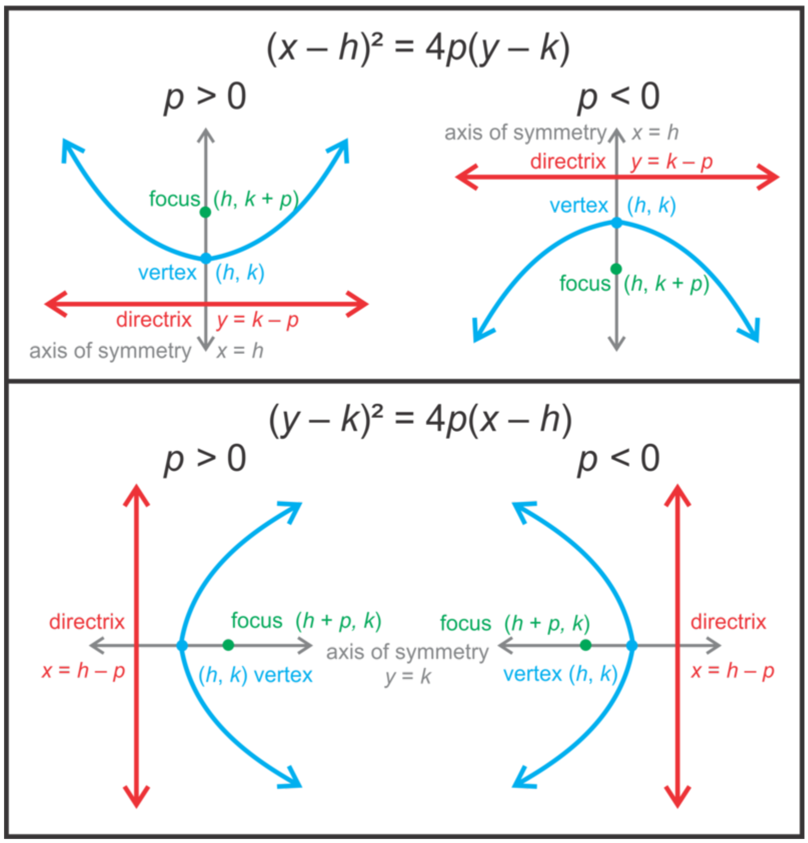
You learned in the *Quadratics* chapter that parabolas don't always have their vertex at $(0,0)$. In this concept, we will address parabolas where the vertex is (h,k), learn how to find the focus, directrix and graph.

Recall from the previous concept that the equation of a parabola is $x^2 = 4py$ or $y^2 = 4px$ and the vertex is on the origin. In the Quadratics chapter, we learned that the vertex form of a parabola is $y = a(x-h)^2 + k$. Combining the two, we can find the vertex form for conics.

$$\begin{aligned} y = a(x-h)^2 + k \textit{ and } x^2 &= 4py && \text{Solve the first for } (x-h)^2. \\ (x-h)^2 &= \frac{1}{a}(y-k) && \text{From \#13 in the previous concept, we found that } 4p = \frac{1}{a}. \\ (x-h)^2 &= 4p(y-k) \end{aligned}$$

If the parabola is horizontal, then the equation will be $(y-k)^2 = 4p(x-h)$. Notice, that even though the orientation is changed, the h and k values remain with the x and y values, respectively.

Finding the focus and directrix are a little more complicated. Use the extended table (from the previous concept) below to help you find these values.



Notice that the way we find the focus and directrix does not change whether p is positive or negative.

Example A

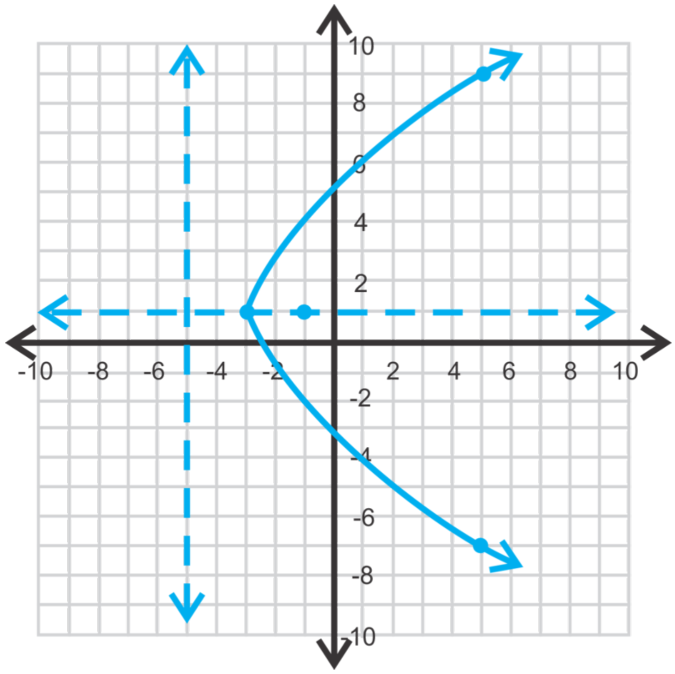
Analyze the equation $(y-1)^2 = 8(x+3)$. Find the vertex, axis of symmetry, focus, and directrix. Then determine if the function opens up, down, left or right.

Solution: First, because y is squared, we know that the parabola will open to the left or right. We can conclude that the parabola will open to the *right* because 8 is positive, meaning that p is positive. Next, find the vertex. Using the general equation, $(y-k)^2 = 4p(x-h)$, the vertex is $(-3,1)$ and the axis of symmetry is $y = 1$. Setting $4p = 8$, we have that $p = 2$. Adding p to the x-value of the vertex, we get the focus, $(-1,1)$. Subtracting p from the x-value of the vertex, we get the directrix, $x = -5$.

Example B

Graph the parabola from Example A. Plot the vertex, axis of symmetry, focus, and directrix.

Solution: First, plot all the critical values we found from Example A. Then, determine a set of symmetrical points that are on the parabola to make sure your curve is correct. If $x = 5$, then y is either -7 or 9. This means that the points $(5,-7)$ and $(5,9)$ are both on the parabola.



It is important to note that parabolas with a horizontal orientation are not functions because they do not pass the vertical line test.

Example C

The vertex of a parabola is $(-2,4)$ and the directrix is $y = 7$. Find the equation of the parabola.

Solution: First, let's determine the orientation of this parabola. Because the directrix is horizontal, we know that the parabola will open up or down (see table/pictures above). We also know that the directrix is *above* the vertex, making the parabola open down and p will be negative (plot this on an $x-y$ plane if you are unsure).

To find p, we can use the vertex, (h,k) and the equation for a horizontal directrix, $y = k - p$.

$$\begin{aligned} 7 &= 4 - p \\ 3 &= -p && \text{Remember, } p \text{ is negative because of the downward orientation of the parabola.} \\ -3 &= p \end{aligned}$$

Now, using the general form, $(x-h)^2 = 4p(y-k)$, we can find the equation of this parabola.

$$(x-(-2))^2 = 4(-3)(y-4)$$
$$(x+2)^2 = -12(y-4)$$

Guided Practice

1. Find the vertex, focus, axis of symmetry and directrix of $(x+5)^2 = 2(y+2)$.

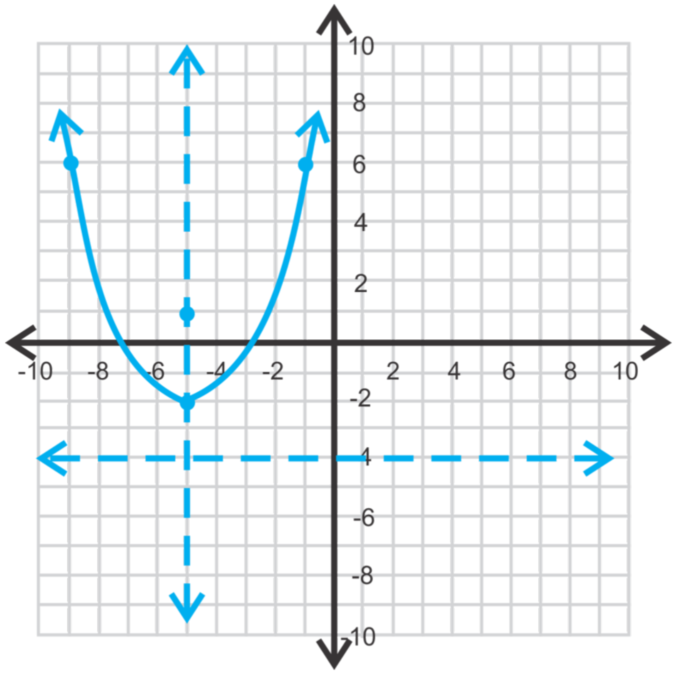
2. Graph the parabola from #1.

3. Find the equation of the parabola with vertex $(-5,-1)$ and focus $(-8,-1)$.

Answers

1. The vertex is $(-5,-2)$ and the parabola opens up because p is positive and x is squared. $4p = 2$, making $p = 2$. The focus is $(-5,-2+2)$ or $(-5,0)$, the axis of symmetry is $x = -5$, and the directrix is $y = -2-2$ or $y = -4$.

2.



3. The vertex is $(-5,-1)$, so $h = -5$ and $k = -1$. The focus is $(-8,-1)$, meaning that that parabola will be horizontal. We know this because the y-values of the vertex and focus are both -1. Therefore, p is added or subtracted to h.

$(h+p,k) \rightarrow (-8,-1)$ we can infer that $h+p = -8 \rightarrow -5+p = -8$ and $p = -3$

Therefore, the equation is $(y-(-1))^2 = 4(-3)(x-(-5)) \rightarrow (y+1)^2 = -12(x+5)$.

Vocabulary

Standard Form (of a Parabola)
$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$ where (h,k) is the vertex.

Problem Set

Find the vertex, focus, axis of symmetry, and directrix of the parabolas below.

1. $(x+1)^2 = -3(y-6)$
2. $(x-3)^2 = y-7$
3. $(y+2)^2 = 8(x+1)$
4. $y^2 = -10(x-3)$
5. $(x+6)^2 = 4(y+8)$
6. $(y-5)^2 = -\frac{1}{2}x$
7. Graph the parabola from #1.
8. Graph the parabola from #2.
9. Graph the parabola from #4.
10. Graph the parabola from #5.

Find the equation of the parabola given the vertex and either the focus or directrix.

11. vertex: $(2,-1)$, focus: $(2,-4)$
12. vertex: $(-3,6)$, directrix: $x=2$
13. vertex: $(6,10)$, directrix: $y=9.5$
14. ***Challenge*** focus: $(-1,-2)$, directrix: $x=3$
15. ***Extension*** Rewrite the equation of the parabola, $x^2-8x+2y+22=0$, in standard form by completing the square. Then, find the vertex. (For a review, see the *Completing the Square when* $a=1$ concept.)

15.7 Analyzing the Graph of a Quadratic Function

Objective

Graphing, finding the vertex and $x-$intercepts, and using all the forms of a quadratic equation.

Review Queue

Solve the following equations using the method of your choice.

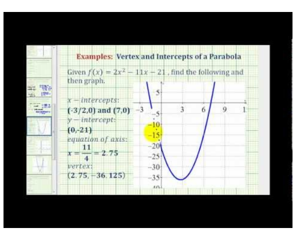
1. $x^2+6x-27=0$
2. $x^2-10x+29=0$
3. $x^2-8x+16=0$
4. In this lesson, we are going to be analyzing the graph of a quadratic function. Using what you know about the solutions of a quadratic equation, what do you think the shape of the function will look like? Draw it on your paper.

Finding the Parts of a Parabola

Objective

Finding the $x-$intercepts, vertex, axis of symmetry, and $y-$intercept of a parabola.

Watch This



MEDIA

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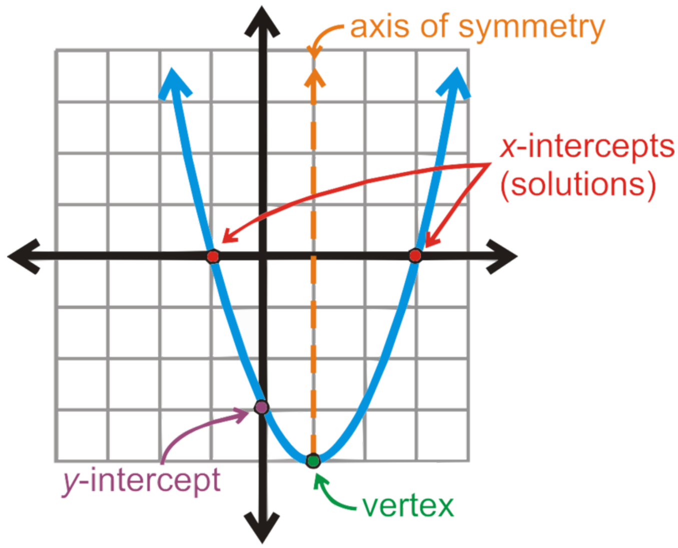
James Sousa: Ex 4: Graph a Quadratic Function in General Form by Finding Key Components

Guidance

Now that we have found the solutions of a quadratic equation we will graph the function. First, we need to introduce y or $f(x)$. A quadratic function is written $y=ax^2+bx+c$ or $f(x)=ax^2+bx+c$ (see *Finding the Domain and Range of Functions* concept). All quadratic equations are also functions.

Recall that the solutions of a quadratic equation are found when the equation is set equal to zero. This is also the same as when $y=0$. Therefore, the solutions of a quadratic equation are also the $x-$ **intercepts** of that function, when graphed.



The graph of a quadratic equation is called a **parabola** and looks like the figure to the left. A parabola always has "U" shape and depending on certain values, can be wider or narrower. The lowest part of a parabola, or minimum, is called the **vertex**. Parabolas can also be flipped upside-down, and in this case, the vertex would be the maximum value. Notice that this parabola is symmetrical about vertical line that passes through the vertex. This line is called the **axis of symmetry.** Lastly, where the parabola passes through the $y-$axis (when $x = 0$), is the **$y-$intercept**.

If you are given (or can find) the $x-$intercepts and the vertex, you can always graph a parabola.

Investigation: Finding the Vertex of a Parabola

1. The equation of the parabola above is $y = x^2 - 2x - 3$. Find a, b, and c. $a = 1, b = -2, c = -3$

2. What are the coordinates of the vertex? (1, -4)

3. Create an expression using a and b (from Step 1) that would be equal to the $x-$coordinate of the vertex. $1 = \frac{-b}{2a}$

4. Plug in $x = 1$ to the equation of the parabola. What do you get for y? $y = -4$

From this investigation, we have introduced how to find the vertex of a parabola. The $x-$coordinate of the vertex is $x = \frac{-b}{2a}$. To find y, plug in this value to the equation, also written $f\left(\frac{-b}{2a}\right)$. $x = \frac{-b}{2a}$ is also the equation of the axis of symmetry.

Example A

Find the vertex, axis of symmetry, $x-$intercepts, and $y-$intercept of $y = -\frac{1}{2}x^2 - 2x + 6$.

Solution: First, let's find the $x-$intercepts. This equation is factorable and $ac = -3$. The factors of -3 that add up to -2 are -3 and 1. Expand the $x-$term and factor.

$$\begin{aligned}
-\frac{1}{2}x^2 - 2x + 6 &= 0 \\
-\frac{1}{2}x^2 - 3x + x + 6 &= 0 \\
-x\left(\frac{1}{2}x + 3\right) + 2\left(\frac{1}{2}x + 3\right) &= 0 \\
\left(\frac{1}{2}x + 3\right)(-x + 2) &= 0
\end{aligned}$$

Solving for x, the intercepts are (-6, 0) and (2, 0).

To find the vertex, use $x = \frac{-b}{2a}$.

$x = \frac{-(-2)}{2 \cdot -\frac{1}{2}} = \frac{2}{-1} = -2$ Plug this into the equation: $y = -\frac{1}{2}(-2)^2 - 2(-2) + 6 = -2 + 4 + 6 = 8$.

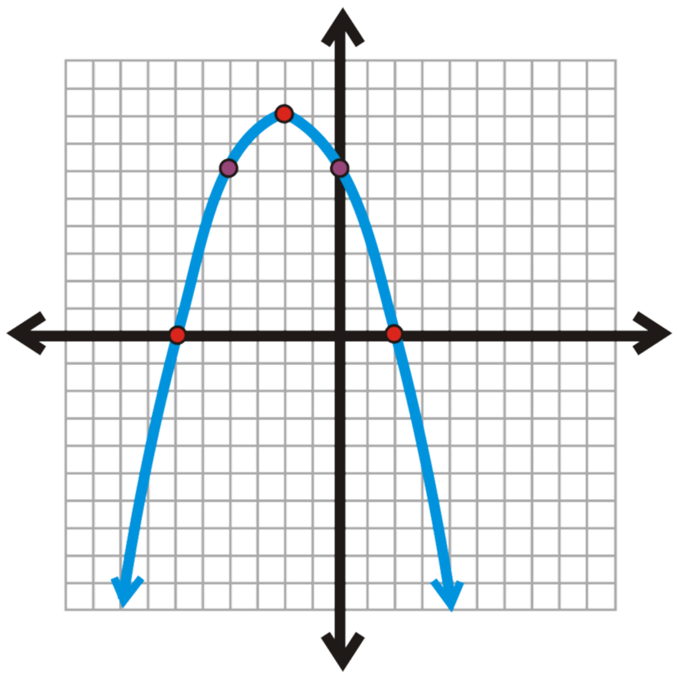
Therefore, the vertex is (-2, 8) and the axis of symmetry is $x = -2$.

To find the $y-$intercept, $x = 0$. $y = -\frac{1}{2}(0)^2 - 2(0) + 6 = 6$. Therefore, the $y-$intercept is (0, 6).

Example B

Sketch a graph of the parabola from Example A.

Solution: Plot the vertex and two $x-$intercepts (red points). Plot the $y-$intercept. Because all parabolas are symmetric, the corresponding point on the other side would be (-4, 6). Connect the five points to form the parabola.



For this parabola, the vertex is the **maximum** value. If you look at the equation, $y = -\frac{1}{2}x^2 - 2x + 6$, we see that the a value is negative. When a is negative, the sides of the parabola, will point down.

Example C

Find the vertex and $x-$intercepts of $y = 2x^2 - 5x - 25$. Then, sketch a graph.

Solution: First, this is a factorable function. $ac = -50$. The factors of -50 that add up to -5 are -10 and 5.

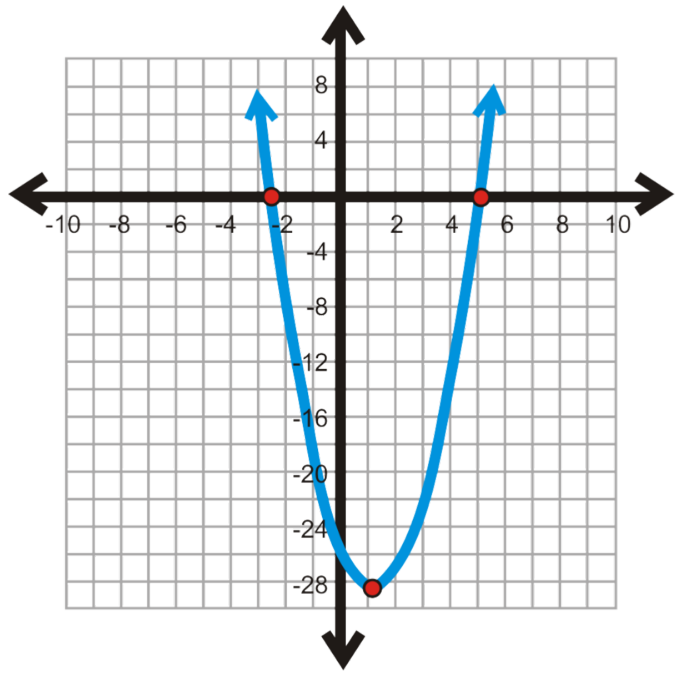
$$\begin{aligned} 2x^2 - 5x - 25 &= 0 \\ 2x^2 - 10x + 5x - 25 &= 0 \\ 2x(x-5) + 5(x-5) &= 0 \\ (2x+5)(x-5) &= 0 \end{aligned}$$

Setting each factor equal to zero, we get $x = 5$ and $-\frac{5}{2}$.

From this, we get that the $x-$intercepts are (5, 0) and $\left(-\frac{5}{2}, 0\right)$. To find the vertex, use $x = \frac{-b}{2a}$.

$x = \frac{5}{2 \cdot 2} = \frac{5}{4}$ Now, find y. $y = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) - 25 = \frac{25}{8} - \frac{25}{4} - 25 = -\frac{225}{8} = -28\frac{1}{8}$

The vertex is $\left(\frac{5}{4}, -28\frac{1}{8}\right)$. To graph this, we will need to estimate the vertex and draw an appropriate scale for the grid. As a decimal, the vertex is (1.25, -28.125).



Guided Practice

1. Find the $x-$intercepts, $y-$intercept, vertex, and axis of symmetry of $y = -x^2 + 7x - 12$.

2. Sketch a graph of the parabola from #1.

3. Find the vertex of $y = -4x^2 + 16x - 17$. Does the parabola open up or down?

Answers

1. This is a factorable quadratic equation.

$$\begin{aligned} -(x^2 - 7x + 12) &= 0 \\ -(x^2 - 3x - 4x + 12) &= 0 \\ -[x(x-3) - 4(x-3)] &= 0 \\ -(x-3)(x-4) &= 0 \end{aligned}$$

The $x-$intercepts are (3, 0) and (4, 0).

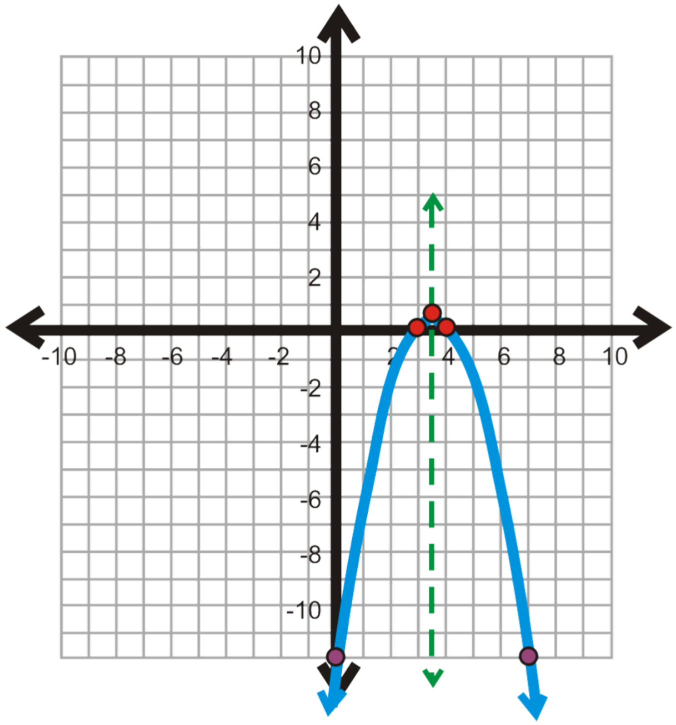
$$\begin{aligned} y &= -0^2 + 7(0) - 12 \\ y &= -12 \end{aligned}$$

The $y-$intercept is (0, -12).

The $x-$coordinate of the vertex is $x = \frac{-7}{2(-1)} = \frac{7}{2}$. The $y-$coordinate is $y = -\left(\frac{7}{2}\right)^2 + 7\left(\frac{7}{2}\right) - 12 = \frac{1}{4}$.

Therefore, the vertex is $\left(\frac{7}{2}, \frac{1}{4}\right)$ and the parabola opens down because $a < 0$. The axis of symmetry is $x = \frac{7}{2}$.

2. Plot all the points you found in #1. Then, connect the points to create the parabola.



3. First, the parabola opens down because a is negative. The $x-$coordinate of the vertex is $x = \frac{-16}{2(-4)} = \frac{-16}{-8} = 2$. The $y-$coordinate is $y = -4(2)^2 + 16(2) - 17 = -16 + 32 - 17 = -1$. This makes the vertex (2, -1).

Even though the problem does not ask, we can infer that this parabola does not cross the $x-$axis because it points down and the vertex is below the $x-$axis. This means that the solutions would be imaginary.

Vocabulary

Parabola
The "U" shaped graph of a quadratic equation.

Vertex
The highest or lowest point of a parabola. The $x-$coordinate of the vertex is $\frac{-b}{2a}$.

Maximum/Minimum
The highest/lowest point of a function.

$x-$intercept(s)
The point(s) where a function crosses the $x-$axis. $x-$intercepts are also called solutions, roots or zeros.

$y-$intercept
The point where a function crosses the $y-$axis. A function will only cross the $y-$axis once.

Axis of Symmetry
The line that a parabola is symmetric about. The vertex always lies on this line.

Problem Set

Find the vertex of each parabola and determine if it is a maximum or minimum.

1. $y = x^2 - 12x + 11$

2. $y = x^2 + 10x - 18$
3. $y = -3x^2 + 4x + 17$
4. $y = 2x^2 - 9x - 11$
5. $y = -x^2 + 6x - 9$
6. $y = -\frac{1}{4}x^2 + 8x - 33$

Find the vertex, $x-$intercepts, $y-$intercept, and axis of symmetry of each *factorable* quadratic equation below. Then, sketch a graph of each one.

7. $y = x^2 - 12x + 11$
8. $y = -2x^2 - 5x + 12$
9. $y = \frac{1}{3}x^2 + 4x - 15$
10. $y = 3x^2 + 26x - 9$
11. $y = -x^2 + 10x - 25$
12. $y = -\frac{1}{2}x^2 + 5x + 28$
13. If a function is not factorable, how would you find the $x-$intercepts?

Find the vertex and $x-$intercepts of the following quadratic equations. Then, sketch the graph. These equations are not factorable.

14. $y = -x^2 + 8x - 9$
15. $y = 2x^2 - x - 8$

Complete the table of values for the quadratic equations below. Then, plot the points and graph.

16. $y = x^2 - 2x + 2$

TABLE 15.2:

x	y
5	
3	
1	
-1	
-3	

17. $y = x^2 + 4x + 13$

TABLE 15.3:

x	y
4	
0	
-2	
-4	
-8	

18. **Writing** What do you notice about the two parabolas from 16 and 17? What type of solutions do these functions have? Solve #16.
19. **Writing** How many different ways can a parabola intersect the $x-$axis? Draw parabolas on an $x - y$ plane to

represent the different solution possibilities.

20. **Challenge** If the $x-$coordinate of the vertex is $\frac{-b}{2a}$ for $y = ax^2 + bx + c$, find the $y-$coordinate in terms of a, b, and c.

Vertex, Intercept, and Standard Form

Objective

To explore the different forms of the quadratic equation.

Guidance

So far, we have only used the **standard form** of a quadratic equation, $y = ax^2 + bx + c$ to graph a parabola. From standard form, we can find the vertex and either factor or use the Quadratic Formula to find the $x-$intercepts. The **intercept form** of a quadratic equation is $y = a(x - p)(x - q)$, where a is the same value as in standard form, and p and q are the $x-$intercepts. This form looks very similar to a factored quadratic equation.

Example A

Change $y = 2x^2 + 9x + 10$ to intercept form and find the vertex. Graph the parabola.

Solution: First, let's change this equation into intercept form by factoring. $ac = 20$ and the factors of 20 that add up to 9 are 4 and 5. Expand the $x-$term.

$$\begin{aligned}
y &= 2x^2 + 9x + 10 \\
y &= 2x^2 + 4x + 5x + 10 \\
y &= 2x(x + 2) + 5(x + 2) \\
y &= (2x + 5)(x + 2)
\end{aligned}$$

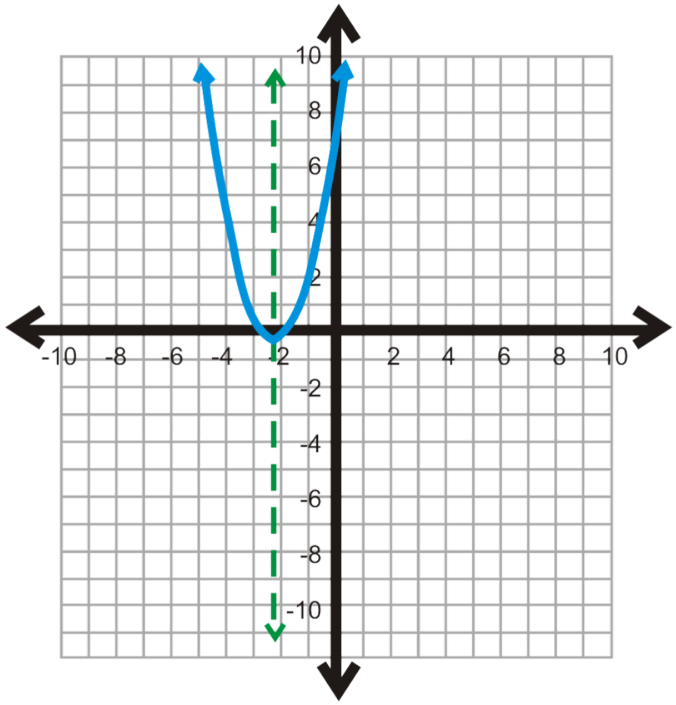
Notice, this does not exactly look like the definition. The factors cannot have a number in front of x. Pull out the 2 from the first factor to get $y = 2\left(x + \frac{5}{2}\right)(x + 2)$. Now, find the vertex. Recall that all parabolas are symmetrical. This means that the axis of symmetry is *halfway* between the $x-$intercepts or their average.

$$\text{axis of symmetry} = \frac{p+q}{2} = \frac{-\frac{5}{2} - 2}{2} = -\frac{9}{2} \div 2 = -\frac{9}{2} \cdot \frac{1}{2} = -\frac{9}{4}$$

This is also the $x-$coordinate of the vertex. To find the $y-$coordinate, plug the $x-$value into either form of the quadratic equation. We will use Intercept form.

$$\begin{aligned}
y &= 2\left(-\frac{9}{4} + \frac{5}{2}\right)\left(-\frac{9}{4} + 2\right) \\
y &= 2 \cdot \frac{1}{4} \cdot -\frac{1}{4} \\
y &= -\frac{1}{8}
\end{aligned}$$

The vertex is $\left(-2\frac{1}{4}, -\frac{1}{8}\right)$. Plot the $x-$intercepts and the vertex to graph.

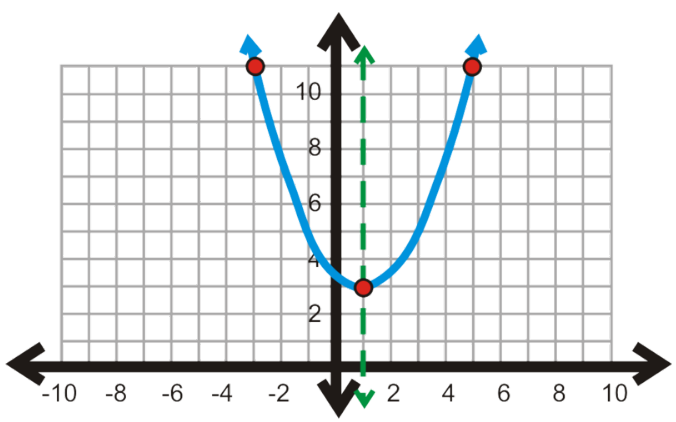


The last form is vertex form. **Vertex form** is written $y = a(x-h)^2 + k$, where (h, k) is the vertex and a is the same is in the other two forms. Notice that h is negative in the equation, but positive when written in coordinates of the vertex.

Example B

Find the vertex of $y = \frac{1}{2}(x-1)^2 + 3$ and graph the parabola.

Solution: The vertex is going to be (1, 3). To graph this parabola, use the symmetric properties of the function. Pick a value on the left side of the vertex. If $x = -3$, then $y = \frac{1}{2}(-3-1)^2 + 3 = 11$. -3 is 4 units away from 1 (the $x-$coordinate of the vertex). 4 units on the *other* side of 1 is 5. Therefore, the $y-$coordinate will be 11. Plot (1, 3), (-3, 11), and (5, 11) to graph the parabola.



Example C

Change $y = x^2 - 10x + 16$ into vertex form.

Solution: To change an equation from standard form into vertex form, you must complete the square. Review the *Completing the Square* Lesson if needed. The major difference is that you will not need to solve this equation.

$$\begin{aligned} y &= x^2 - 10x + 16 \\ y - 16 + 25 &= x^2 - 10x + 25 && \text{Move 16 to the other side and add } \left(\frac{b}{2}\right)^2 \text{ to both sides.} \\ y + 9 &= (x-5)^2 && \text{Simplify left side and factor the right side} \\ y &= (x-5)^2 - 9 && \text{Subtract 9 from both sides to get } y \text{ by itself.} \end{aligned}$$

To solve an equation in vertex form, set $y = 0$ and solve for x.

$$\begin{aligned} (x-5)^2 - 9 &= 0 \\ (x-5)^2 &= 9 \\ x - 5 &= \pm 3 \\ x &= 5 \pm 3 \textit{ or } 8 \textit{ and } 2 \end{aligned}$$

Guided Practice

1. Find the intercepts of $y = 2(x-7)(x+2)$ and change it to standard form.

2. Find the vertex of $y = -\frac{1}{2}(x+4)^2 - 5$ and change it to standard form.

3. Change $y = x^2 + 18x + 45$ to intercept form and graph.

4. Change $y = x^2 - 6x - 7$ to vertex form and graph.

Answers

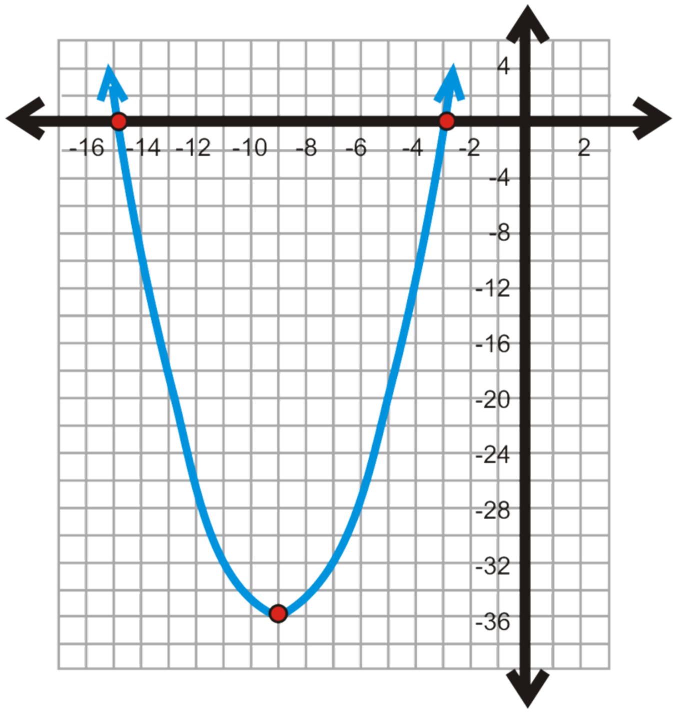
1. The intercepts are the opposite sign from the factors; (7, 0) and (-2, 0). To change the equation into standard form, FOIL the factors and distribute a.

$$\begin{aligned} y &= 2(x-7)(x+2) \\ y &= 2(x^2 - 5x - 14) \\ y &= 2x^2 - 10x - 28 \end{aligned}$$

2. The vertex is (-4, -5). To change the equation into standard form, FOIL $(x+4)^2$, distribute a, and then subtract 5.

$$\begin{aligned} y &= -\frac{1}{2}(x+4)(x+4) - 5 \\ y &= -\frac{1}{2}(x^2 + 8x + 16) - 5 \\ y &= -\frac{1}{2}x^2 - 4x - 21 \end{aligned}$$

3. To change $y = x^2 + 18x + 45$ into intercept form, factor the equation. The factors of 45 that add up to 18 are 15 and 3. Intercept form would be $y = (x+15)(x+3)$. The intercepts are (-15, 0) and (-3, 0). The $x-$coordinate of the vertex is halfway between -15 and -3, or -9. The $y-$coordinate of the vertex is $y = (-9)^2 + 18(-9) + 45 = -36$. Here is the graph:

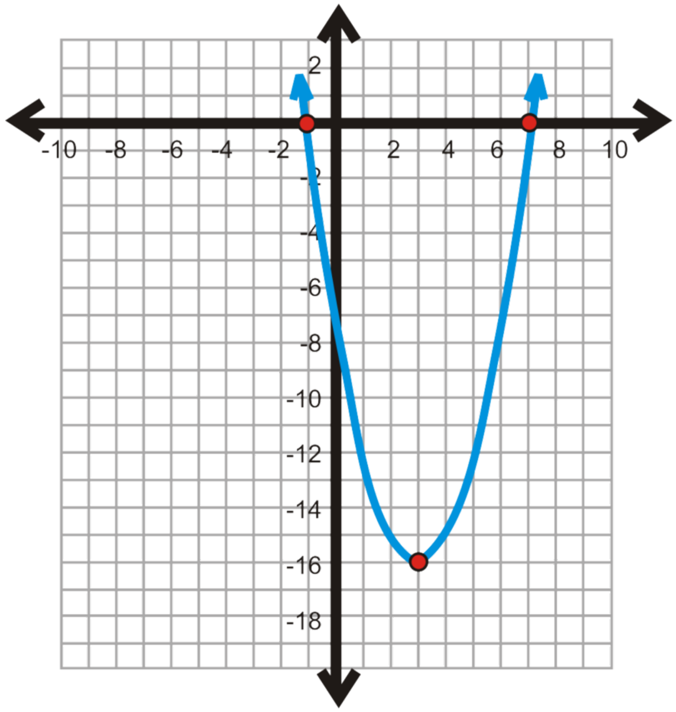


4. To change $y = x^2 - 6x - 7$ into vertex form, complete the square.

$$\begin{aligned} y + 7 + 9 &= x^2 - 6x + 9 \\ y + 16 &= (x - 3)^2 \\ y &= (x - 3)^2 - 16 \end{aligned}$$

The vertex is (3, -16).

For vertex form, we could solve the equation by using square roots or we could factor the standard form. Either way, we will get that the $x-$intercepts are (7, 0) and (-1, 0).



Vocabulary

Standard form
$y = ax^2 + bx + c$

Intercept form
$y = a(x - p)(x - q)$, where p and q are the $x-$intercepts.

Vertex form
$y = a(x - h)^2 + k$, where (h, k) is the vertex.

Problem Set

1. Fill in the table below.

TABLE 15.4:

	Equation	**Vertex**	**Intercepts** (or how to find the intercepts)
Standard form			
Intercept form			
Vertex form			

Find the vertex and $x-$intercepts of each function below. Then, graph the function. If a function does not have any $x-$intercepts, use the symmetry property of parabolas to find points on the graph.

2. $y = (x - 4)^2 - 9$
3. $y = (x + 6)(x - 8)$
4. $y = x^2 + 2x - 8$

5. $y = -(x-5)(x+7)$
6. $y = 2(x+1)^2 - 3$
7. $y = 3(x-2)^2 + 4$
8. $y = \frac{1}{3}(x-9)(x+3)$
9. $y = -(x+2)^2 + 7$
10. $y = 4x^2 - 13x - 12$

Change the following equations to intercept form.

11. $y = x^2 - 3x + 2$
12. $y = -x^2 - 10x + 24$
13. $y = 4x^2 + 18x + 8$

Change the following equations to vertex form.

14. $y = x^2 + 12x - 28$
15. $y = -x^2 - 10x + 24$
16. $y = 2x^2 - 8x + 15$

Change the following equations to standard form.

17. $y = (x-3)^2 + 8$
18. $y = 2\left(x - \frac{3}{2}\right)(x-4)$
19. $y = -\frac{1}{2}(x+6)^2 - 11$

Using the Graphing Calculator to Graph Quadratic Equations

Objective

To use the graphing calculator to graph parabolas, find their intercepts, and the vertex.

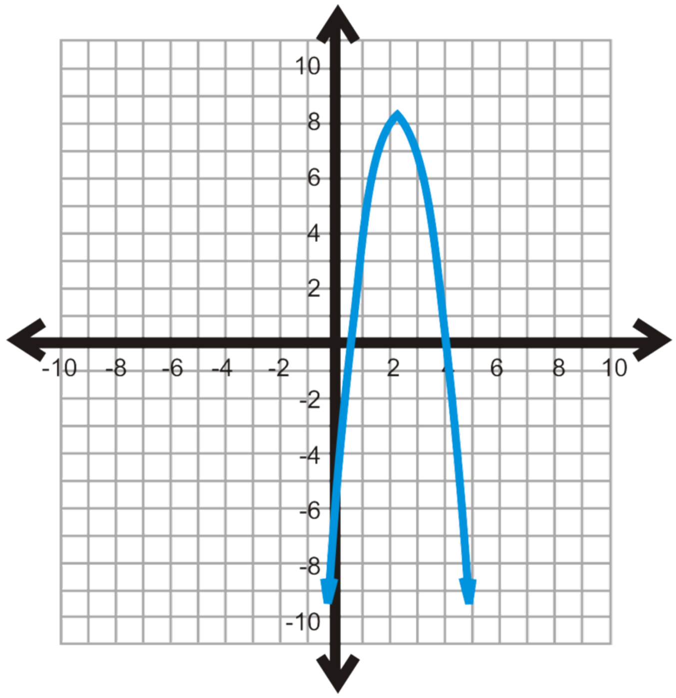
Guidance

A graphing calculator can be a very helpful tool when graphing parabolas. This concept outlines how to use the TI-83/84 to graph and find certain points on a parabola.

Example A

Graph $y = -3x^2 + 14x - 8$ using a graphing calculator.

Solution: Using a TI-83/84, press the $Y =$ button. Enter in the equation. Be careful not to confuse the negative sign and the subtraction sign. The equation should look like $y = -3x^2 + 14x - 8$ or $y = -3x^2 + 14x - 8$. Press GRAPH.



If your graph does not look like this one, there may be an issue with your window. Press ZOOM and then **6:ZStandard,** ENTER. This should give you the standard window.

Example B

Using your graphing calculator, find the vertex of the parabola from Example A.

Solution: To find the vertex, press 2^{nd} TRACE (CALC). The Calculate menu will appear. In this case, the vertex is a maximum, so select **4:maximum,** ENTER. The screen will return to your graph. Now, you need to tell the calculator the Left Bound. Using the arrows, arrow over to the left side of the vertex, press ENTER. Repeat this for the Right Bound. The calculator then takes a guess, press ENTER again. It should give you that the maximum is $X = 2.3333333$ and $Y = 8.3333333$. As fractions, the coordinates of the vertex are $\left(2\frac{1}{3}, 8\frac{1}{3}\right)$. Make sure to write the coordinates of the vertex as a point.

Example C

Using your graphing calculator, find the $x-$intercepts of the parabola from Example A.

Solution: To find the $x-$intercepts, press 2^{nd} TRACE (CALC). The Calculate menu will appear. Select **2:Zero,** ENTER. The screen will return to your graph. Let's focus on the left-most intercept. Now, you need to tell the calculator the Left Bound. Using the arrows, arrow over to the left side of the vertex, press ENTER. Repeat this for the Right Bound (keep the bounds close to the intercept). The calculator then takes a guess, press ENTER again. This intercept is $X = .666667$, or $\left(\frac{2}{3}, 0\right)$. Repeat this process for the second intercept. You should get (4, 0).

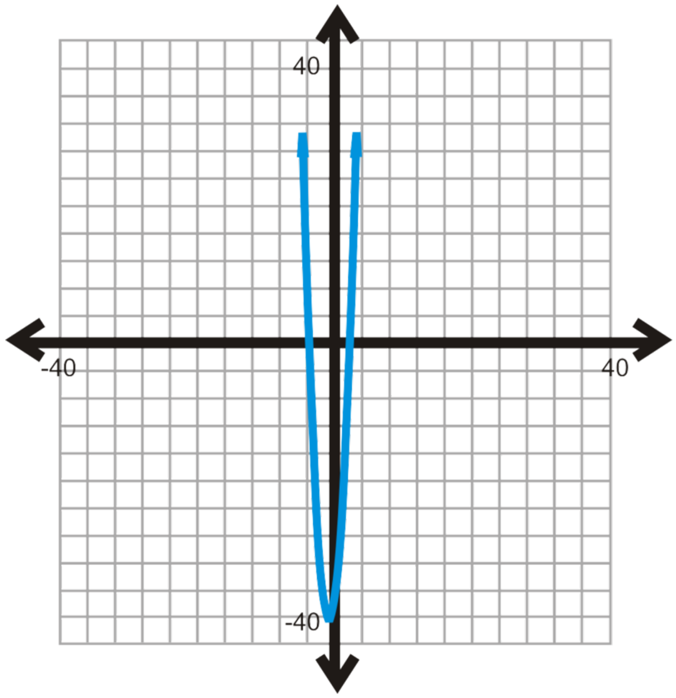
NOTE: When graphing parabolas and the vertex does not show up on the screen, you will need to zoom out. The calculator will not find the value(s) of any $x-$intercepts or the vertex that do not appear on screen. To zoom out, press ZOOM, **3:Zoom Out**, ENTER, ENTER.

Guided Practice

1. Graph $y = 6x^2 + 11x - 35$ using a graphing calculator. Find the vertex and $x-$intercepts. Round your answers to the nearest hundredth.

Answers

1. Using the steps above, the vertex is (-0.917, -40.04) and is *a minimum*. The $x-$intercepts are (1.67, 0) and (-3.5, 0).



Problem Set

Graph the quadratic equations using a graphing calculator. Find the vertex and $x-$intercepts, if there are any. If there are no $x-$intercepts, use algebra to find the imaginary solutions. Round all real answers to the nearest hundredth.

1. $y = x^2 - x - 6$
2. $y = -x^2 + 3x + 28$
3. $y = 2x^2 + 11x - 40$
4. $y = x^2 - 6x + 7$
5. $y = x^2 + 8x + 13$
6. $y = x^2 + 6x + 34$
7. $y = 10x^2 - 13x - 3$
8. $y = -4x^2 + 12x - 3$
9. $y = \frac{1}{3}(x-4)^2 + 12$
10. **Calculator Investigation** The *parent graph* of a quadratic equation is $y = x^2$.

    a. Graph $y = x^2, y = 3x^2$, and $y = \frac{1}{2}x^2$ on the same set of axes in the calculator. Describe how a effects the shape of the parabola.
    b. Graph $y = x^2, y = -x^2$, and $y = -2x^2$ on the same set of axes in the calculator. Describe how a effects the shape of the parabola.
    c. Graph $y = x^2, y = (x-1)^2$, and $y = (x+4)^2$ on the same set of axes in the calculator. Describe how h effects the location of the parabola.
    d. Graph $y = x^2, y = x^2 + 2$, and $y = x^2 - 5$ on the same set of axes in the calculator. Describe how k effects the location of the parabola.

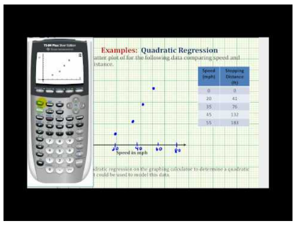
11. The path of a baseball hit by a bat follows a parabola. A batter hits a home run into the stands that can be modeled by the equation $y = -0.003x^2 + 1.3x + 4$, where x is the horizontal distance and y is the height (in feet) of the ball. Find the maximum height of the ball and its total distance travelled.

Modeling with Quadratic Functions

Objective

To find the quadratic equation that fits to a data set.

Watch This



MEDIA
Click image to the left for more content.

James Sousa: Ex: Quadratic Regression on the TI84 - Stopping Distance

Guidance

When finding the equation of a parabola, you can use any of the three forms. If you are given the vertex and any other point, you only need two points to find the equation. However, if you are not given the vertex you must have at least three points to find the equation of a parabola.

Example A

Find the equation of the parabola with vertex (-1, -4) and passes through (2, 8).

Solution: Use vertex form and substitute -1 for h and -4 for k.

$$y = a(x-(-1))^2 - 4$$
$$y = a(x+1)^2 - 4$$

Now, take the second point and plug it for x and y and solve for a.

$$8 = a(2+4)^2 - 4$$
$$12 = 36a$$
$$\frac{1}{3} = a$$

The equation is $y = \frac{1}{3}(x+1)^2 - 4$.

Like in the *Analyzing Scatterplots* lesson, we can also fit a set of data to a quadratic equation. In this concept, we will be using **quadratic regression** and a TI-83/84.

Example B

Determine the quadratic equation of best fit for the data set below.

TABLE 15.5:

x	0	4	7	12	17
y	7	9	10	8	3

Solution: We need to enter the $x-$coordinates as a list of data and the $y-$coordinates as another list.

1. Press STAT.

2. In EDIT, select **1:Edit....** Press ENTER.

3. The List table appears. If there are any current lists, you will need to clear them. To do this, arrow up to L1 so that it is highlighted (black). Press CLEAR, then ENTER. Repeat with L2, if necessary.

4. Now, enter the data into the lists. Enter all the entries into L1 (x) first and press enter between each entry. Then, repeat with L2 and y.

5. Press 2^{nd} MODE (QUIT).

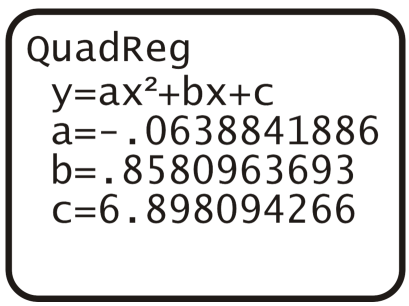
Now that we have everything in the lists, we can use *quadratic* regression to determine the equation of best fit.

6. press STAT and then arrow over to the CALC menu.

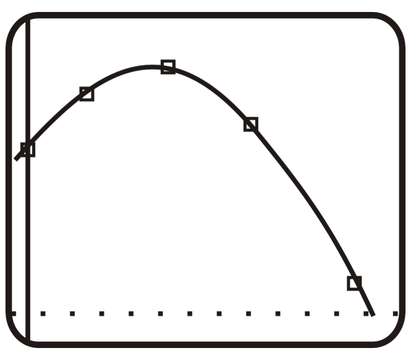
7. Select **5:QuadReg.** Press ENTER.

8. You will be taken back to the main screen. Type **(L1,L2)** and press ENTER. L1 is 2^{nd} 1, L2 is 2^{nd} 2.

9. The following screen appears. The equation of best fit is $y = -0.64x^2 + 0.86x + 6.90$.



If you would like to plot the equation on the scatterplot follow the steps from the *Finding the Equation of Best Fit using a Graphing Calculator* concept. The scatterplot and parabola are to the right.



This technique can be applied to real-life problems. You can also use technique to find the equation of any parabola, given three points.

Example C

Find the equation of the parabola that passes through (1, 11), (2, 20), (-3, 75).

Solution: You can use the same steps from Example B to find the equation of the parabola. Doing this, you should get the equation is $y = 5x^2 - 6x + 12$.

This problem can also be done by solving three equations, with three unknowns. If we plug in (x, y) to $y = ax^2 + bx + c$, we would get:

$$\begin{aligned} 11 &= a + b + c \\ 20 &= 4a + 2b + c \\ 75 &= 9a - 3b + c \end{aligned}$$

Use linear combinations to solve this system of equations (see *Solving a System in Three Variables Using Linear Combinations* concept). This problem will be finished in the Problem Set.

Guided Practice

1. Find the equation of the parabola with $x-$intercepts (4, 0) and (-5, 0) that passes through (-3, 8).

2. A study compared the speed, x (in miles per hour), and the average fuel economy, y (in miles per gallon) of a sports car. Here are the results.

TABLE 15.6:

speed	30	40	50	55	60	65	70	80
fuel economy	11.9	16.1	21.1	22.2	25.0	26.1	25.5	23.2

Plot the scatterplot and use your calculator to find the equation of best fit.

Answers

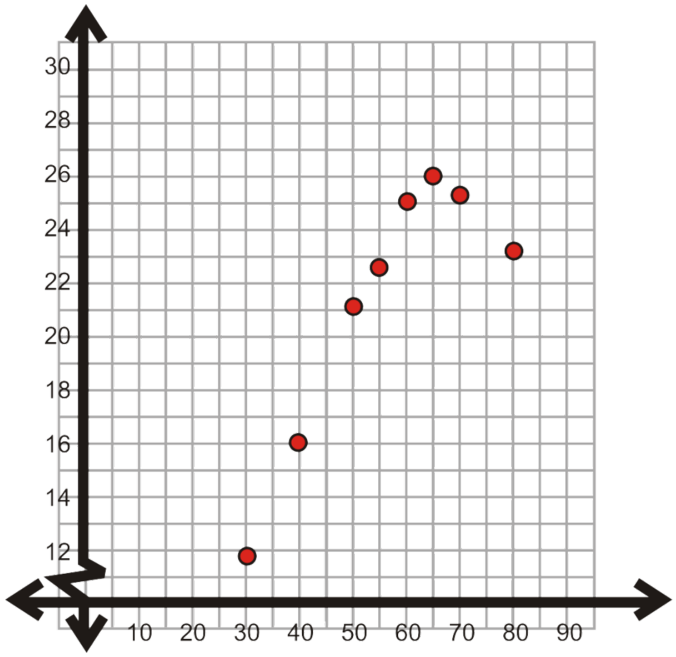
1. Because we are given the intercepts, use intercept form to find the equation.

$y = a(x-4)(x+5)$ Plug in (-3,8) and solve for a

$$\begin{aligned} 8 &= a(-3-4)(-3+5) \\ 8 &= -14a \\ -\frac{4}{7} &= a \end{aligned}$$

The equation of the parabola is $y = -\frac{4}{7}(x-4)(x+5)$.

2. Plotting the points, we have:



Using the steps from Example B, the quadratic regression equation is $y = -0.009x^2 + 1.24x - 18.23$.

Vocabulary

Quadratic Regression
The process through which the equation of best fit is a quadratic equation.

Problem Set

Find the equation of the parabola given the following points. No decimal answers.

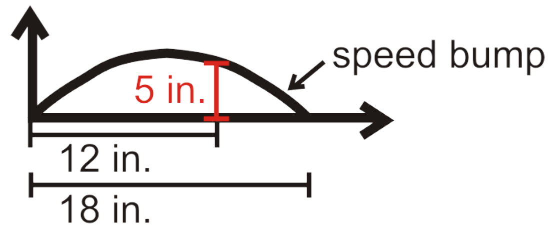
1. vertex: (-1, 1) point: (1, -7)
2. $x-$intercepts: -2, 2 point: (4, 3)
3. vertex: (9, -4) point: (5, 12)
4. $x-$intercepts: 8, -5 point: (3, 20)
5. $x-$intercepts: -9, -7 point: (-3, 36)
6. vertex: (6, 10) point: (2, -38)
7. vertex: (-4, -15) point: (-10, 1)
8. vertex: (0, 2) point: (-4, -12)
9. $x-$intercepts: 3, 16 point: (7, 24)

Use a graphing calculator to find the quadratic equation (in standard form) that passes through the given three points. No decimal answers.

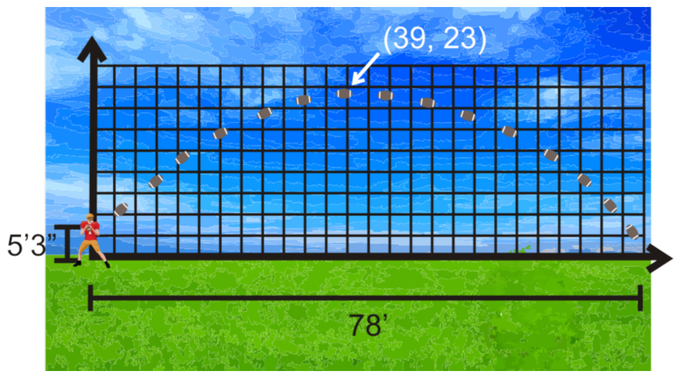
10. (-4, -51), (-1, -18), (4, -43)
11. (-5, 131), (-1, -5), (3, 51)
12. (-2, 9), (2, 13), (6, 41)
13. **Challenge** Finish computing Example C using linear combinations.

For the quadratic modeling questions below, use a graphing calculator. Round any decimal answers to the nearest hundredth.

14. The surface of a speed bump is shaped like a parabola. Write a quadratic model for the surface of the speed bump shown.



15. **Physics and Photography Connection** Your physics teacher gives you a project to analyze parabolic motion. You know that when a person throws a football, the path is a parabola. Using your camera, you take an long exposure picture of a friend throwing a football. A sketch of the picture is below.



You put the path of the football over a grid, with the $x-$axis as the horizontal distance and the $y-$axis as the height, both in 3 feet increments. The release point, or shoulder height, of your friend is 5 ft, 3 in and you estimate that the maximum height is 23 feet. Find the equation of the parabola.

16. An independent study was done linking advertising to the purchase of an object. 400 households were used in the survey and the commercial exposure was over a one week period. See the data set below.

TABLE 15.7:

# of times commercial was shown, x	1	7	14	21	28	35	42	49
# of households bought item, y	2	25	96	138	88	37	8	6

a) Find the quadratic equation of best fit.

b) Why do you think the amount of homes that purchased the item went down after more exposure to the commercial?

15.8 Parabolas with Vertex at the Origin

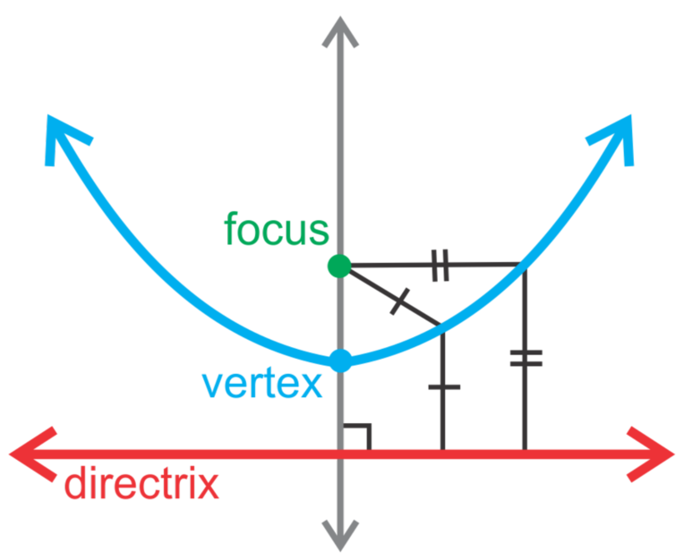
Here you'll write and graph the equation of a parabola, with vertex $(0,0)$, and find the focus, directrix, and vertex.

The area of a square is represented by the equation $y = 9x^2$. What are the focus and directrix of this equation?

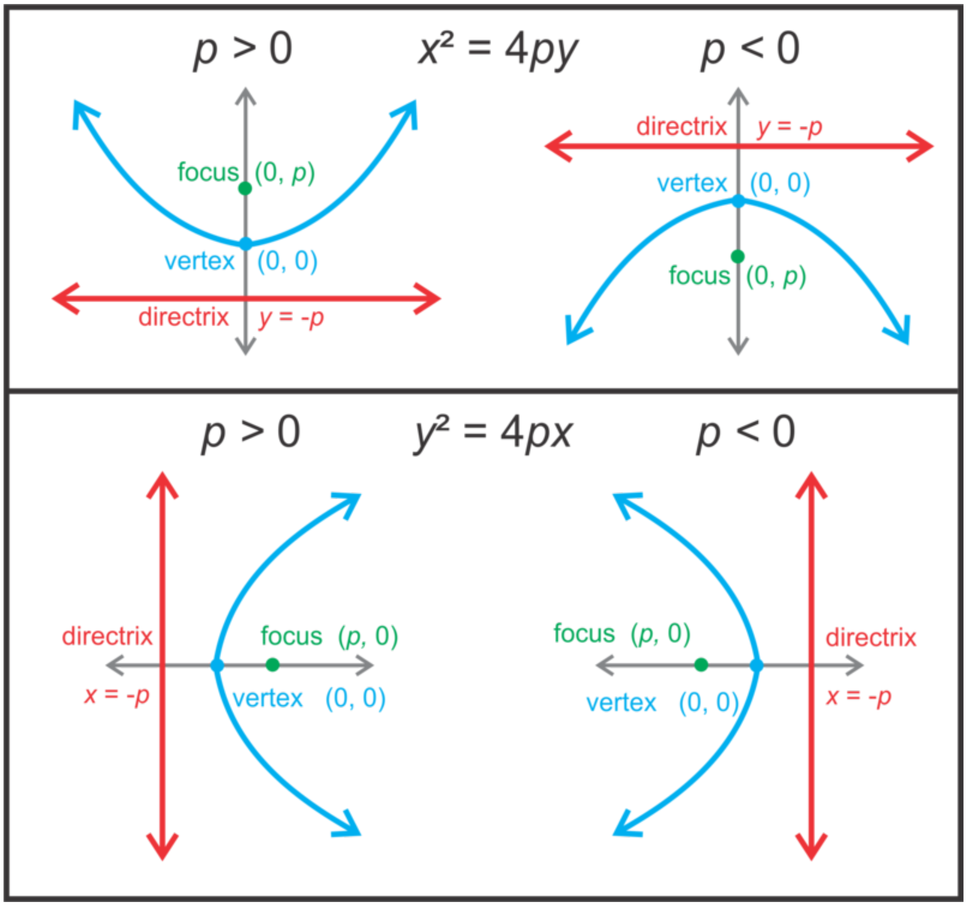
Guidance

You already know that the graph of a parabola has the parent graph $y = x^2$, with a vertex of $(0,0)$ and an axis of symmetry of $x = 0$. A parabola can also be defined in a different way. It has a property such that any point on it is equidistant from another point, called the **focus**, and a line called the **directrix**.

The focus is on the axis of symmetry and the vertex is halfway between it and the directrix. The directrix is perpendicular to the axis of symmetry.



Until now, we have been used to seeing the equation of a parabola like $y = ax^2$. In this concept, we will rewrite the equation to look like $x^2 = 4py$ where p is used to find the focus and directrix. We will also draw the parabola with a horizontal orientation, such that the equation will be $y^2 = 4px$.



Notice, that when the parabola opens to the left or right, the y is squared. In this concept, the vertex will be $(0,0)$.

Example A

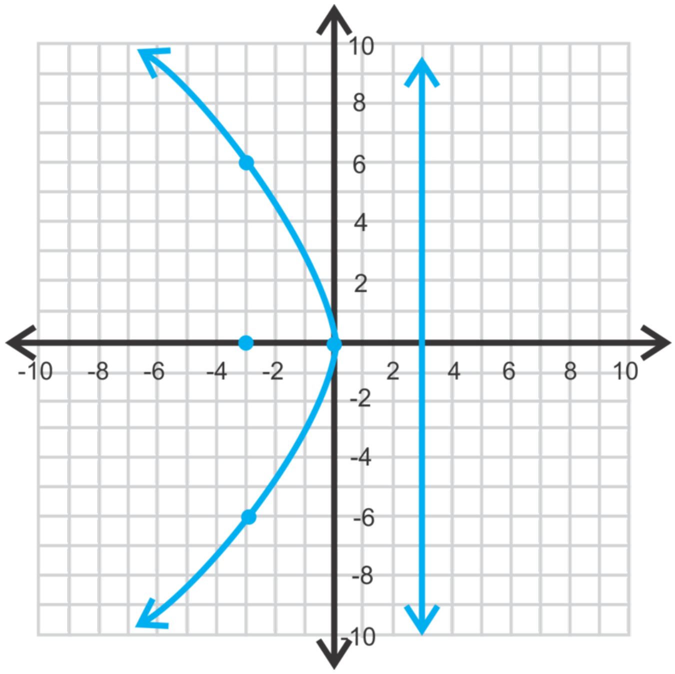
Analyze the equation $y^2 = -12x$. Find the focus, directrix, and determine if the function opens up, down, to the left or right. Then graph the parabola.

Solution: To find the focus and directrix, we need to find p. We can set $-12 = 4p$ and solve for p.

$$-12 = 4p$$
$$-3 = p$$

Because y is squared, we know that the parabola opens to the left or right. Because p is negative, we know it is going to open to the left, towards the negative side of the x-axis. Using the pictures above, this parabola is like the second one under $y^2 = 4px$. Therefore, the focus is $(-3,0)$ and the directrix is $x = 3$. To graph the parabola, plot the vertex, focus, directrix, and sketch the curve. Find at least one or two points on the curve to make sure your sketch is accurate. For example, because $(-3,6)$ is on the parabola, then $(-3,-6)$ is also on the parabola because it is symmetrical.

Notice that the points $(-3,6)$ and $(-3,-6)$ are equidistant from the focus and the directrix. They are both 6 units from each.



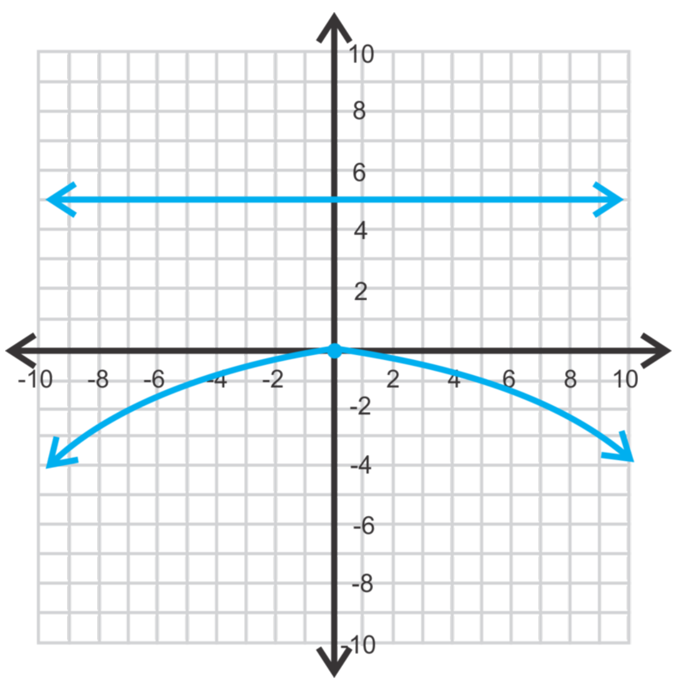
Example B

The focus of a parabola is $\left(0, \frac{1}{2}\right)$. Find the equation of the parabola.

Solution: Because the p value is the y-value and positive, this parabola is going to open up. So, the general equation is $x^2 = 4py$. Plugging in $\frac{1}{2}$ for p, we have $x^2 = 4 \cdot \frac{1}{2}y$ or $x^2 = 2y$.

Example C

Find the equation of the parabola below.



Solution: The equation of the directrix is $y = 5$, which means that $p = -5$ and the general equation will be $x^2 = 4py$. Plugging in -5 for p, we have $x^2 = -20y$.

Intro Problem Revisit To find the focus and directrix, we need to solve for x^2 and then find p.

$$y = 9x^2$$
$$\frac{1}{9}y = x^2$$

We can now set $\frac{1}{9} = 4p$ and solve for p.

$$\frac{1}{9} = 4p$$
$$\frac{1}{36} = p$$

Therefore, the focus is $(0, \frac{1}{36})$ and the directrix is $y = -\frac{1}{36}$.

Guided Practice

1. Determine if the parabola $x^2 = -2y$ opens up, down, left or right.

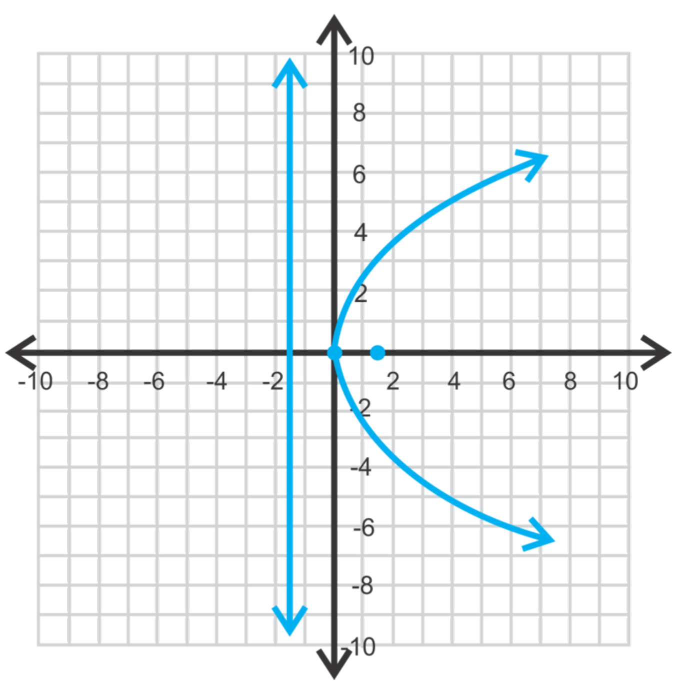
2. Find the focus and directrix of $y^2 = 6x$. Then, graph the parabola.

3. Find the equation of the parabola with directrix $x = -\frac{3}{8}$.

Answers

1. Down; p is negative and x is squared.

2. Solving for p, we have $4p = 6 \to p = \frac{3}{2}$. Because y is squared and p is positive, the parabola will open to the right. The focus is $\left(\frac{3}{2}, 0\right)$ and the directrix is $x = -\frac{3}{2}$.



3. If the directrix is negative and vertical ($x =$), we know that the equation is going to be $y^2 = 4px$ and the parabola will open to the right, making p positive; $p = \frac{3}{8}$. Therefore, the equation will be $y^2 = 4 \cdot \frac{3}{8} \cdot x \rightarrow y^2 = \frac{3}{2}x$.

Vocabulary

Parabola
The set of points that are equidistant from a fixed point on the interior of the curve, called the **focus**, and a line on the exterior, called the **directrix**. The directrix is vertical or horizontal, depending on the orientation of the parabola.

Equation of a Parabola, Vertex at the Origin
$y^2 = 4px$ or $^2 = 4py$

Practice

Determine if the parabola opens to the left, right, up or down.

1. $x^2 = 4y$
2. $y^2 = -\frac{1}{2}x$
3. $x^2 = -y$

Find the focus and directrix of the following parabolas.

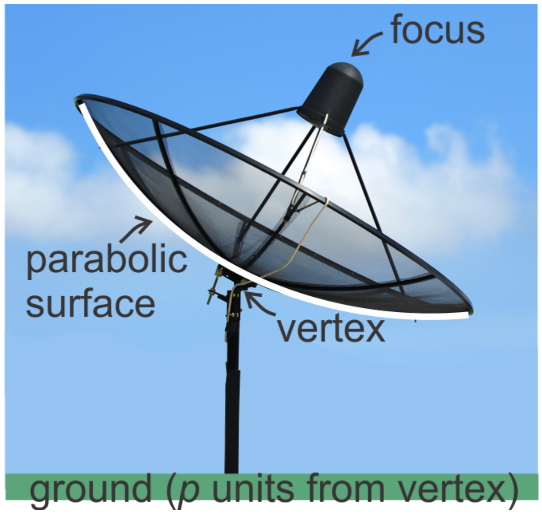
4. $x^2 = -2y$
5. $y^2 = \frac{1}{4}x$
6. $y^2 = -5x$

Graph the following parabolas. Identify the focus and directrix as well.

7. $x^2 = 8y$
8. $y^2 = \frac{1}{2}x$
9. $x^2 = -3y$

Find the equation of the parabola given that the vertex is $(0,0)$ and the focus or directrix.

10. focus: $(4,0)$
11. directrix: $x = 10$
12. focus: $\left(0, \frac{7}{2}\right)$
13. In the *Quadratics* chapter, the basic parabolic equation was $y = ax^2$. Now, we write $x^2 = 4py$. Rewrite p in terms of a and determine how they affect each other.
14. ***Challenge*** Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$, to prove that the point $(4,2)$ is on the parabola $x^2 = 8y$.
15. ***Real World Application*** A satellite dish is a 3-dimensional parabola used to retrieve sound, TV, or other waves. Assuming the vertex is $(0,0)$, where would the focus have to be on a satellite dish that is 4 feet wide and 9 inches deep? You may assume the parabola has a vertical orientation (opens up).


focus
parabolic
surface
vertex
ground (p units from vertex)

15.9 Parabolas with Vertex at (h, k)

Here you'll write and graph the equation of a parabola with vertex (h,k) and find the focus, directrix, and vertex.

Your homework assignment is to find the focus of the parabola $(x+4)^2 = -12(y-5)$. You say the focus is $(-4,5)$. Banu says the focus is $(0,-3)$. Carlos says the focus is $(-4,2)$. Which one of you is correct?

Guidance

You learned in the *Quadratic Functions* chapter that parabolas don't always have their vertex at $(0,0)$. In this concept, we will address parabolas where the vertex is (h,k), learn how to find the focus, directrix and graph.

Recall from the previous concept that the equation of a parabola is $x^2 = 4py$ or $y^2 = 4px$ and the vertex is on the origin. In the *Quadratic Functions* chapter, we learned that the vertex form of a parabola is $y = a(x-h)^2 + k$. Combining the two, we can find the vertex form for conics.

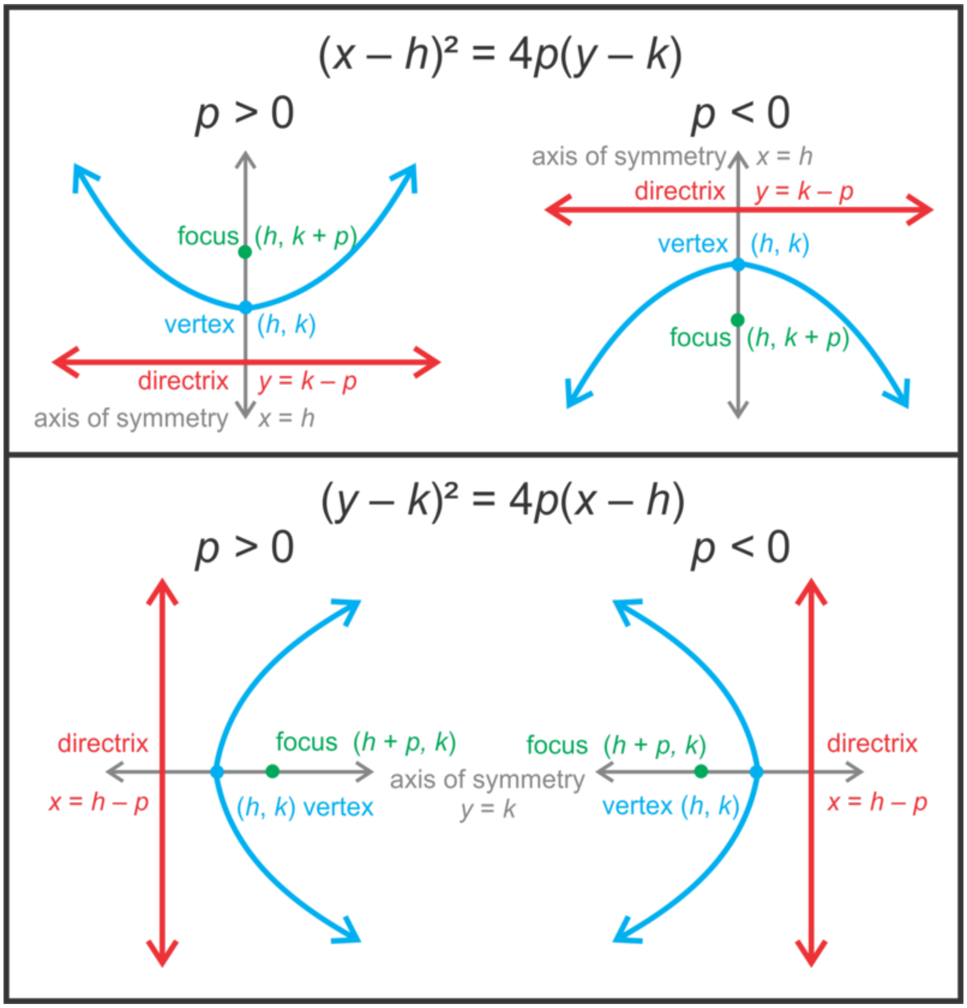
$$y = a(x-h)^2 + k \text{ and } x^2 = 4py \qquad \text{Solve the first for } (x-h)^2.$$

$$(x-h)^2 = \frac{1}{a}(y-k) \qquad \text{From \#13 in the previous concept, we found that } 4p = \frac{1}{a}.$$

$$(x-h)^2 = 4p(y-k)$$

If the parabola is horizontal, then the equation will be $(y-k)^2 = 4p(x-h)$. Notice, that even though the orientation is changed, the h and k values remain with the x and y values, respectively.

Finding the focus and directrix are a little more complicated. Use the extended table (from the previous concept) below to help you find these values.



Notice that the way we find the focus and directrix does not change whether p is positive or negative.

Example A

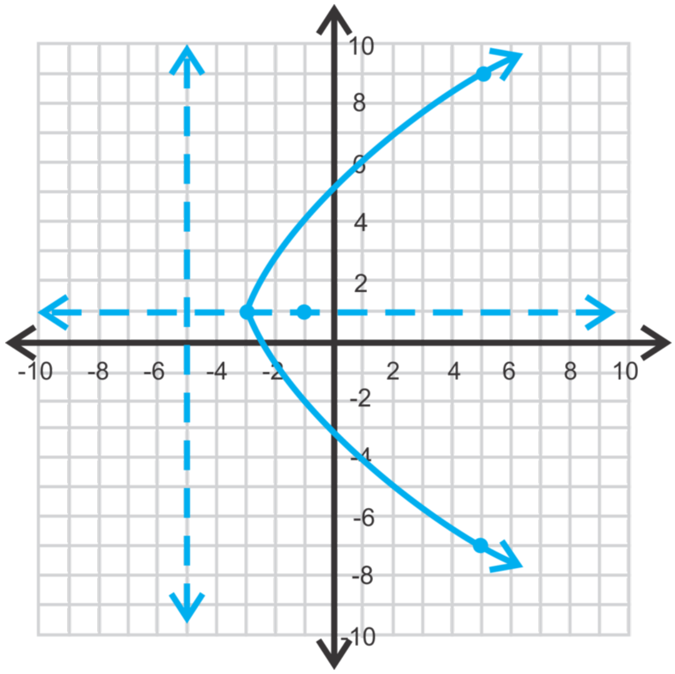
Analyze the equation $(y-1)^2 = 8(x+3)$. Find the vertex, axis of symmetry, focus, and directrix. Then determine if the function opens up, down, left or right.

Solution: First, because y is squared, we know that the parabola will open to the left or right. We can conclude that the parabola will open to the *right* because 8 is positive, meaning that p is positive. Next, find the vertex. Using the general equation, $(y-k)^2 = 4p(x-h)$, the vertex is $(-3,1)$ and the axis of symmetry is $y = 1$. Setting $4p = 8$, we have that $p = 2$. Adding p to the x-value of the vertex, we get the focus, $(-1,1)$. Subtracting p from the x-value of the vertex, we get the directrix, $x = -5$.

Example B

Graph the parabola from Example A. Plot the vertex, axis of symmetry, focus, and directrix.

Solution: First, plot all the critical values we found from Example A. Then, determine a set of symmetrical points that are on the parabola to make sure your curve is correct. If $x = 5$, then y is either -7 or 9. This means that the points $(5,-7)$ and $(5,9)$ are both on the parabola.



It is important to note that parabolas with a horizontal orientation are not functions because they do not pass the vertical line test.

Example C

The vertex of a parabola is $(-2,4)$ and the directrix is $y = 7$. Find the equation of the parabola.

Solution: First, let's determine the orientation of this parabola. Because the directrix is horizontal, we know that the parabola will open up or down (see table/pictures above). We also know that the directrix is *above* the vertex, making the parabola open down and p will be negative (plot this on an $x-y$ plane if you are unsure).

To find p, we can use the vertex, (h,k) and the equation for a horizontal directrix, $y = k - p$.

$$\begin{aligned} 7 &= 4 - p \\ 3 &= -p \qquad\qquad \text{Remember, } p \text{ is negative because of the downward orientation of the parabola.} \\ -3 &= p \end{aligned}$$

Now, using the general form, $(x-h)^2 = 4p(y-k)$, we can find the equation of this parabola.

$$\begin{aligned} (x-(-2))^2 &= 4(-3)(y-4) \\ (x+2)^2 &= -12(y-4) \end{aligned}$$

Intro Problem Revisit This parabola is of the form $(x-h)^2 = 4p(y-k)$. From the table earlier in this lesson, we can see that the focus of a parabola of this form is $(h, k+p)$. So now we have to find h, k, and p.

If we compare $(x+4)^2 = -12(y-5)$ to $(x-h)^2 = 4p(y-k)$, we see that:

1. $4 = -h$ or $h = -4$

2. $-12 = 4p$ or $p = -3$

3. $5 = k$

From these facts we can find $k + p = 5 + (-3) = 2$.

Therefore, the focus of the parabola is $(-4, 2)$ and Carlos is correct.

Guided Practice

1. Find the vertex, focus, axis of symmetry and directrix of $(x+5)^2 = 2(y+2)$.

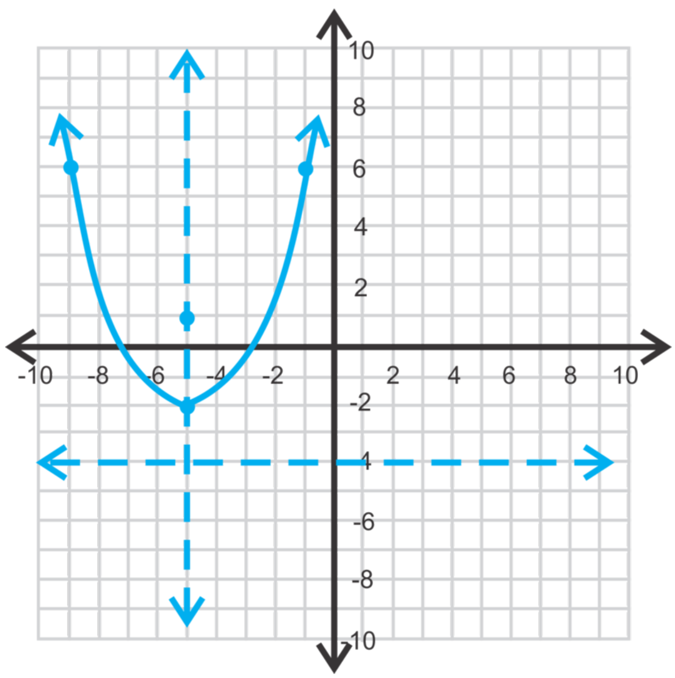
2. Graph the parabola from #1.

3. Find the equation of the parabola with vertex $(-5, -1)$ and focus $(-8, -1)$.

Answers

1. The vertex is $(-5, -2)$ and the parabola opens up because p is positive and x is squared. $4p = 2$, making $p = 2$. The focus is $(-5, -2+2)$ or $(-5, 0)$, the axis of symmetry is $x = -5$, and the directrix is $y = -2-2$ or $y = -4$.

2.



3. The vertex is $(-5, -1)$, so $h = -5$ and $k = -1$. The focus is $(-8, -1)$, meaning that that parabola will be horizontal. We know this because the y-values of the vertex and focus are both -1. Therefore, p is added or subtracted to h.

$(h+p, k) \rightarrow (-8, -1)$ we can infer that $h + p = -8 \rightarrow -5 + p = -8$ and $p = -3$

Therefore, the equation is $(y-(-1))^2 = 4(-3)(x-(-5)) \rightarrow (y+1)^2 = -12(x+5)$.

Vocabulary

Standard Form (of a Parabola)

$(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$ where (h, k) is the vertex.

Practice

Find the vertex, focus, axis of symmetry, and directrix of the parabolas below.

1. $(x+1)^2 = -3(y-6)$
2. $(x-3)^2 = y-7$
3. $(y+2)^2 = 8(x+1)$
4. $y^2 = -10(x-3)$
5. $(x+6)^2 = 4(y+8)$
6. $(y-5)^2 = -\frac{1}{2}x$
7. Graph the parabola from #1.
8. Graph the parabola from #2.
9. Graph the parabola from #4.
10. Graph the parabola from #5.

Find the equation of the parabola given the vertex and either the focus or directrix.

11. vertex: $(2,-1)$, focus: $(2,-4)$
12. vertex: $(-3,6)$, directrix: $x = 2$
13. vertex: $(6,10)$, directrix: $y = 9.5$
14. ***Challenge*** focus: $(-1,-2)$, directrix: $x = 3$
15. ***Extension*** Rewrite the equation of the parabola, $x^2 - 8x + 2y + 22 = 0$, in standard form by completing the square. Then, find the vertex. (For a review, see the *Completing the Square When the Leading Coefficient Equals 1* concept.)

15.10 Circles

Objective

To identify the parts of and graph a circle, centered on the origin or not.

Review Queue

1. Given that 6 and 8 are the legs of a right triangle, what is the hypotenuse?
2. Given that 5 is one leg of a right triangle and 13 is the hypotenuse, what is the length of the other leg?
3. What is the area and circumference of a circle with radius 4?
4. What is the area of circle with circumference 20π?

Circles Centered at the Origin

Objective

To find the radius and graph a circle centered at the origin.

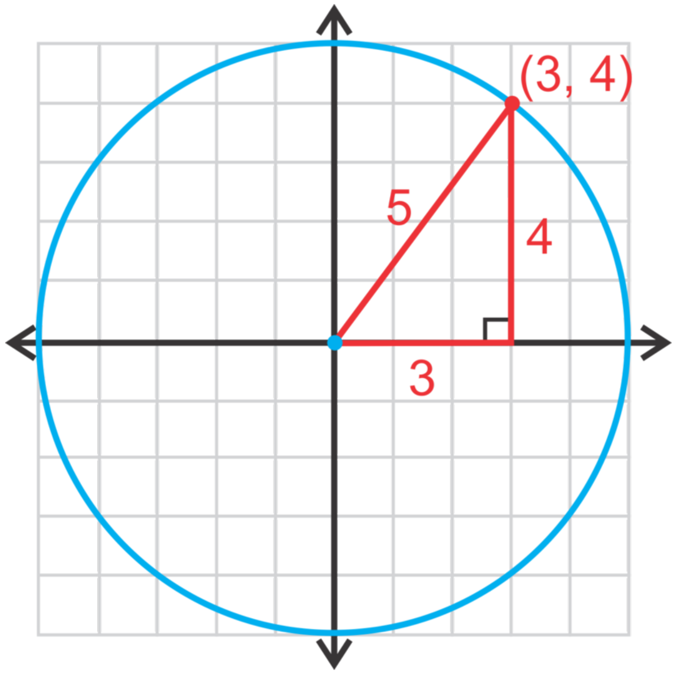
Guidance

Until now, your only reference to circles was from geometry. A **circle** is the set of points that are equidistant (the **radius**) from a given point (the **center**). A line segment that passes through the center and has endpoints on the circle is a **diameter**.

Now, we will take a circle and place it on the $x-y$ plane to see if we can find its equation. In this concept, we are going to place the center of the circle on the origin.

Investigation: Finding the Equation of a Circle

1. On a piece of graph paper, draw an $x-y$ plane. Using a compass, draw a circle, centered at the origin that has a radius of 5. Find the point $(3,4)$ on the circle and draw a right triangle with the radius as the hypotenuse.



2. Using the length of each side of the right triangle, show that the Pythagorean Theorem is true.

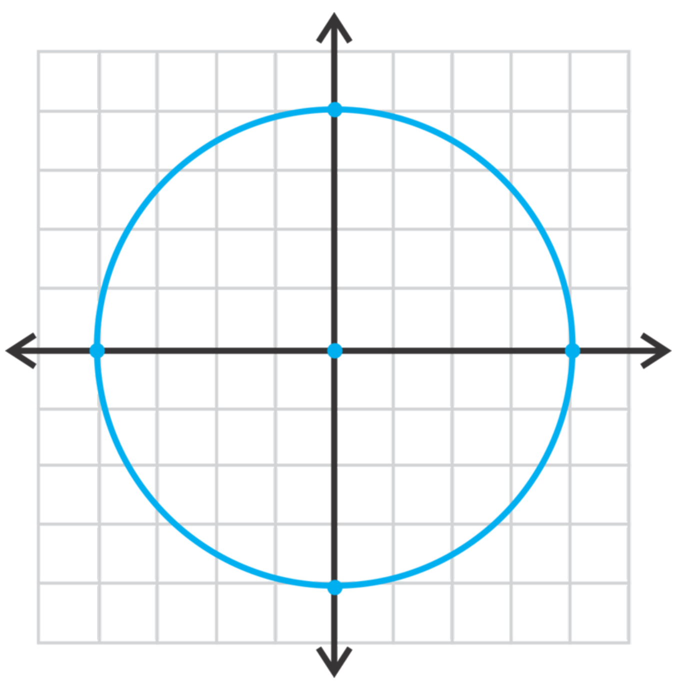
3. Now, instead of using $(3,4)$, change the point to (x,y) so that it represents any point on the circle. Using r to represent the radius, rewrite the Pythagorean Theorem.

The **equation of a circle**, centered at the origin, is $x^2+y^2=r^2$, where r is the radius and (x,y) is any point on the circle.

Example A

Find the radius of $x^2+y^2=16$ and graph.

Solution: To find the radius, we can set $16=r^2$, making $r=4$. r is not -4 because it is a distance and distances are always positive. To graph the circle, start at the origin and go out 4 units in each direction and connect.



Example B

Find the equation of the circle with center at the origin and passes through $(-7,-7)$.

Solution: Using the equation of the circle, we have: $(-7)^2+(-7)^2=r^2$. Solve for r^2.

$$\begin{aligned}(-7)^2+(-7)^2&=r^2\\ 49+49&=r^2\\ 98&=r^2\end{aligned}$$

So, the equation is $x^2+y^2=98$. The radius of the circle is $r=\sqrt{98}=7\sqrt{2}$.

Example C

Determine if the point $(9,-11)$ is on the circle $x^2+y^2=225$.

Solution: Substitute the point in for x and y and see if it equals 225.

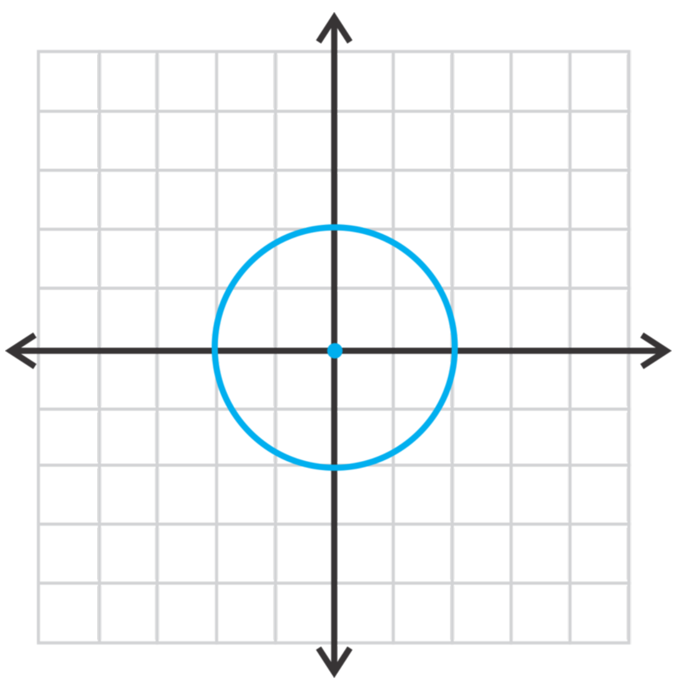
$$\begin{aligned}9^2+(-11)^2&=225\\ 81+121&\stackrel{?}{=}225 \quad \text{The point is not on the circle.}\\ 202&\neq 225\end{aligned}$$

Guided Practice

1. Graph and find the radius of $x^2+y^2=4$.
2. Find the equation of the circle with a radius of $6\sqrt{5}$.
3. Find the equation of the circle that passes through $(5,8)$.
4. Determine if $(-10,7)$ in on the circle $x^2+y^2=149$.

Answers

1. $r=\sqrt{4}=2$



2. Plug in $6\sqrt{5}$ for r in $x^2+y^2=r^2$

$$\begin{aligned}
x^2+y^2 &= \left(6\sqrt{5}\right)^2 \\
x^2+y^2 &= 6^2\cdot\left(\sqrt{5}\right)^2 \\
x^2+y^2 &= 36\cdot 5 \\
x^2+y^2 &= 180
\end{aligned}$$

3. Plug in $(5,8)$ for x and y, respectively.

$$\begin{aligned}
5^2+8^2 &= r^2 \\
25+64 &= r^2 \\
89 &= r^2
\end{aligned}$$

The equation is $x^2+y^2=89$

4. Plug in $(-10,7)$ to see if it is a valid equation.

$$\begin{aligned}
(-10)^2+7^2 &= 149 \\
100+49 &= 149
\end{aligned}$$

Yes, the point is on the circle.

Vocabulary

Circle
The set of points that are a given distance, the **radius**, from a given point, the **center**.

Diameter
A line segment with its endpoints on the circle that passes through the center.

Equation of a Circle
If (x,y) is on the circle, then $x^2+y^2=r^2$ is its equation, where r is the radius.

Problem Set

Graph the following circles and find the radius.

1. $x^2+y^2=9$
2. $x^2+y^2=64$
3. $x^2+y^2=8$
4. $x^2+y^2=50$
5. $2x^2+2y^2=162$
6. $5x^2+5y^2=150$

Write the equation of the circle with the given radius and centered at the origin.

7. 14
8. 6
9. $9\sqrt{2}$

Write the equation of the circle that passes through the given point and is centered at the origin.

10. $(7,-24)$
11. $(2,2)$
12. $(-9,-10)$

Determine if the following points are on the circle, $x^2+y^2=74$.

13. $(-8,0)$
14. $(7,-5)$
15. $(6,-6)$

Challenge In Geometry, you learned about tangent lines to a circle. Recall that the tangent line touches a circle at one point and is perpendicular to the radius at that point, called the point of tangency.

16. The equation of a circle is $x^2+y^2=10$ with point of tangency $(-3,1)$.

    a. Find the slope of the radius from the center to $(-3,1)$.
    b. Find the perpendicular slope to (a). This is the slope of the tangent line.
    c. Use the slope from (b) and the given point to find the equation of the tangent line.

17. Repeat the steps in #16 to find the equation of the tangent line to $x^2+y^2=34$ with a point of tangency of $(3,5)$.

Circles Centered at (h, k)

Objective

To find the equation of and graph circles with a center of (h,k).

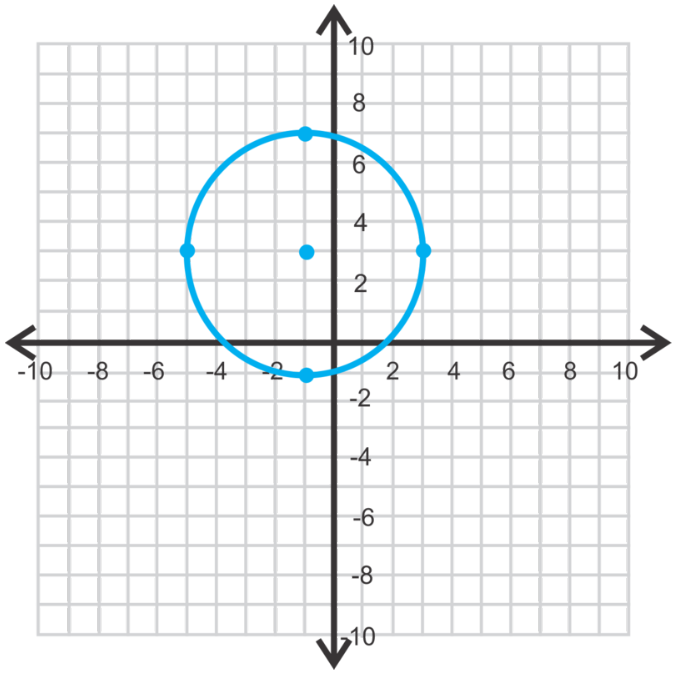
Guidance

When a circle is centered at the origin (as in the last concept), the equation is $x^2+y^2=r^2$. If we rewrite this equation, using the center, it would look like $(x-0)^2+(y-0)^2=r^2$. Extending this idea to any point as the center, we would have $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center.

Example A

Find the center and radius of $(x+1)^2+(y-3)^2=16$ and graph.

Solution: Using the general equation above, the center would be $(-1,3)$ and the radius is $\sqrt{16}$ or 4. To graph, plot the center and then go out 4 units up, down, to the left, and to the right.



Example B

Find the equation of the circle with center $(2,4)$ and radius 5.

Solution: Plug in the center and radius to the equation and simplify.

$$(x-2)^2+(y-4)^2 = 5^2$$
$$(x-2)^2+(y-4)^2 = 25$$

Example C

Find the equation of the circle with center $(6,-1)$ and $(5,2)$ is on the circle.

Solution: In this example, we are not given the radius. To find the radius, we must use the distance formula, $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

$$\begin{aligned} r &= \sqrt{(5-6)^2+(2-(-1))^2} \\ &= \sqrt{(-1)^2+3^2} \\ &= \sqrt{1+9} \\ &= \sqrt{10} \end{aligned}$$

Therefore, the equation of this circle is $(x-6)^2+(y-(-1))^2 = \left(\sqrt{10}\right)^2$ or $(x-6)^2+(y+1)^2 = 10$.

Guided Practice

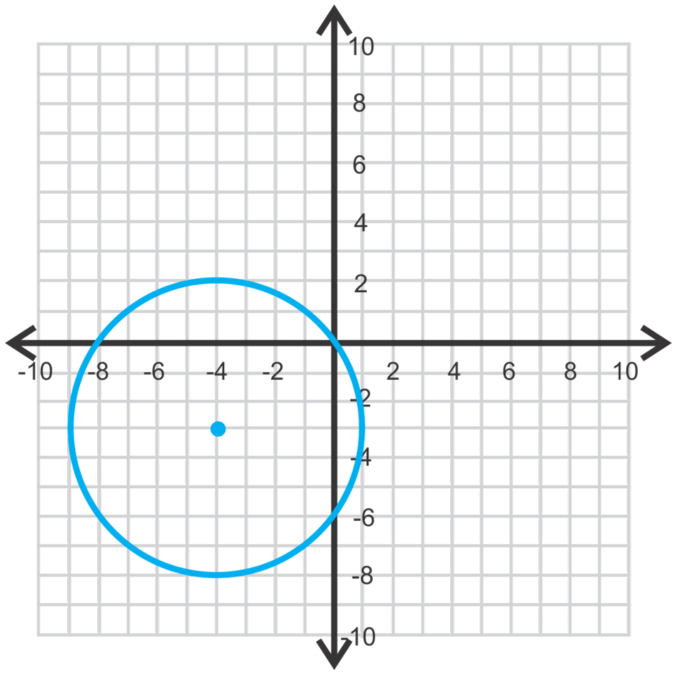
1. Graph $(x+4)^2+(y+3)^2 = 25$ and find the center and radius.

2. Find the equation of the circle with center $(-8,3)$ and $(2,-5)$ is on the circle.

3. The endpoints of a diameter of a circle are $(-3,1)$ and $(9,6)$. Find the equation.

Answers

1. The center is $(-4,-3)$ and the radius is 5.



2. Use the distance formula to find the radius.

$$\begin{aligned} r &= \sqrt{(2-(-8))^2+(-5-3)^2} \\ &= \sqrt{10^2+(-8)^2} \\ &= \sqrt{100+64} \\ &= \sqrt{164} \end{aligned}$$

The equation of this circle is $(x+8)^2+(y-3)^2=164$.

3. In this problem, we are not given the center or radius. We can find the length of the diameter using the distance formula and then divide it by 2.

$$\begin{aligned} d &= \sqrt{(9-(-3))^2+(6-1)^2} \\ &= \sqrt{12^2+5^2} \qquad \text{The radius is } 13 \div 2 = \frac{13}{2}. \\ &= \sqrt{144+25} \\ &= \sqrt{169} = 13 \end{aligned}$$

Now, we need to find the center. Use the midpoint formula with the endpoints.

$$\begin{aligned} c &= \left(\frac{-3+9}{2}, \frac{1+6}{2}\right) \\ &= \left(3, \frac{7}{2}\right) \end{aligned}$$

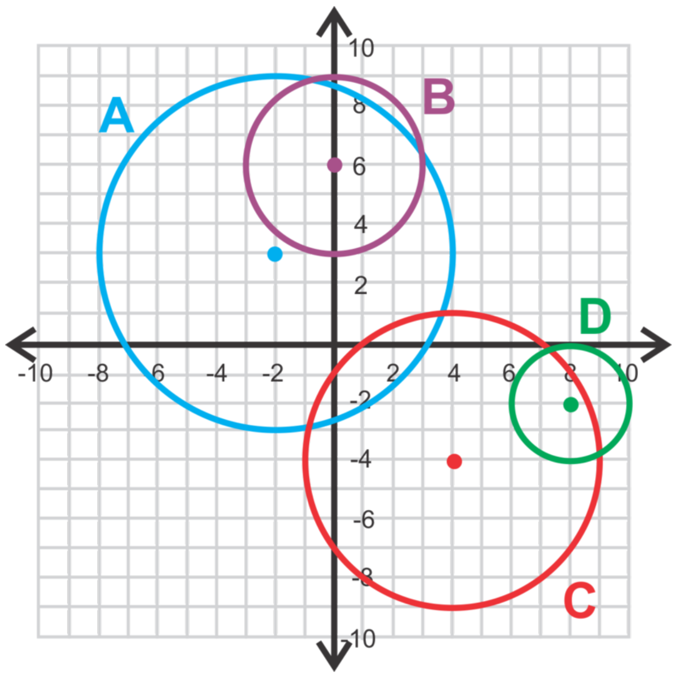
Therefore, the equation is $(x-3)^2+\left(y-\frac{7}{2}\right)^2=\frac{169}{4}$.

Vocabulary

Standard Form (of a Circle)
$(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius.

Problem Set

For questions 1-4, match the equation with the graph.



1. $(x-8)^2+(y+2)^2=4$
2. $x^2+(y-6)^2=9$
3. $(x+2)^2+(y-3)^2=36$
4. $(x-4)^2+(y+4)^2=25$

Graph the following circles. Find the center and radius.

5. $(x-2)^2+(y-5)^2=16$
6. $(x+4)^2+(y+3)^2=18$
7. $(x+7)^2+(y-1)^2=8$

Find the equation of the circle, given the information below.

8. center: $(-3,-3)$ radius: 7
9. center: $(-7,6)$ radius: $\sqrt{15}$
10. center: $(8,-1)$ point on circle: $(0,14)$
11. center: $(-2,-5)$ point on circle: $(3,2)$
12. diameter endpoints: $(-4,1)$ and $(6,3)$
13. diameter endpoints: $(5,-8)$ and $(11,2)$
14. Is $(-9,12)$ on the circle $(x+5)^2+(y-6)^2=54$? How do you know?
15. ***Challenge*** Using the steps from #16 in the previous concept, find the equation of the tangent line to the circle with center $(3,-4)$ and the point of tangency is $(-1,8)$.

16. ***Extension*** Rewrite the equation of the circle, $x^2 + y^2 + 4x - 8y + 11 = 0$ in standard form by completing the square for both the x and y terms. Then, find the center and radius.

CHAPTER 16 Getting Reading for Algebra II

Chapter Outline

16.1 The Unit Circle

Objective

To understand the concept of angles of rotation, co-terminal angles, reference angles and to find the exact value trigonometric ratios for angles that are multiples of 30 and 45 degrees given in degrees and radians.

Review Queue

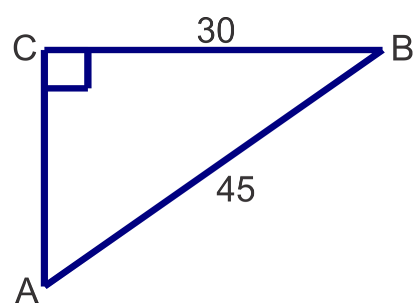
1. Find the $m\angle A$ given the ratio.

a. $\sin A = 0.9371$

b. $\tan A = 3.2191$

c. $\cos A = 0.7145$

2. Solve the triangle.



3. A 20 ft ladder leaning against the side of a building makes an angle of 75° with the ground. How high up the building does it reach?

Introduction to Angles of Rotations, Coterminal Angles and Reference Angles

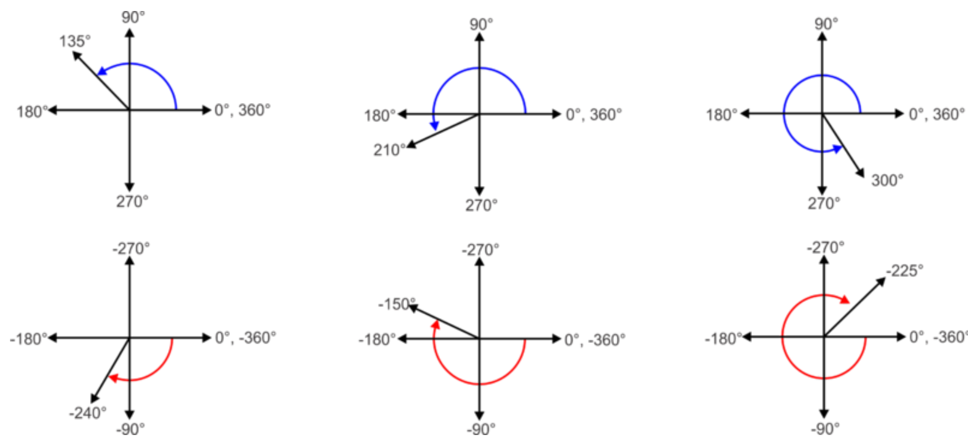
Objective

Understand the concept of an angle of rotation in the coordinate plane, identify coterminal and reference angles and find trigonometric ratios for any angle measure.

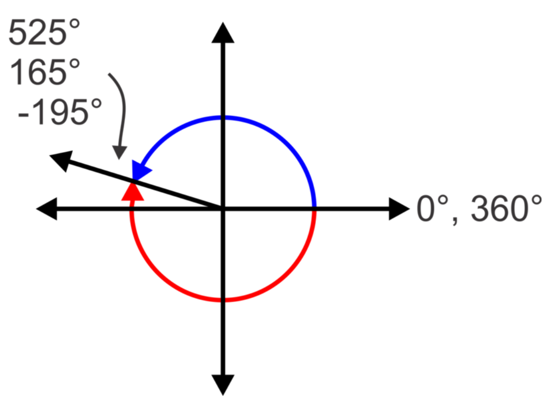
Guidance

Angles of rotation are formed in the coordinate plane between the positive x-axis (**initial side**) and a ray (**terminal side**). Positive angle measures represent a counterclockwise rotation while negative angles indicate a clockwise rotation.

Since the x and y axes are perpendicular, each axis then represents an increment of ninety degrees of rotation. The diagrams below show a variety of angles formed by rotating a ray through the quadrants of the coordinate plane.



An angle of rotation can be described infinitely many ways. It can be described by a positive or negative angle of rotation or by making multiple full circle rotations through 360°. The example below illustrates this concept.



For the angle 525°, an entire 360° rotation is made and then we keep going another 165° to 525°. Therefore, the resulting angle is equivalent to $525^\circ - 360^\circ$, or 165°. In other words, the terminal side is in the same location as the terminal side for a 165° angle. If we subtract 360° again, we get a negative angle, -195°. Since they all share the same terminal side, they are called **coterminal angles**.

Example A

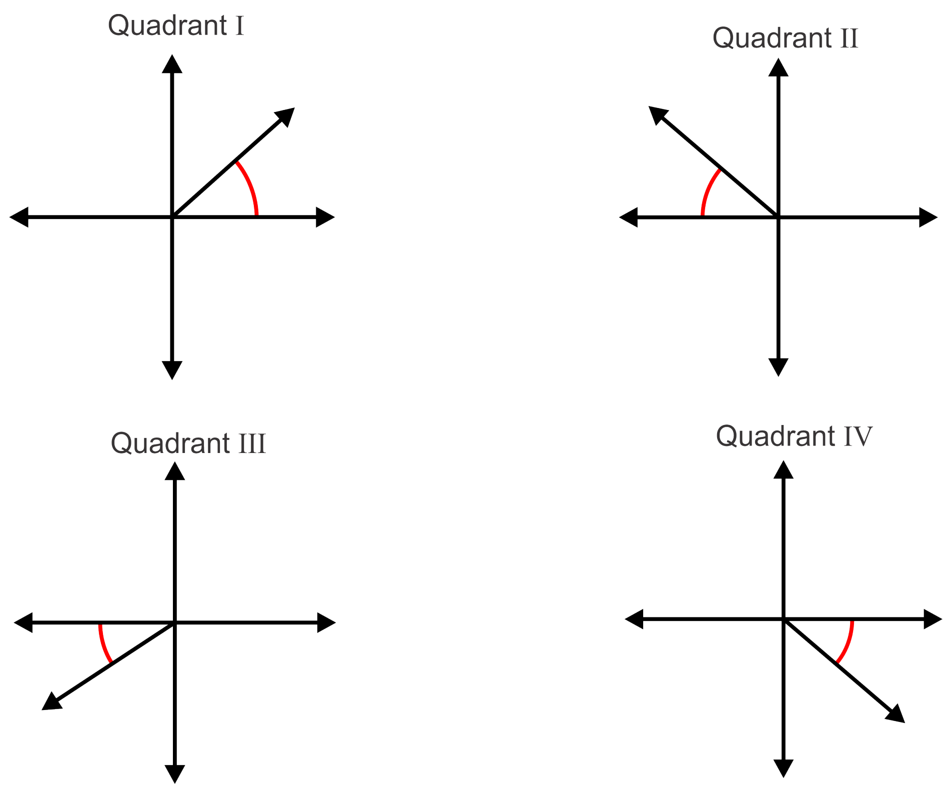
Determine two coterminal angles to 837°, one positive and one negative.

Solution: To find coterminal angles we simply add or subtract 360° multiple times to get the angles we desire. $837^\circ - 360^\circ = 477^\circ$, so we have a positive coterminal angle. Now we can subtract 360° again to get $477^\circ - 360^\circ = 117^\circ$.

More Guidance

A **reference angle** is the acute angle between the terminal side of an angle and the x –axis. The diagram below shows the reference angles for terminal sides of angles in each of the four quadrants.

Note: A reference angle is never determined by the angle between the terminal side and the y –axis. This is a common error for students, especially when the terminal side appears to be closer to the y –axis than the x –axis.



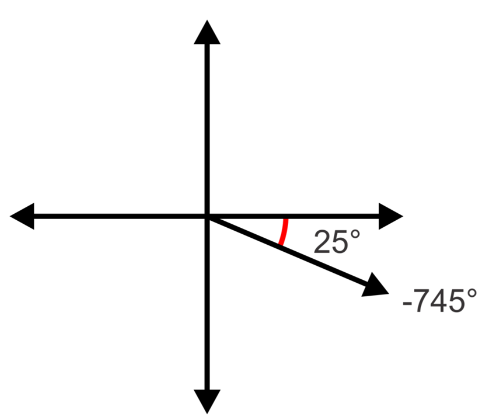
Example B

Determine the quadrant in which -745° lies and hence determine the reference angle.

Solution: Since our angle is more than one rotation, we need to add 360° until we get an angle whose absolute value is less than 360°: $-745^\circ + 360^\circ = -385^\circ$, again $-385^\circ + 360^\circ = -25^\circ$.

Now we can plot the angle and determine the reference angle:

Note that the reference angle is positive 25°. All reference angles will be positive as they are acute angles (between 0° and 90°).



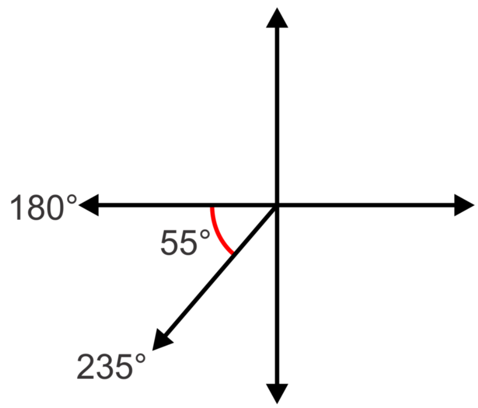
Example C

Give two coterminal angles to 595°, one positive and one negative, find the reference angle.

Solution: To find the coterminal angles we can add/subtract 360°. In this case, our angle is greater than 360° so it makes sense to subtract 360° to get a positive coterminal angle: $595^\circ - 360^\circ = 235^\circ$. Now subtract again to get a negative angle: $235^\circ - 360^\circ = -125^\circ$.

By plotting any of these angles we can see that the terminal side lies in the third quadrant as shown.

Since the terminal side lies in the third quadrant, we need to find the angle between 180° and 235°, so $235^\circ - 180^\circ = 55^\circ$.



Guided Practice

1. Find two coterminal angles to 138°, one positive and one negative.

2. Find the reference angle for 895°.

3. Find the reference angle for 343°.

Answers

1. $138^\circ + 360^\circ = 498^\circ$ and $138^\circ - 360^\circ = -222^\circ$

2. $895^\circ - 360^\circ = 535^\circ, 535^\circ - 360^\circ = 175^\circ$. The terminal side lies in the second quadrant, so we need to determine the angle between 175° and 180°, which is 5°.

3. 343° is in the fourth quadrant so we need to find the angle between 343° and 360° which is 17°.

Problem Set

Find two coterminal angles to each angle measure, one positive and one negative.

1. -98°
2. 475°
3. -210°
4. 47°
5. -1022°

Determine the quadrant in which the terminal side lies and find the reference angle for each of the following angles.

6. 102°
7. -400°
8. 1307°
9. -820°
10. 304°
11. Explain why the reference angle for an angle between 0° and 90° is equal to itself.

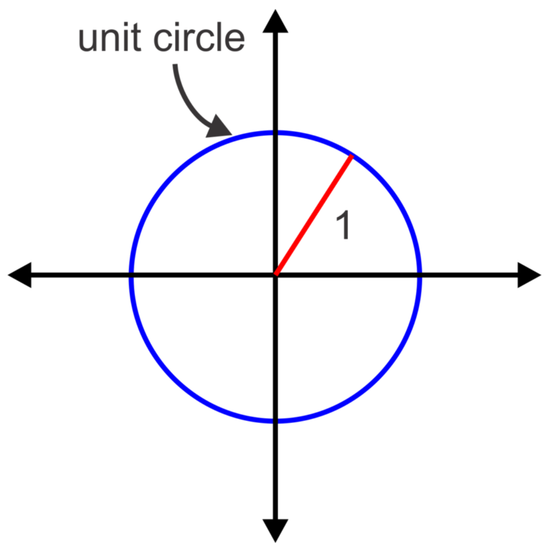
Introduction to the Unit Circle and Radian Measure

Objective

Understand the concept of a unit circle, the meaning of one radian and to convert between radians and degrees.

Guidance

The unit circle is the circle centered at the origin with radius equal to one unit. This means that the distance from the origin to any point on the circle is equal to one unit.



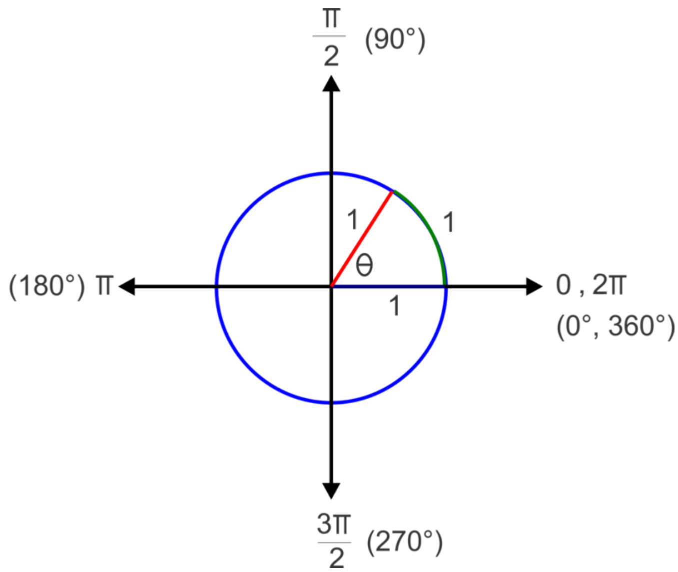
Using the unit circle, we can define another unit of measure for angles, radians. Radian measure is based upon the circumference of the unit circle. The circumference of the unit circle is 2π ($2\pi r$, where $r = 1$). So a full revolution, or 360°, is equal to 2π radians. Half a rotation, or 180° is equal to π radians.

One radian is equal to the measure of θ, the rotation required for the arc length intercepted by the angle to be equal to the radius of the circle. In other words the arc length is 1 unit for $\theta = 1$ radian.

We can use the equality, $\pi = 180^\circ$ to convert from degrees to radians and vice versa.

To convert from degrees to radians, multiply by $\frac{\pi}{180^\circ}$.

To convert from radians to degrees, multiply by $\frac{180^\circ}{\pi}$.



Example A

a. Convert 250° to radians.

b. Convert 3π to degrees.

Solution:

a. To convert from degrees to radians, multiply by $\frac{\pi}{180^\circ}$. So, $\frac{250\pi}{180} = \frac{25\pi}{18}$.

b. To convert from radians to degrees, multiply by $\frac{180^\circ}{\pi}$. So, $3\pi \times \frac{180^\circ}{\pi} = 3 \times 180^\circ = 540^\circ$.

Example B

Find two angles, one positive and one negative, coterminal to $\frac{5\pi}{3}$ and find its reference angle, in radians.

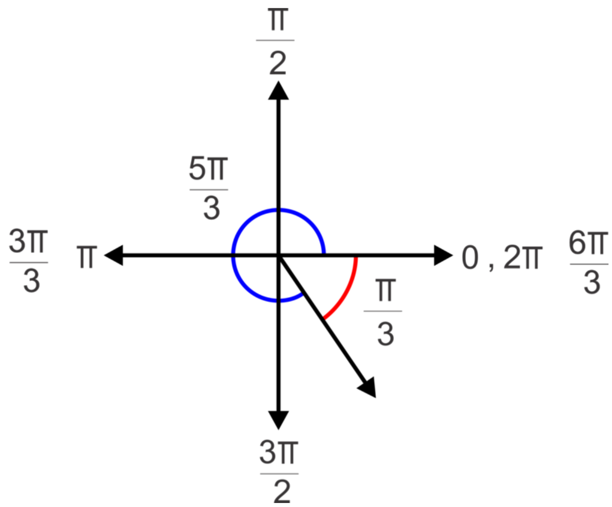
Solution: Since we are working in radians now we will add/subtract multiple of 2π instead of 360°. Before we can add, we must get a common denominator of 3 as shown below.

$$\frac{5\pi}{3} + 2\pi = \frac{5\pi}{3} + \frac{6\pi}{3} = \frac{11\pi}{3} \quad \textit{and} \quad \frac{5\pi}{3} - 2\pi = \frac{5\pi}{3} - \frac{6\pi}{3} = -\frac{\pi}{3}$$

Now, to find the reference angle, first determine in which quadrant $\frac{5\pi}{3}$ lies. If we think of the measures of the angles on the axes in terms of π and more specifically, in terms of $\frac{\pi}{3}$, this task becomes a little easier.

Consider π is equal to $\frac{3\pi}{3}$ and 2π is equal to $\frac{6\pi}{3}$ as shown in the diagram. Now we can see that the terminal side of $\frac{5\pi}{3}$ lies in the fourth quadrant and thus the reference angle will be:

$$\frac{6\pi}{3} - \frac{5\pi}{3} = \frac{\pi}{3}$$



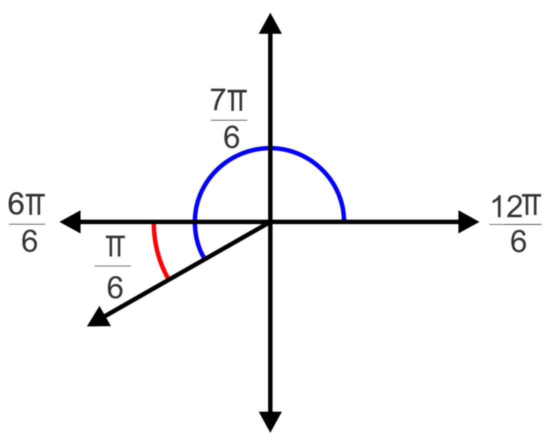
Example C

Find two angles coterminal to $\frac{7\pi}{6}$, one positive and one negative, and find its reference angle, in radians.

Solution: This time we will add multiples of 2π with a common denominator of 6, or $\frac{2\pi}{1} \times \frac{6}{6} = \frac{12\pi}{6}$. For the positive angle, we add to get $\frac{7\pi}{6} + \frac{12\pi}{6} = \frac{19\pi}{6}$. For the negative angle, we subtract to get $\frac{7\pi}{6} - \frac{12\pi}{6} = \frac{5\pi}{6}$.

In this case π is equal to $\frac{6\pi}{6}$ and 2π is equal to $\frac{12\pi}{6}$ as shown in the diagram. Now we can see that the terminal side of $\frac{7\pi}{6}$ lies in the third quadrant and thus the reference angle will be:

$$\frac{7\pi}{6} - \frac{6\pi}{6} = \frac{\pi}{6}$$



Guided Practice

1. Convert the following angle measures from degrees to radians.

a. -45°

b. 120°

c. 330°

2. Convert the following angle measures from radians to degrees.

a. $\frac{5\pi}{6}$

b. $\frac{13\pi}{4}$

c. $-\frac{5\pi}{2}$

3. Find two coterminal angles to $\frac{11\pi}{4}$, one positive and one negative, and its reference angle.

Answers

1. a. $-45^\circ \times \frac{\pi}{180^\circ} = -\frac{\pi}{4}$

b. $120^\circ \times \frac{\pi}{180^\circ} = \frac{2\pi}{3}$

c. $330^\circ \times \frac{\pi}{180^\circ} = \frac{11\pi}{6}$

2. a. $\frac{5\pi}{6} \times \frac{180^\circ}{\pi} = 150^\circ$

b. $\frac{13\pi}{4} \times \frac{180^\circ}{\pi} = 585^\circ$

c. $-\frac{5\pi}{2} \times \frac{180^\circ}{\pi} = -450^\circ$

3. There are many possible coterminal angles, here are some possibilities:

positive coterminal angle: $\frac{11\pi}{4} + \frac{8\pi}{4} = \frac{19\pi}{4}$ or $\frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}$,

negative coterminal angle: $\frac{11\pi}{4} - \frac{16\pi}{4} = -\frac{5\pi}{4}$ or $\frac{11\pi}{4} - \frac{24\pi}{4} = -\frac{13\pi}{4}$

Using the coterminal angle, $\frac{3\pi}{4}$, which is $\frac{\pi}{4}$ from $\frac{4\pi}{4}$. So the terminal side lies in the second quadrant and the reference angle is $\frac{\pi}{4}$.

Problem Set

For problems 1-5, convert the angle from degrees to radians. Leave answers in terms of π.

1. 135°
2. 240°
3. -330°
4. 450°
5. -315°

For problems 6-10, convert the angle measure from radians to degrees.

6. $\frac{7\pi}{3}$
7. $-\frac{13\pi}{6}$
8. $\frac{9\pi}{2}$
9. $-\frac{3\pi}{4}$
10. $\frac{5\pi}{6}$

For problems 11-14, find two coterminal angles (one positive, one negative) and the reference angle for each angle in radians.

11. $\frac{8\pi}{3}$
12. $\frac{11\pi}{4}$
13. $-\frac{\pi}{6}$
14. $\frac{4\pi}{3}$

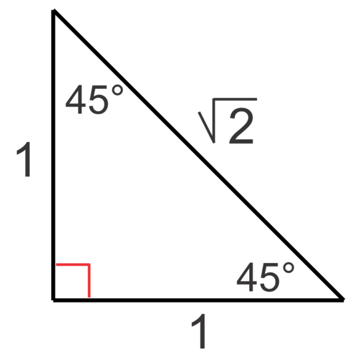
Trigonometric Ratios on the Unit Circle

Objective

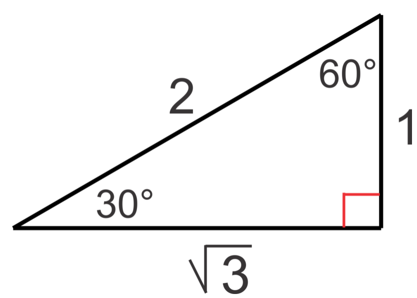
Determine exact value of trigonometric ratios for multiples of $0^\circ, 30^\circ$ and 45° (or $0, \frac{\pi}{6}, \frac{\pi}{4}$ radians).

Guidance

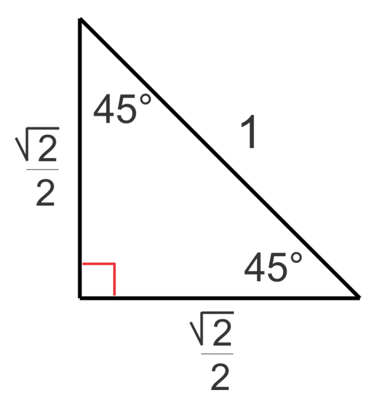
Recall special right triangles from Geometry. In a $(30^\circ - 60^\circ - 90^\circ)$ triangle, the sides are in the ratio $1 : \sqrt{3} : 2$. In an isosceles triangle $(45^\circ - 45^\circ - 90^\circ)$, the congruent sides and the hypotenuse are in the ratio $1 : 1 : \sqrt{2}$.

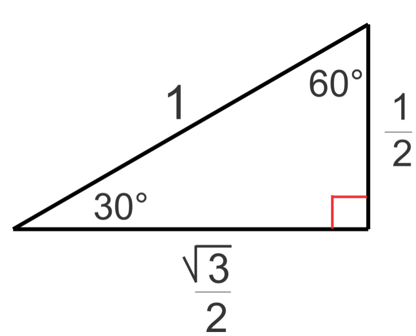


In a $(30^\circ - 60^\circ - 90^\circ)$ triangle, the sides are in the ratio $1 : \sqrt{3} : 2$.



Now let's make the hypotenuse equal to 1 in each of the triangles so we'll be able to put them inside the unit circle. Using the appropriate ratios, the new side lengths are:

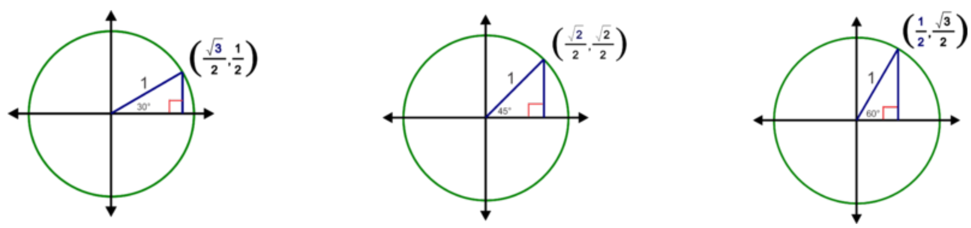




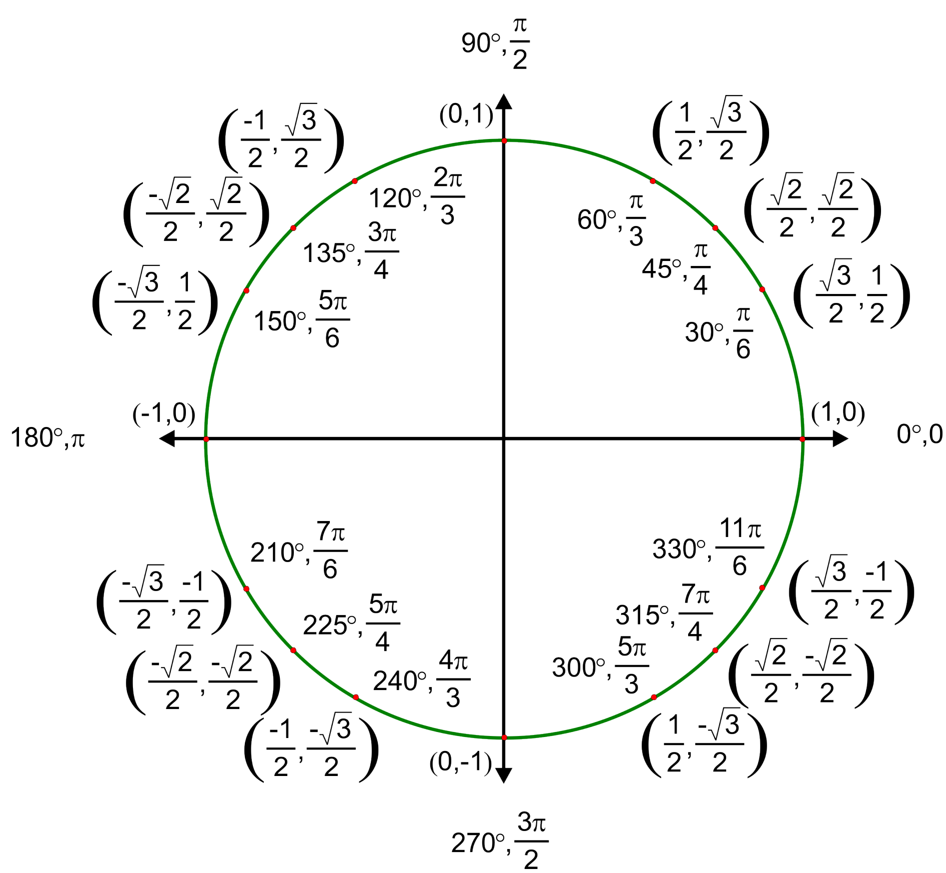
Using these triangles, we can evaluate sine, cosine and tangent for each of the angle measures.

$$\begin{array}{lll}
\sin 45^\circ = \frac{\sqrt{2}}{2} & \sin 60^\circ = \frac{\sqrt{3}}{2} & \sin 30^\circ = \frac{1}{2} \\
\cos 45^\circ = \frac{\sqrt{2}}{2} & \cos 60^\circ = \frac{1}{2} & \cos 30^\circ = \frac{\sqrt{3}}{2} \\
\tan 45^\circ = 1 & \tan 60^\circ = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} & \tan 30^\circ = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}
\end{array}$$

These triangles can now fit inside the unit circle.



Putting together the trigonometric ratios and the coordinates of the points on the circle, which represent the lengths of the legs of the triangles, $(\Delta x, \Delta y)$, we can see that each point is actually $(\cos\theta, \sin\theta)$, where θ is the reference angle. For example, $\sin 60^\circ = \frac{\sqrt{3}}{2}$ is the y –coordinate of the point on the unit circle in the triangle with reference angle 60°. By reflecting these triangles across the axes and finding the points on the axes, we can find the trigonometric ratios of all multiples of $0^\circ, 30^\circ$ and 45° (or $0, \frac{\pi}{6}, \frac{\pi}{4}$ radians).



Example A

Find $\sin \frac{3\pi}{2}$.

Solution: Find $\frac{3\pi}{2}$ on the unit circle and the corresponding point is $(0, -1)$. Since each point on the unit circle is $(\cos\theta, \sin\theta)$, $\sin \frac{3\pi}{2} = -1$.

Example B

Find $\tan \frac{7\pi}{6}$.

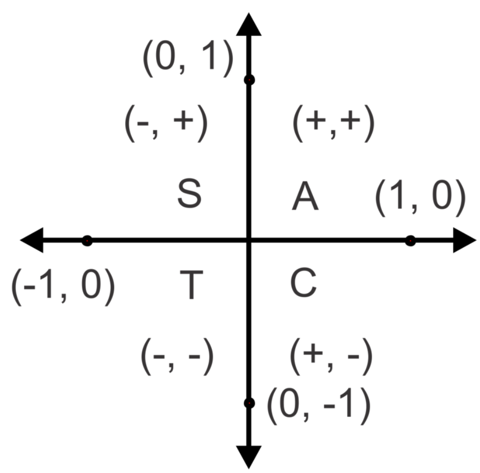
Solution: This time we need to look at the ratio $\frac{\sin\theta}{\cos\theta}$. We can use the unit circle to find $\sin \frac{7\pi}{6} = -\frac{1}{2}$ and $\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$. Now, $\tan \frac{7\pi}{6} = \frac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

More Guidance

Another way to approach these exact value problems is to use the reference angles and the special right triangles. The benefit of this method is that there is no need to memorize the entire unit circle. If you memorize the special right triangles, can determine reference angles and know where the ratios are positive and negative you can put the pieces together to get the ratios. Looking at the unit circle above, we see that all of the ratios are positive in Quadrant I, sine is the only positive ratio in Quadrant II, tangent is the only positive ratio in Quadrant III and cosine is the only positive ratio in Quadrant IV.

Keeping this diagram in mind will help you remember where cosine, sine and tangent are positive and negative. You can also use the pneumonic device - **A**ll **S**tudents **T**ake **C**alculus, or **ASTC**, to recall which is positive (all the others would be negative) in which quadrant.

The coordinates on the vertices will help you determine the ratios for the multiples of 90° or $\frac{\pi}{2}$.



Example C

Find the exact values for the following trigonometric functions using the alternative method.

a. $\cos 120^\circ$

b. $\sin \frac{5\pi}{3}$

c. $\tan \frac{7\pi}{2}$

Solution:

a. First, we need to determine in which quadrant the angles lies. Since 120° is between 90° and 180° it will lie in Quadrant II. Next, find the reference angle. Since we are in QII, we will subtract from 180° to get 60°. We can use the reference angle to find the ratio, $\cos 60^\circ = \frac{1}{2}$. Since we are in QII where only sine is positive, $\cos 120^\circ = -\frac{1}{2}$.

b. This time we will need to work in terms of radians but the process is the same. The angle $\frac{5\pi}{3}$ lies in QIV and the reference angle is $\frac{\pi}{3}$. This means that our ratio will be negative. Since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$.

c. The angle $\frac{7\pi}{2}$ represents more than one entire revolution and it is equivalent to $2\pi + \frac{3\pi}{2}$. Since our angle is a multiple of $\frac{\pi}{2}$ we are looking at an angle on an axis. In this case, the point is $(0,-1)$. Because $\tan\theta = \frac{\sin\theta}{\cos\theta}, \tan\frac{7\pi}{2} = \frac{-1}{0}$, which is undefined. Thus, $\tan\frac{7\pi}{2}$ is undefined.

Guided Practice

Find the exact trigonometric ratios. You may use either method.

1. $\cos\frac{7\pi}{3}$

2. $\tan\frac{9\pi}{2}$

3. $\sin 405^\circ$

4. $\tan\frac{11\pi}{6}$

5. $\cos\frac{2\pi}{3}$

Answers

1. $\frac{7\pi}{3}$ has a reference angle of $\frac{\pi}{3}$ in QI. $\cos\frac{\pi}{3} = \frac{1}{2}$ and since cosine is positive in QI, $\cos\frac{7\pi}{3} = \frac{1}{2}$.

2. $\frac{9\pi}{2}$ is coterminal to $\frac{\pi}{2}$ which has coordinates (0, 1). So $\tan\frac{9\pi}{2} = \frac{\sin\frac{9\pi}{2}}{\cos\frac{9\pi}{2}} = \frac{1}{0}$ which is undefined.

3. 405° has a reference angle of 45° in QI. $\sin 45^\circ = \frac{\sqrt{2}}{2}$ and since sine is positive in QI, $\sin 405^\circ = \frac{\sqrt{2}}{2}$.

4. $\frac{11\pi}{6}$ is coterminal to $\frac{\pi}{6}$ in QIV. $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ and since tangent is negative in QIV, $\tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3}$.

5. $\frac{2\pi}{3}$ is coterminal to $\frac{\pi}{3}$ in QII. $\cos\frac{\pi}{3} = \frac{1}{2}$ and since cosine is negative in QII, $\cos\frac{2\pi}{3} = \frac{1}{2}$.

Problem Set

Find the exact values for the following trigonometric functions.

1. $\sin\frac{3\pi}{4}$
2. $\cos\frac{3\pi}{2}$
3. $\tan 300^\circ$
4. $\sin 150^\circ$
5. $\cos\frac{4\pi}{3}$
6. $\tan\pi$
7. $\cos\left(-\frac{15\pi}{4}\right)$
8. $\sin 225^\circ$
9. $\tan\frac{7\pi}{6}$
10. $\sin 315^\circ$
11. $\cos 450^\circ$
12. $\sin\left(-\frac{7\pi}{2}\right)$
13. $\cos\frac{17\pi}{6}$
14. $\tan 270^\circ$
15. $\sin(-210^\circ)$

Reciprocal Trigonometric Functions

Objective

Determine the ratios of the reciprocal trigonometric ratios cosecant, secant and cotangent for angles that are multiples of $0^\circ, 30^\circ$ and 45° (or $0, \frac{\pi}{6}, \frac{\pi}{4}$ radians) without a calculator and evaluate the reciprocal trigonometric functions for all other angles using the calculator.

Guidance

Each of the trigonometric ratios has a reciprocal function associated with it as shown below.

The reciprocal of sine is cosecant: $\frac{1}{\sin\theta} = \csc\theta$, so $\csc\theta = \frac{H}{O}$ (hypotenuse over opposite)

The reciprocal of cosine is secant: $\frac{1}{\cos\theta} = \sec\theta$, so $\sec\theta = \frac{H}{A}$ (hypotenuse over adjacent)

The reciprocal of tangent is cotangent: $\frac{1}{\tan\theta} = \cot\theta$, so $\cot\theta = \frac{A}{O}$ (adjacent over opposite)

Example A

Use your calculator to evaluate $\sec\frac{2\pi}{5}$.

Solution: First, be sure that your calculator is in radian mode. To check/change the mode, press the MODE button and make sure RADIAN is highlighted. If it is not, use the arrow keys to move the cursor to RADIANS and press enter to select RADIAN as the mode. Now we are ready to use the calculator to evaluate the reciprocal trig function. Since the calculator does not have a button for secant, however, we must utilize the reciprocal relationship between cosine and secant:

$$\text{Since } \sec\theta = \frac{1}{\cos\theta}, \sec\frac{2\pi}{5} = \frac{1}{\cos\frac{2\pi}{5}} = 3.2361.$$

Example B

Use you calculator to evaluate $\cot 100^\circ$.

Solution: This time we will need to be in degree mode. After the mode has been changed we can use the reciprocal of cotangent, which is tangent, to evaluate as shown:

$$\text{Since } \cot\theta = \frac{1}{\tan\theta}, \cot 100^\circ = \frac{1}{\tan 100^\circ} \approx -0.1763.$$

Example C

Find the exact value of $\csc\frac{5\pi}{3}$ without using a calculator. Give your answer in exact form.

Solution: The reciprocal of cosecant is sine so we will first find $\sin\frac{5\pi}{3}$ Using either the unit circle or the alternative method, we can determine that $\sin\frac{5\pi}{3}$ is $-\frac{\sqrt{3}}{2}$ using a 60° reference angle in the fourth quadrant. Now, find its reciprocal: $\frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$.

Guided Practice

Use your calculator to evaluate the following reciprocal trigonometric functions.

1. $\csc\frac{7\pi}{8}$
2. $\cot 85^\circ$

Evaluate the following without using a calculator. Give all answers in exact form.

3. $\sec 225^\circ$
4. $\csc\frac{5\pi}{6}$

Answers

1. $\csc \frac{7\pi}{8} = \frac{1}{\sin \frac{7\pi}{8}} = 2.6131$

2. $\cot 85^\circ = \frac{1}{\tan 85^\circ} = 0.0875$

3. $\sec 225^\circ$ is the reciprocal of $\cot 225^\circ$, a 45° reference angle in quadrant three where cosine is negative. Because $\cos 45^\circ = \frac{1}{\sqrt{2}}, \cos 225^\circ = -\frac{1}{\sqrt{2}}$, and $\sec 225^\circ = -\sqrt{2}$.

4. $\csc \frac{5\pi}{6}$ is the reciprocal of $\sin \frac{5\pi}{6}$, a $\frac{\pi}{6}$ or 30° reference angle in the second quadrant where sine is positive. Because $\sin \frac{\pi}{6} = \frac{1}{2}, \sin \frac{5\pi}{6} = \frac{1}{2}$, and $\csc \frac{5\pi}{6} = 2$.

Problem Set

Use your calculator to evaluate the reciprocal trigonometric functions. Round your answers to four decimal places.

1. $\csc 95^\circ$
2. $\cot 278^\circ$
3. $\sec \frac{14\pi}{5}$
4. $\cot(-245^\circ)$
5. $\sec \frac{6\pi}{7}$
6. $\csc \frac{23\pi}{13}$
7. $\cot 333^\circ$
8. $\csc \frac{9\pi}{5}$

Evaluate the following trigonometric functions without using a calculator. Give your answers exactly.

9. $\sec \frac{5\pi}{6}$
10. $\csc \left(-\frac{3\pi}{2}\right)$
11. $\cot 225^\circ$
12. $\sec \frac{11\pi}{3}$
13. $\csc \frac{7\pi}{6}$
14. $\sec 270^\circ$
15. $\cot \frac{5\pi}{3}$
16. $\csc 315^\circ$

Inverse Trigonometric Functions

Objective

Determine the angle(s) given the exact value trigonometric ratios for angles that are multiples of $0^\circ, 30^\circ$ and 45° (or $0, \frac{\pi}{6}, \frac{\pi}{4}$ radians).

Guidance

Earlier in the unit we learned how to find the measure of an acute angle in a right triangle using the inverse trigonometric ratios on the calculator. Now we will extend this inverse concept to finding the possible angle measures given a trigonometric ratio on the unit circle. We say possible, because there are an infinite number of possible angles with the same ratios. Think of the unit circle. For which angles does $\sin\theta = \frac{1}{2}$? From the special right triangles, we know that the reference angle must be 30° or $\frac{\pi}{6}$. But because sine is positive in the first and second quadrants, the angle could also be 150° or $\frac{5\pi}{6}$. In fact, we could take either of these angles and add or subtract 360° or 2π to it any number of times and still have a coterminal angle for which the sine ratio would remain $\frac{1}{2}$. For problems in this concept we will specify a finite interval for the possible angles measures. In general, this interval will be $0 \le \theta < 360^\circ$ for degree measures and $0 \le \theta < 2\pi$ for radian measures.

Inverse Trigonometric Ratios on the Calculator

When you use the calculator to find an angle given a ratio, the calculator can only give one angle measure. The answers for the respective functions will always be in the following quadrants based on the sign of the ratio.

TABLE 16.1:

Trigonometric Ratio	Positive Ratios	Negative Ratios
Sine	$0 \leq \theta \leq 90$ or $0 \leq \theta \leq \frac{\pi}{2}$	$-90 \leq \theta \leq 0$ or $-\frac{\pi}{2} \leq \theta \leq 0$
Cosine	$0 \leq \theta \leq 90$ or $0 \leq \theta \leq \frac{\pi}{2}$	$90 < \theta \leq 180^\circ$ or $\frac{\pi}{2} < \theta \leq \pi$
Tangent	$0 \leq \theta \leq 90$ or $0 \leq \theta \leq \frac{\pi}{2}$	$-90 \leq \theta < 0$ or $-\frac{\pi}{2} \leq \theta < 0$

Example A

Use your calculator to find all solutions on the interval $0 \leq \theta < 360^\circ$. Round your answers to the nearest tenth.

a. $\cos^{-1}(0.5437)$

b. $\tan^{-1}(-3.1243)$

c. $\csc^{-1}(3.0156)$

Solution: For all of these, we must first make sure the calculator is in degree mode.

a. Type in 2^{nd}COS, to get $\cos^{-1}($ on your calculator screen. Next, type in the ratio to get $\cos^{-1}(0.5437)$ on the calculator and press ENTER. The result is 57.1°. This is an angle in the first quadrant and a reference angle. We want to have all the possible angles on the interval $0 \leq \theta < 360^\circ$. To find the second angle, we need to think about where else cosine is positive. This is in the fourth quadrant. Since the reference angle is 57.1°, we can find the angle by subtracting 57.1° from 360° to get 302.9° as our second angle. So $\cos^{-1}(0.5437) = 57.1^\circ, 302.9^\circ$.

b. Evaluate $\tan^{-1}(-3.1243)$ on the calculator using the same process to get -72.3°. This is a 72.3° reference angle in the fourth quadrant. Since we want all possible answers on the interval $0 \leq \theta < 360^\circ$, we need angles with reference angles of 72.3° in the second and fourth quadrants where tangent is negative.

2^{nd} quadrant: $180^\circ - 72.3^\circ = 107.7^\circ$ and 4^{th} quadrant: $360^\circ - 72.3^\circ = 287.7^\circ$

So, $\tan^{-1}(-3.1243) = 107.7^\circ, 287.7^\circ$

c. This time we have a reciprocal trigonometric function. Recall that $\sin\theta = \frac{1}{\csc\theta}$. In this case, $\csc\theta = 3.0156$ so $\sin\theta = \frac{1}{3.0156}$ and therefore $\csc^{-1}(3.0156) = \sin^{-1}\left(\frac{1}{3.0156}\right) = 19.4^\circ$ from the calculator. Now, we need to find our second possible angle measure. Since sine (and subsequently, cosecant) is positive in the second quadrant, that is where our second answer lies. The reference angle is 19.4° so the angle is $180^\circ - 19.4^\circ = 160.6^\circ$. So, $\csc^{-1}(3.0156) = 19.4^\circ, 160.6^\circ$.

Example B

Use your calculator to find θ, to two decimal places, where $0 \leq \theta < 2\pi$.

a. $\sec\theta = 2.1647$

b. $\sin\theta = -1.0034$

c. $\cot\theta = -1.5632$

Solution: For each these, we will need to be in radian mode on the calculator.

a. Since $\cos\theta = \frac{1}{\sec\theta}, \sec^{-1}(2.1647) = \cos^{-1}\left(\frac{1}{2.1647}\right) = 1.09$ radians. This is a first quadrant value and thus the reference angle as well. Since cosine (and subsequently, secant) is also positive in the fourth quadrant, we can find the second answer by subtracting from 2π: $2\pi - 1.09 = 5.19$.

Hence, $\sec^{-1}(2.1647) = 1.09, 5.19$

b. From the calculator, $\sin^{-1}(-0.3487) = -0.36$ radians, a fourth quadrant reference angle of 0.36 radians. Now we can use this reference angle to find angles in the third and fourth quadrants within the interval given for θ.

3^{rd} quadrant: $\pi + 0.36 = 3.50$ and 4^{th} quadrant: $2\pi - 0.36 = 5.92$

So, $\sin^{-1}(-0.3487) = 3.50, 5.92$

c. Here, $\tan\theta = \frac{1}{\cot\theta}$, so $\cot^{-1}(-1.5632) = \tan^{-1}\left(-\frac{1}{1.5632}\right) = -0.57$, a fourth quadrant reference angle of 0.57 radians. Since the ratio is negative and tangent and cotangent are both negative in the 2^{nd} and 4^{th} quadrants, those are the angles we must find.

2^{nd} quadrant: $\pi - 0.57 = 2.57$ and 4^{th} quadrant: $2\pi - 0.57 = 5.71$

So, $\cot^{-1}(-1.5632) = 2.57, 5.71$

Example C

Without using a calculator, find θ, where $0 \leq \theta < 2\pi$.

a. $\sin\theta = -\frac{\sqrt{3}}{2}$

b. $\cos\theta = \frac{\sqrt{2}}{2}$

c. $\tan\theta = -\frac{\sqrt{3}}{3}$

d. $\csc\theta = -2$

Solution:

a. From the special right triangles, sine has the ratio $\frac{\sqrt{3}}{2}$ for the reference angle $\frac{\pi}{3}$. Now we can use this reference angle to find angles in the 3^{rd} and 4^{th} quadrant where sine is negative.

3^{rd} quadrant: $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ and 4^{th} quadrant: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

So, $\theta = \frac{4\pi}{3}, \frac{5\pi}{3}$.

b. From the special right triangles, cosine has the ratio $\frac{\sqrt{2}}{2}$ for the reference angle $\frac{\pi}{4}$. Since cosine is positive in the first and fourth quadrants, one answer is $\frac{\pi}{4}$ and the second answer (4^{th} quadrant) will be $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$. So, $\theta = \frac{\pi}{4}, \frac{7\pi}{4}$.

c. From the special right triangles, tangent has the ratio $\frac{\sqrt{3}}{3}$ for the reference angle $\frac{\pi}{6}$. Since tangent is negative in the second and fourth quadrants, we will subtract $\frac{\pi}{6}$ from π and 2π to find the angles.

$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ and $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. So, $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$.

d. First, consider that if $\csc\theta = -2$, then $\sin\theta = -\frac{1}{2}$. Next, from special right triangles, we know that sine is $\frac{1}{2}$ for a $\frac{\pi}{6}$ reference angle. Finally, find the angles with a reference angle of $\frac{\pi}{6}$ in the third and fourth quadrants where sine is negative. $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. So, $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

Guided Practice

1. Use your calculator to find all solutions on the interval $0 \leq \theta < 360^\circ$. Round your answers to the nearest tenth.

a. $\sin^{-1}(0.7821)$

b. $\cot^{-1}(-0.6813)$

c. $\sec^{-1}(4.0159)$

2. Use your calculator to find θ, to two decimal places, where $0 \leq \theta < 2\pi$.

a. $\cos\theta = -0.9137$

b. $\tan\theta = 5.0291$

c. $\csc\theta = 2.1088$

3. Without using a calculator, find θ, where $0 \leq \theta < 2\pi$.

a. $\cos\theta = -\frac{\sqrt{3}}{2}$

b. $\cot\theta = \frac{\sqrt{3}}{3}$

c. $\sin\theta = -1$

Answers

1. a. 51.5° and $180^\circ - 51.5^\circ = 128^\circ$

b. $\cot^{-1}(-0.6813) = \tan^{-1}\left(-\frac{1}{0.6813}\right) = -55.7^\circ, 180^\circ - 55.7^\circ = 124.3^\circ$ and $360^\circ - 55.7^\circ = 304.3^\circ$

c. $\sec^{-1}(4.0159) = \cos^{-1}\left(\frac{1}{4.0159}\right) = 75.6^\circ$ and $360^\circ - 75.6^\circ = 284.4^\circ$

2. a. $\cos^{-1}(-0.9137) = 2.72$ and $\pi + 2.72 = 30.34$

b. $\tan^{-1}(5.0291) = 1.37$ and $\pi + 1.37 = 4.51$

c. $\csc^{-1}(2.1088) = \sin^{-1}\left(\frac{1}{2.1088}\right) = 0.49$ and $\pi - 0.49 = 2.65$

3. a. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, since the ratio is negative, $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ and $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$

b. $\cot^{-1}\left(\frac{\sqrt{3}}{3}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}, \theta = \frac{\pi}{3}$, and $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$

c. $\sin^{-1}(-1) = \frac{3\pi}{2}, \theta = \frac{3\pi}{2}$

Problem Set

For problems 1-6, use your calculator to find all solutions on the interval $0 \leq \theta < 360^\circ$. Round your answers to the nearest tenth.

1. $\cos^{-1}(-0.2182)$
2. $\sec^{-1}(10.8152)$
3. $\tan^{-1}(-20.2183)$
4. $\sin^{-1}(0.8785)$
5. $\csc^{-1}(-6.9187)$
6. $\cot^{-1}(0.8316)$

For problems 7-12, use your calculator to find θ, to two decimal places, where $0 \leq \theta < 2\pi$.

7. $\sin\theta = -0.6153$
8. $\cos\theta = 0.1382$
9. $\cot\theta = -2.8135$
10. $\sec\theta = -8.8775$
11. $\tan\theta = 0.9990$
12. $\csc\theta = 12.1385$

For problems 13-18, find θ, without using a calculator, where $0 \leq \theta < 2\pi$.

13. $\sin\theta = 0$
14. $\cos\theta = -\frac{\sqrt{2}}{2}$
15. $\tan\theta = -1$
16. $\sec\theta = \frac{2\sqrt{3}}{3}$
17. $\sin\theta = \frac{1}{2}$
18. $\cot\theta = \text{undefined}$
19. $\cos\theta = -\frac{1}{2}$
20. $\csc\theta = \sqrt{2}$
21. $\tan\theta = \frac{\sqrt{3}}{3}$

16.2 Solving Radical Equations

Objective

To solve radical equations.

Review Queue

Solve for x.

1. $x^2 - 9x + 14 = 0$
2. $3x^2 - 11x - 20 = 0$
3. $\sqrt{x} = 4$

Solving Simple Radical Equations

Objective

To solve basic radical equations.

Guidance

Solving radical equations are very similar to solving other types of equations. The objective is to get x by itself. However, now there are radicals within the equations. Recall that the opposite of the square root of something is to square it.

Example A

Is $x = 5$ the solution to $\sqrt{2x+15} = 8$?

Solution: Plug in 5 for x to see if the equation holds true. If it does, then 5 is the solution.

$$\begin{aligned}\sqrt{2(5)+15} &= 8\\ \sqrt{10+15} &= 9\\ \sqrt{25} &\neq 8\end{aligned}$$

We know that $\sqrt{25} = 5$, so $x = 5$ is not the solution.

Example B

Solve $\sqrt{2x-5}+7 = 16$.

Solution: To solve for x, we need to isolate the radical. Subtract 7 from both sides.

$$\begin{aligned}\sqrt{2x-5}+7 &= 16\\ \sqrt{2x-5} &= 9\end{aligned}$$

Now, we can square both sides to eliminate the radical. Only square both sides when the radical is alone on one side of the equals sign.

$$\sqrt{2x-5}^2 = 9^2$$
$$2x-5 = 81$$
$$2x = 86$$
$$x = 43$$

Check: $\sqrt{2(43)-5}+7 = \sqrt{86-5}+7 = \sqrt{81}+7 = 9+7 = 16 \boxed{\checkmark}$

ALWAYS check your answers when solving radical equations. Sometimes, you will solve an equation, get a solution, and then plug it back in and it will not work. These types of solutions are called **extraneous solutions** and are not actually considered solutions to the equation.

Example C

Solve $3\sqrt[3]{x-8}-2 = -14$.

Solution: Again, isolate the radical first. Add 2 to both sides and divide by 3.

$$3\sqrt[3]{x-8}-2 = -14$$
$$3\sqrt[3]{x-8} = -12$$
$$\sqrt[3]{x-8} = -4$$

Now, cube both sides to eliminate the radical.

$$\sqrt[3]{x-8}^3 = (-4)^3$$
$$x-8 = -64$$
$$x = -56$$

Check: $3\sqrt[3]{-56-8}-2 = 3\sqrt[3]{-64}-2 = 3\cdot -4-2 = -12-2 = -14 \boxed{\checkmark}$

Guided Practice

Solve the equations and check your answers.

1. $\sqrt{x+5} = 6$
2. $5\sqrt{2x-1}+1 = 26$
3. $\sqrt[4]{3x+11}-2 = 3$

Answers

1. The radical is already isolated here. Square both sides and solve for x.

$$\sqrt{x+5}^2 = 6^2$$
$$x+5 = 36$$
$$x = 31$$

Check: $\sqrt{31+5} = \sqrt{36} = 6 \boxed{\checkmark}$

2. Isolate the radical by subtracting 1 and then dividing by 5.

$$
\begin{aligned}
5\sqrt{2x-1}+1 &= 26 \\
5\sqrt{2x-1} &= 25 \\
\sqrt{2x-1} &= 5
\end{aligned}
$$

Square both sides and continue to solve for x.

$$
\begin{aligned}
\sqrt{2x-1}^2 &= 5^2 \\
2x-1 &= 25 \\
2x &= 26 \\
x &= 13
\end{aligned}
$$

Check: $5\sqrt{2(13)-1}+1 = 5\sqrt{26-1} = 5\sqrt{25}+1 = 5\cdot 5+1 = 25+1 = 26 \boxed{\checkmark}$

3. In this problem, we have a fourth root. That means, once we isolate the radical, we must raise both sides to the fourth power to eliminate it.

$$
\begin{aligned}
\sqrt[4]{3x+11}-2 &= 3 \\
\sqrt[4]{3x-11}^4 &= 5^4 \\
3x-11 &= 625 \\
3x &= 636 \\
x &= 212
\end{aligned}
$$

Check: $\sqrt[4]{3(212)+11}-2 = \sqrt[4]{636-11}-2 = \sqrt[4]{625}-2 = 5-2 = 3 \boxed{\checkmark}$

Vocabulary

Extraneous Solution
A solved-for value of x, that when checked, is not actually a solution.

Problem Set

Solve the equations and check your answers.

1. $\sqrt{x+5} = 6$
2. $2-\sqrt{x+1} = 0$
3. $4\sqrt{5-x} = 12$
4. $\sqrt{x+9}+7 = 11$
5. $\frac{1}{2}\sqrt[3]{x-2} = 1$
6. $\sqrt[3]{x+3}+5 = 9$
7. $5\sqrt{15-x}+2 = 17$
8. $-5 = \sqrt[5]{x-5}-7$
9. $\sqrt[4]{x-6}+10 = 13$
10. $\frac{8}{5}\sqrt[3]{x+5} = 8$
11. $3\sqrt{x+7}-2 = 25$
12. $\sqrt[4]{235+x}+9 = 14$

Solving Radical Equations with Variables on Both Sides

Objective

To solve more complicated radical equations.

Guidance

In this concept, we will continue solving radical equations. Here, we will address variables and radicals on both sides of the equation.

Example A

Solve $\sqrt{4x+1}-x=-1$

Solution: Now we have an x that is not under the radical. We will still isolate the radical.

$$\begin{aligned} \sqrt{4x+1}-x &= -1 \\ \sqrt{4x-1} &= x-1 \end{aligned}$$

Now, we can square both sides. Be careful when squaring $x-1$, the answer is not x^2-1.

$$\begin{aligned} \sqrt{4x+1}^2 &= (x-1)^2 \\ 4x+1 &= x^2-2x+1 \end{aligned}$$

This problem is now a quadratic. To solve quadratic equations, we must either factor, when possible or use the Quadratic Formula. Combine like terms and set one side equal to zero.

$$\begin{aligned} 4x+1 &= x^2-2x+1 \\ 0 &= x^2-6x \\ 0 &= x(x-6) \\ x &= 0 \textit{ or } 6 \end{aligned}$$

Check both solutions: $\sqrt{4(0)+1}-1=\sqrt{0+1}-1=1-1=0\neq-1$. 0 is an extraneous solution. $\sqrt{4(6)+1}-6=\sqrt{24+1}-6=5-6=-1$ $\boxed{\checkmark}$ Therefore, 6 is the only solution.

Example B

Solve $\sqrt{8x-11}-\sqrt{3x+19}=0$.

Solution: In this example, you need to isolate both radicals. To do this, subtract the second radical from both sides. Then, square both sides to eliminate the variable.

$$\begin{aligned} \sqrt{8x-11}-\sqrt{3x+19} &= 0 \\ \sqrt{8x-11}^2 &= \sqrt{3x+19}^2 \\ 8x-11 &= 3x+19 \\ 5x &= 30 \\ x &= 6 \end{aligned}$$

Check: $\sqrt{8(6)-11}-\sqrt{3(6)+19}=\sqrt{48-11}-\sqrt{18+19}=\sqrt{37}-\sqrt{37}=0\ \boxed{\checkmark}$

Example C

Solve $\sqrt[4]{4x+1}=x$

Solution: The radical is isolated. To eliminate it, we must raise both sides to the fourth power.

$$\begin{aligned}
\sqrt[4]{2x^2-1}^{\,4} &= x^4\\
2x^2-1 &= x^4\\
0 &= x^4-2x^2+1\\
0 &= (x^2-1)(x^2-1)\\
0 &= (x-1)(x+1)(x-1)(x+1)\\
x &= 1 \ or \ -1
\end{aligned}$$

Check: $\sqrt[4]{2(1)^2-1}=\sqrt[4]{2-1}=\sqrt[4]{1}=1\ \boxed{\checkmark}$ and $\sqrt[4]{2(-1)^2-1}=\sqrt[4]{2-1}=\sqrt[4]{1}=1\ \boxed{\checkmark}$

Guided Practice

Solve the following radical equations. Check for extraneous solutions.

1. $\sqrt[3]{4x^3-24}=x$
2. $\sqrt{5x-3}=\sqrt{3x+19}$
3. $\sqrt{6x-5}-x=-10$

Answers

1. The radical is isolated. Cube both sides to eliminate the cubed root.

$$\begin{aligned}
\sqrt[3]{4x^3-24}^{\,3} &= x^3\\
4x^3-24 &= x^3\\
-24 &= -3x^3\\
8 &= x^3\\
2 &= x
\end{aligned}$$

Check: $\sqrt[3]{4(2)^3-24}=\sqrt[3]{32-24}=\sqrt[3]{8}=2\ \boxed{\checkmark}$

2. Square both sides to solve for x.

$$\begin{aligned}
\sqrt{5x-3}^{\,2} &= \sqrt{3x+19}^{\,2}\\
5x-3 &= 3x+19\\
2x &= 22\\
x &= 11
\end{aligned}$$

Check:

$$\begin{aligned}
\sqrt{5(11)-3} &= \sqrt{3(11)+19}\\
\sqrt{55-3} &= \sqrt{33+19} \quad \boxed{\checkmark}\\
\sqrt{52} &= \sqrt{52}
\end{aligned}$$

3. Add x to both sides and square to eliminate the radical.

$$\begin{aligned}\sqrt{6x-5}^2 &= (x-10)^2 \\ 6x-5 &= x^2-20x+100 \\ 0 &= x^2-26x+105 \\ 0 &= (x-21)(x-5) \\ x &= 21 \text{ or } 5\end{aligned}$$

Check both solutions:

$$\begin{aligned}x=21: &\ \sqrt{6(21)-5}-21 = \sqrt{126-5}-21 = \sqrt{121}-21 = 11-21 = -10 \ \boxed{\checkmark} \\ x=5: &\ \sqrt{6(5)-5}-21 = \sqrt{30-5}-21 = \sqrt{25}-21 = 5-21 \neq -10\end{aligned}$$

5 is an extraneous solution.

Problem Set

Solve the following radical equations. Be sure to check for extraneous solutions.

1. $\sqrt{x-3} = x-5$
2. $\sqrt{x+3+15} = x-12$
3. $\sqrt[4]{3x^2+54} = x$
4. $\sqrt{x^2+60} = 4\sqrt{x}$
5. $\sqrt{x^4+5x^3} = 2\sqrt{2x+10}$
6. $x = \sqrt{5x-6}$
7. $\sqrt{3x+4} = x-2$
8. $\sqrt{x^3+8x} - \sqrt{9x^2-60} = 0$
9. $x = \sqrt[3]{4x+4-x^2}$
10. $\sqrt[4]{x^3+3} = 2\sqrt[4]{x+3}$
11. $x^2 - \sqrt{42x^2+343} = 0$
12. $x\sqrt{x^2-21} = 2\sqrt{x^3-25x+25}$

For questions 13 and 14, you will need to use the method illustrated in the example below.

$$\begin{aligned}\sqrt{x-15} &= \sqrt{x}-3 \\ \left(\sqrt{x-15}\right)^2 &= (\sqrt{x}-3)^2 \\ x-15 &= x-6\sqrt{x}+9 \\ -24 &= -6\sqrt{x} \\ (4)^2 &= (\sqrt{x})^2 \\ 16 &= x\end{aligned}$$

1. Square both sides
2. Combine like terms to isolate the remaining radical
3. Square both sides again to solve

Check:

$$\begin{aligned}\sqrt{16-15} &= \sqrt{16}-3\\ \sqrt{1} &= 4-3\\ 1 &= 1\end{aligned}$$

13. $\sqrt{x+11}-2=\sqrt{x-21}$
14. $\sqrt{x-6}=\sqrt{7x}-22$

Solving Rational Exponent Equations

Objective

To solve equations where the variable has a rational exponent.

Guidance

This concept is very similar to the previous two. When solving a rational exponent equation, isolate the variable. Then, to eliminate the exponent, you will need to raise everything to the reciprocal power.

Example A

Solve $3x^{\frac{5}{2}}=96$.

Solution: First, divide both sides by 3 to isolate x.

$$\begin{aligned}3x^{\frac{5}{2}} &= 96\\ x^{\frac{5}{2}} &= 32\end{aligned}$$

x is raised to the five-halves power. To cancel out this exponent, we need to raise everything to the two-fifths power.

$$\begin{aligned}\left(x^{\frac{5}{2}}\right)^{\frac{2}{5}} &= 32^{\frac{2}{5}}\\ x &= 32^{\frac{2}{5}}\\ x &= \sqrt[5]{32}^2=2^2=4\end{aligned}$$

Check: $3(4)^{\frac{5}{2}}=3\cdot 2^5=3\cdot 32=96\ \boxed{\checkmark}$

Example B

Solve $-2(x-5)^{\frac{3}{4}}+48=-202$.

Solution: Isolate $(x-5)^{\frac{3}{4}}$ by subtracting 48 and dividing by -2.

$$\begin{aligned}-2(x-5)^{\frac{3}{4}}+48 &= -202\\ -2(x-5)^{\frac{3}{4}} &= -250\\ (x-5)^{\frac{3}{4}} &= -125\end{aligned}$$

To undo the three-fourths power, raise everything to the four-thirds power.

$$\begin{aligned}
\left[(x-5)^{\frac{3}{4}}\right]^{\frac{4}{3}} &= (-125)^{\frac{4}{3}} \\
x-5 &= 625 \\
x &= 630
\end{aligned}$$

Check: $-2(630-5)^{\frac{3}{4}}+48 = -2 \cdot 625^{\frac{3}{4}}+48 = -2 \cdot 125+48 = -250+48 = -202 \boxed{\checkmark}$

Guided Practice

Solve the following rational exponent equations and check for extraneous solutions.

1. $8(3x-1)^{\frac{2}{3}} = 200$

2. $6x^{\frac{3}{2}} - 141 = 1917$

Answers

1. Divide both sides by 8 and raise everything to the three-halves power.

$$\begin{aligned}
8(3x-1)^{\frac{2}{3}} &= 200 \\
\left[(3x-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} &= (25)^{\frac{3}{2}} \\
3x-1 &= 125 \\
3x &= 126 \\
x &= 42
\end{aligned}$$

Check: $8(3(42)-1)^{\frac{2}{3}} = 8(126-1)^{\frac{2}{3}} = 8(125)^{\frac{2}{3}} = 8 \cdot 25 = 200 \boxed{\checkmark}$

2. Here, only the x is raised to the three-halves power. Subtract 141 from both sides and divide by 6. Then, eliminate the exponent by raising both sides to the two-thirds power.

$$\begin{aligned}
6x^{\frac{3}{2}} - 141 &= 1917 \\
6x^{\frac{3}{2}} &= 2058 \\
x^{\frac{3}{2}} &= 343 \\
x &= 343^{\frac{2}{3}} = 7^2 = 49
\end{aligned}$$

Check: $6(49)^{\frac{3}{2}} - 141 = 6 \cdot 343 - 141 = 2058 - 141 = 1917 \boxed{\checkmark}$

Problem Set

1. $2x^{\frac{3}{2}} = 54$
2. $3x^{\frac{1}{3}} + 5 = 17$
3. $(7x-3)^{\frac{2}{5}} = 4$
4. $(4x+5)^{\frac{1}{2}} = x-4$
5. $x^{\frac{5}{2}} = 16x^{\frac{1}{2}}$
6. $(5x+7)^{\frac{3}{5}} = 8$
7. $5x^{\frac{2}{3}} = 45$

8. $(7x-8)^{\frac{2}{3}} = 4(x-5)^{\frac{2}{3}}$
9. $7x^{\frac{3}{7}} + 9 = 65$
10. $4997 = 5x^{\frac{3}{2}} - 3$
11. $2x^{\frac{3}{4}} = 686$
12. $x^3 = (4x-3)^{\frac{3}{2}}$

16.3 Properties of Logarithms

Objective

To simplify expressions involving logarithms.

Review Queue

Simplify the following exponential expressions.

1. $(3x^{\frac{1}{2}})^4$
2. $5x^2y \cdot 8x^{-1}y^6$
3. $\left(\frac{2x^{-1}y^2z^8}{5x^0y^{12}z}\right)^{-1}$

Product and Quotient Properties

Objective

To use and apply the product and quotient properties of logarithms.

Guidance

Just like exponents, logarithms have special properties, or shortcuts, that can be applied when simplifying expressions. In this lesson, we will address two of these properties.

Example A

Simplify $\log_b x + \log_b y$.

Solution: First, notice that these logs have the same base. If they do not, then the properties do not apply.

$\log_b x = m$ and $\log_b y = n$, then $b^m = x$ and $b^n = y$.

Now, multiply the latter two equations together.

$$\begin{aligned} b^m \cdot b^n &= xy \\ b^{m+n} &= xy \end{aligned}$$

Recall, that when two exponents with the same base are multiplied, we can add the exponents. Now, reapply the logarithm to this equation.

$b^{m+n} = xy \rightarrow \log_b xy = m + n$

Recall that $m = \log_b x$ and $n = \log_b y$, therefore $\log_b xy = \log_b x + \log_b y$.

This is the **Product Property of Logarithms**.

Example B

Expand $\log_{12} 4y$.

Solution: Applying the Product Property from Example A, we have:

$\log_{12} 4y = \log_{12} 4 + \log_{12} y$

Example C

Simplify $\log_3 15 - \log_3 5$.

Solution: As you might expect, the **Quotient Property of Logarithms** is $\log_b \frac{x}{y} = \log_b x - \log_b y$ (proof in the Problem Set). Therefore, the answer is:

$$\begin{aligned} \log_3 15 - \log_3 5 &= \log_3 \frac{15}{5} \\ &= \log_3 3 \\ &= 1 \end{aligned}$$

Guided Practice

Simplify the following expressions.

1. $\log_7 8 + \log_7 x^2 + \log_7 3y$
2. $\log y - \log 20 + \log 8x$
3. $\log_2 32 - \log_2 z$
4. $\log_8 \frac{16x}{y^2}$

Answers

1. Combine all the logs together using the Product Property.

$$\begin{aligned} \log_7 8 + \log_7 x^2 + \log_7 3y &= \log_7 8x^2 3y \\ &= \log_7 24x^2 y \end{aligned}$$

2. Use both the Product and Quotient Property to condense.

$$\begin{aligned} \log y - \log 20 + \log 8x &= \log \frac{y}{20} \cdot 8x \\ &= \log \frac{2xy}{5} \end{aligned}$$

3. Be careful; you do not have to use either rule here, just the definition of a logarithm.

$$\log_2 32 - \log_2 z = 5 - \log_2 z$$

4. When expanding a log, do the division first and then break the numerator apart further.

$$\begin{aligned} \log_8 \frac{16x}{y^2} &= \log_8 16x - \log_8 y^2 \\ &= \log_8 16 + \log_8 x - \log_8 y^2 \\ &= \frac{4}{3} + \log_8 x - \log_8 y^2 \end{aligned}$$

To determine $\log_8 16$, use the definition and powers of 2: $8^n = 16 \rightarrow 2^{3n} = 2^4 \rightarrow 3n = 4 \rightarrow n = \frac{4}{3}$.

Vocabulary

Product Property of Logarithms

As long as $b \neq 1$, then $\log_b xy = \log_b x + \log_b y$

Quotient Property of Logarithms

As long as $b \neq 1$, then $\log_b \frac{x}{y} = \log_b x - \log_b y$

Problem Set

Simplify the following logarithmic expressions.

1. $\log_3 6 + \log_3 y - \log_3 4$
2. $\log 12 - \log x + \log y^2$
3. $\log_6 x^2 - \log_6 x - \log_6 y$
4. $\ln 8 + \ln 6 - \ln 12$
5. $\ln 7 - \ln 14 + \ln 10$
6. $\log_{11} 22 + \log_{11} 5 - \log_{11} 55$

Expand the following logarithmic functions.

7. $\log_3(abc)$
8. $\log\left(\frac{a^2}{b}\right)$
9. $\log_9\left(\frac{xy}{5}\right)$
10. $\log\left(\frac{2x}{y}\right)$
11. $\log\left(\frac{8x^2}{15}\right)$
12. $\log_4\left(\frac{5}{9y}\right)$
13. Write an algebraic proof of the quotient property. Start with the expression $\log_a x - \log_a y$ and the equations $\log_a x = m$ and $\log_a y = n$ in your proof. Refer to the proof of the product property in Example A as a guide for your proof.

Power Property of Logarithms

Objective

To use the Power Property of logarithms.

Guidance

The last property of logs is the **Power Property**.

$\log_b x = y$

Using the definition of a log, we have $b^y = x$. Now, raise both sides to the n power.

$$(b^y)^n = x^n$$
$$b^{ny} = x^n$$

Let's convert this back to a log with base b, $\log_b x^n = ny$. Substituting for y, we have $\log_b x^n = n \log_b x$.

Therefore, the Power Property says that if there is an exponent within a logarithm, we can pull it out in front of the logarithm.

Example A

Expand $\log_6 17x^5$.

Solution: To expand this log, we need to use the Product Property and the Power Property.

$$\begin{aligned}\log_6 17x^5 &= \log_6 17 + \log_6 x^5 \\ &= \log_6 17 + 5\log_6 x\end{aligned}$$

Example B

Expand $\ln\left(\frac{2x}{y^3}\right)^4$.

Solution: We will need to use all three properties to expand this example. Because the expression within the natural log is in parenthesis, start with moving the 4^{th} power to the front of the log.

$$\begin{aligned}\ln\left(\frac{2x}{y^3}\right)^4 &= 4\ln\frac{2x}{y^3} \\ &= 4(\ln 2x - \ln y^3) \\ &= 4(\ln 2 + \ln x - 3\ln y) \\ &= 4\ln 2 + 4\ln x - 12\ln y\end{aligned}$$

Depending on how your teacher would like your answer, you can evaluate $4\ln 2 \approx 2.77$, making the final answer $2.77 + 4\ln x - 12\ln y$.

Example C

Condense $\log 9 - 4\log 5 - 4\log x + 2\log 7 + 2\log y$.

Solution: This is the opposite of the previous two examples. Start with the Power Property.

$$\begin{aligned}&\log 9 - 4\log 5 - 4\log x + 2\log 7 + 2\log y \\ &\log 9 - \log 5^4 - \log x^4 + \log 7^2 + \log y^2\end{aligned}$$

Now, start changing things to division and multiplication within one log.

$\log\frac{9\cdot 7^2 y^2}{5^4 x^4}$

Lastly, combine like terms.

$\log\frac{441y^2}{625x^4}$

Guided Practice

Expand the following logarithmic expressions.

1. $\ln x^3$

2. $\log_{16}\frac{x^2 y}{32z^5}$

3. $\log(5c^4)^2$

4. Condense into one log: $\ln 5 - 7\ln x^4 + 2\ln y$.

Answers

1. The only thing to do here is apply the Power Property: $3\ln x$.
2. Let's start with using the Quotient Property.

$$\log_{16}\frac{x^2y}{32z^5} = \log_{16}x^2y - \log_{16}32z^5$$

Now, apply the Product Property, followed by the Power Property.

$$\begin{aligned}&= \log_{16}x^2 + \log_{16}y - \left(\log_{16}32 + \log_{16}z^5\right)\\&= 2\log_{16}x + \log_{16}y - \frac{5}{4} - 5\log_{16}z\end{aligned}$$

Simplify $\log_{16}32 \rightarrow 16^n = 32 \rightarrow 2^{4n} = 2^5$ and solve for n. Also, notice that we put parenthesis around the second log once it was expanded to ensure that the z^5 would also be subtracted (because it was in the denominator of the original expression).

3. For this problem, you will need to apply the Power Property twice.

$$\begin{aligned}\log(5c^4)^2 &= 2\log 5c^4\\&= 2(\log 5 + \log c^4)\\&= 2(\log 5 + 4\log c)\\&= 2\log 5 + 8\log c\end{aligned}$$

Important Note: You can write this particular log several different ways. Equivalent logs are: $\log 25 + 8\log c, \log 25 + \log c^8$ and $\log 25c^8$. Because of these properties, there are several different ways to write one logarithm.

4. To condense this expression into one log, you will need to use all three properties.

$$\begin{aligned}\ln 5 - 7\ln x^4 + 2\ln y &= \ln 5 - \ln x^{28} + \ln y^2\\&= \ln\frac{5y^2}{x^{28}}\end{aligned}$$

Important Note: If the problem was $\ln 5 - (7\ln x^4 + 2\ln y)$, then the answer would have been $\ln\frac{5}{x^{28}y^2}$. But, because there are no parentheses, the y^2 is in the numerator.

Vocabulary

Power Property
As long as $b \neq 1$, then $\log_b x^n = n\log_b x$.

Problem Set

Expand the following logarithmic expressions.

1. $\log_4(9x)^3$
2. $\log\left(\frac{3x}{y}\right)^2$

3. $\log_8 \frac{x^3y^2}{z^4}$
4. $\log_5 \left(\frac{25x^4}{y}\right)^2$
5. $\ln \left(\frac{6x}{y^3}\right)^{-2}$
6. $\ln \left(\frac{e^5x^{-2}}{y^3}\right)^6$

Condense the following logarithmic expressions.

7. $2\log_6 x + 5\log_6 y$
8. $3(\log x - \log y)$
9. $\frac{1}{2}\log(x+1) - 3\log y$
10. $4\log_2 y + \frac{1}{3}\log_2 x^3$
11. $\frac{1}{5}\left[10\log_2(x-3) + \log_2 32 - \log_2 y\right]$
12. $4\left[\frac{1}{2}\log_3 y - \frac{1}{3}\log_3 x - \log_3 z\right]$

16.4 Logarithmic Functions

Objective

To learn about the inverse of an exponential function, the logarithm.

Review Queue

Find the inverse of the following functions.

1. $f(x) = \frac{1}{2}x - 5$

2. $g(x) = \sqrt{x+5}$

3. $h(x) = 6x^2 - 1$

Solve the equations below.

4. $3^x = 27$

5. $2^x = \frac{1}{8}$

6. $5^x = 1$

Defining Logarithms

Objective

To define and use logarithms.

Guidance

You can probably guess that $x = 3$ in $2^x = 8$ and $x = 4$ in $2^x = 16$. But, what is x if $2^x = 12$? Until now, we did not have an inverse to an exponential function. But, because we have variables in the exponent, we need a way to get them out of the exponent. Introduce the logarithm. A **logarithm** is defined as the inverse of an exponential function. It is written $\log_b a = x$ such that $b^x = a$. Therefore, if $5^2 = 25$ (**exponential form**), then $\log_5 25 = 2$ (**logarithmic form**).

There are two special logarithms, or logs. One has base 10, and rather that writing $\log_{10}$, we just write log. The other is the **natural log**, the inverse of the natural number. The natural log has base e and is written ln This is the only log that is not written using log.

Example A

Rewrite $\log_3 27 = 3$ in exponential form.

Solution: Use the definition above, also called the "key".

$$\log_b a = x \leftrightarrow b^x = a$$
$$\log_3 27 = 3 \leftrightarrow 3^3 = 27$$

Example B

Find:

a) $\log 1000$

b) $\log_7 \frac{1}{49}$

c) $\log_{\frac{1}{2}}(-8)$

Solution: Using the key, we can rearrange all of these in terms of exponents.

a) $\log 1000 = x \Rightarrow 10^x = 1000, x = 3$.

b) $\log_7 \frac{1}{49} = x \Rightarrow 7^x = \frac{1}{49}, x = -2$.

c) $\log_{\frac{1}{2}}(-8) = x \Rightarrow \left(\frac{1}{2}\right)^x = -8$. There is no solution. A positive number when raised to any power will never be negative.

There are two special logarithms that you may encounter while writing them into exponential form.

The first is $\log_b 1 = 0$, because $b^0 = 1$. The second is $\log_b b = 1$ because $b^1 = b \cdot b$ can be any number except 1.

Example C

Use your calculator to find the following logarithms. Round your answer to the nearest hundredth.

a) $\ln 7$

b) $\log 35$

c) $\log_5 226$

Solution:

a) Locate the LN button on your calculator. Depending on the brand, you may have to input the number first. For a TI-83 or 84, press the LN, followed by the 7 and ENTER. The answer is 1.95.

b) The LOG button on the calculator is base 10. Press LOG, 35, ENTER. The answer is 1.54.

c) To use the calculator for a base other than 10 or the natural log, you need to use the change of base formula.

Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$, such that x, a, and $b > 0$ and a and $b \neq 1$.

So, to use this for a calculator, you can use either LN or LOG.

$\log_5 226 = \frac{\log 226}{\log 5}$ or $\frac{\ln 226}{\ln 5} \approx 3.37$

In the TI-83 or 84, the keystrokes would be $\frac{\text{LOG}(226)}{\text{LOG}(5)}$, ENTER.

Guided Practice

1. Write $6^2 = 36$ in logarithmic form.

2. Evaluate the following expressions without a calculator.

a) $\log_{\frac{1}{2}} 16$

b) $\log 100$

c) $\log_{64} \frac{1}{8}$

3. Use a calculator to evaluate each expression. Round your answers to the hundredths place.

a) $\ln 32$

b) $\log_7 94$

c) $\log 65$

4. Use the change of base formula to evaluate $\log_8 \frac{7}{9}$ in a calculator.

Answers

1. Using the key, we have: $6^2 = 36 \rightarrow \log_6 36 = 2$.

2. Change each logarithm into exponential form and solve for x.

a) $\log_{\frac{1}{2}} 16 \rightarrow \left(\frac{1}{2}\right)^x = 16$. x must be negative because the answer is not a fraction, like the base.

$2^4 = 16$, so $\left(\frac{1}{2}\right)^{-4} = 16$. Therefore, $\log_{\frac{1}{2}} 16 = -4$.

b) $\log 100 \rightarrow 10^x = 100$. $x = 2$, therefore, $\log 100 = 2$.

c) $\log_{64} \frac{1}{8} \rightarrow 64^x = \frac{1}{8}$. First, $\sqrt{64} = 8$, so $64^{\frac{1}{2}} = 8$. To make this a fraction, we need to make the power negative. $64^{-\frac{1}{2}} = \frac{1}{8}$, therefore $\log_{64} \frac{1}{8} = -\frac{1}{2}$.

3. Using a calculator, we have:

a) 3.47

b) 2.33

c) 1.81

4. Rewriting $\log_8 \frac{7}{9}$ using the change of base formula, we have: $\frac{\log \frac{7}{9}}{\log 8}$. Plugging it into a calculator, we get $\frac{\log\left(\frac{7}{9}\right)}{\log 8} \approx -0.12$.

Vocabulary

Logarithm
The inverse of an exponential function and is written $\log_b a = x$ such that $b^x = a$.

Exponential Form
$b^x = a$, such that b is the base and x is the exponent.

Logarithmic Form
$\log_b a = x$, such that b is the base.

Natural Log
The inverse of the natural number, e, written ln.

Change of Base Formula
Let b, x, and y be positive numbers, $b \neq 1$ and $y \neq 1$. Then, $\log_y x = \frac{\log_b x}{\log_b y}$. More specifically, $\log_y x = \frac{\log x}{\log y}$ and $\log_y x = \frac{\ln x}{\ln y}$, so that expressions can be evaluated using a calculator.

Problem Set

Convert the following exponential equations to logarithmic equations.

1. $3^x = 5$
2. $a^x = b$
3. $4(5^x) = 10$

Convert the following logarithmic equations to exponential equations.

4. $\log_2 32 = x$
5. $\log_{\frac{1}{3}} x = -2$
6. $\log_a y = b$

convert the following logarithmic expressions without a calculator.

7. $\log_5 25$
8. $\log_{\frac{1}{3}} 27$
9. $\log \frac{1}{10}$
10. $\log_2 64$

Evaluate the following logarithmic expressions using a calculator. You may need to use the Change of Base Formula for some problems.

11. $\log 72$
12. $\ln 8$
13. $\log_2 12$
14. $\log_3 9$

Inverse Properties of Logarithmic Functions

Objective

To understand the inverse properties of a logarithmic function.

Guidance

By the definition of a logarithm, it is the inverse of an exponent. Therefore, a logarithmic function is the inverse of an exponential function. Recall what it means to be an inverse of a function. When two inverses are composed (see the *Inverse of a Function* concept), they equal x. Therefore, if $f(x) = b^x$ and $g(x) = \log_b x$, then:

$f \circ g = b^{\log_b x} = x$ and $g \circ f = \log_b b^x = x$

These are called the Inverse Properties of Logarithms.

Example A

Find:

a) $10^{\log 56}$

b) $e^{\ln 6} \cdot e^{\ln 2}$

Solution: For each of these examples, we will use the Inverse Properties.

a) Using the first property, we see that the bases cancel each other out. $10^{\log 56} = 56$

b) Here, e and the natural log cancel out and we are left with $6 \cdot 2 = 12$.

Example B

Find $\log_4 16^x$

Solution: We will use the second property here. Also, rewrite 16 as 4^2.

$\log_4 16^x = \log_4 (4^2)^x = \log_4 4^{2x} = 2x$

Example C

Find the inverse of $f(x) = 2e^{x-1}$.

Solution: See the *Finding the Inverse* concept for the steps on how to find the inverse.

Change $f(x)$ to y. Then, switch x and y.

$$y = 2e^{x-1}$$
$$x = 2e^{y-1}$$

Now, we need to isolate the exponent and take the logarithm of both sides. First divide by 2.

$$\frac{x}{2} = e^{y-1}$$
$$\ln\left(\frac{x}{2}\right) = \ln e^{y-1}$$

Recall the Inverse Properties from earlier in this concept. $\log_b b^x = x$; applying this to the right side of our equation, we have $\ln e^{y-1} = y-1$. Solve for y.

$$\ln\left(\frac{x}{2}\right) = y-1$$
$$\ln\left(\frac{x}{2}\right) + 1 = y$$

Therefore, $\ln\left(\frac{x}{2}\right)+1$ is the inverse of $2e^{y-1}$.

Guided Practice

1. Simplify $5^{\log_5 6x}$.

2. Simplify $\log_9 81^{x+2}$.

3. Find the inverse of $f(x) = 4^{x+2} - 5$.

Answers

1. Using the first inverse property, the log and the base cancel out, leaving $6x$ as the answer.

$$5^{\log_5 6x} = 6x$$

2. Using the second inverse property and changing 81 into 9^2 we have:

$$\begin{aligned}\log_9 81^{x+2} &= \log_9 9^{2(x+2)}\\ &= 2(x+2)\\ &= 2x+4\end{aligned}$$

3. Follow the steps from Example C to find the inverse.

$$\begin{aligned}f(x) &= 4^{x+2}-5\\ y &= 4^{x+2}-5\\ x &= 4^{y+2}-5\\ x+5 &= 4^{y+2}\\ \log_4(x+5) &= y+2\\ \log_4(x+5)-2 &= y\end{aligned}$$

Vocabulary

Inverse Properties of Logarithms

$\log_b b^x = x$ and $b^{\log_b x} = x, b \neq 1$

Problem Set

Use the Inverse Properties of Logarithms to simplify the following expressions.

1. $\log_3 27^x$
2. $\log_5 \left(\frac{1}{5}\right)^x$
3. $\log_2 \left(\frac{1}{32}\right)^x$
4. $10^{\log(x+3)}$
5. $\log_6 36^{(x-1)}$
6. $9^{\log_9(3x)}$
7. $e^{\ln(x-7)}$
8. $\log\left(\frac{1}{100}\right)^{3x}$
9. $\ln e^{(5x-3)}$

Find the inverse of each of the following exponential functions.

10. $y = 3e^{x+2}$
11. $f(x) = \frac{1}{5}e^{\frac{x}{7}}$
12. $y = 2 + e^{2x-3}$
13. $f(x) = 7^{\frac{3}{x}+1-5}$
14. $y = 2(6)^{\frac{x-5}{2}}$
15. $f(x) = \frac{1}{3}(8)^{\frac{x}{2}-5}$

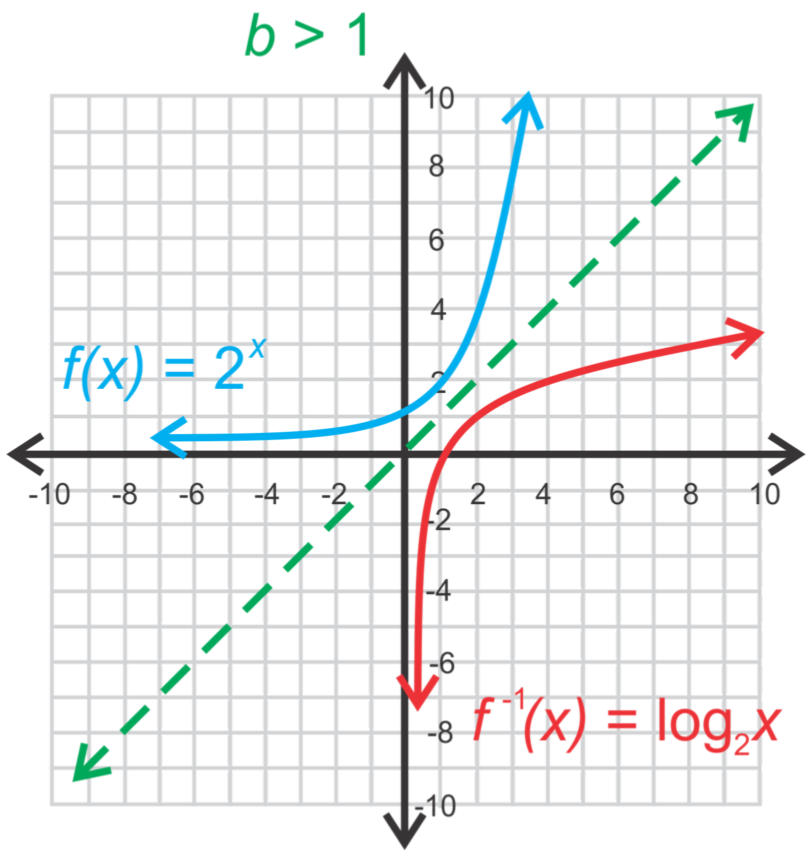
Graphing Logarithmic Functions

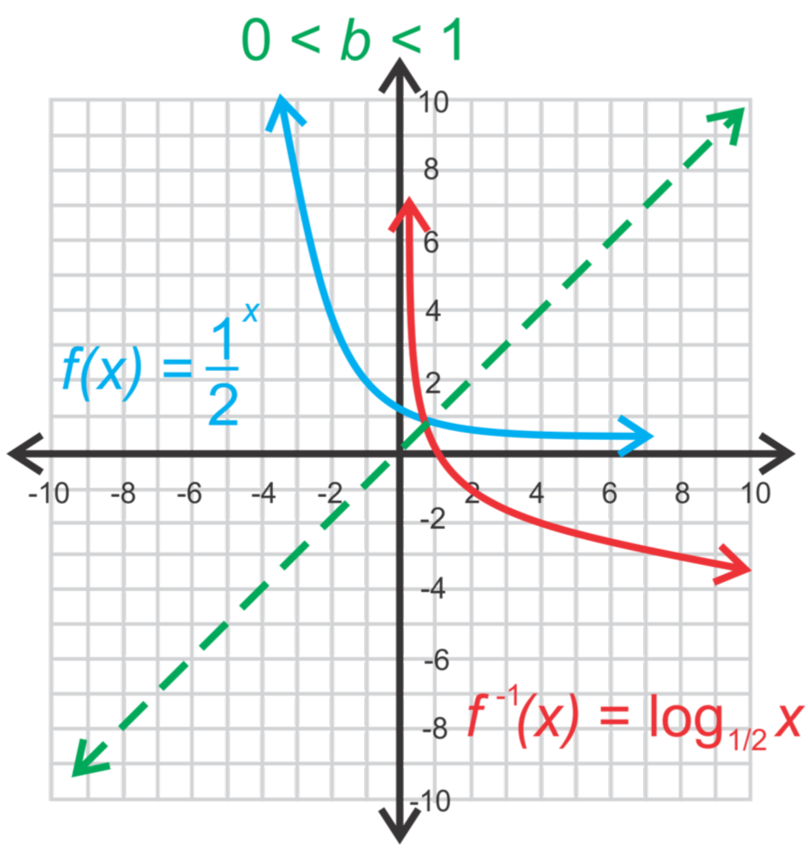
Objective

To graph a logarithmic function by hand and using a calculator.

Guidance

Now that we are more comfortable with using these functions as inverses, let's use this idea to graph a logarithmic function. Recall that functions are inverses of each other when they are mirror images over the line $y = x$. Therefore, if we reflect $y = b^x$ over $y = x$, then we will get the graph of $y = \log_b x$.



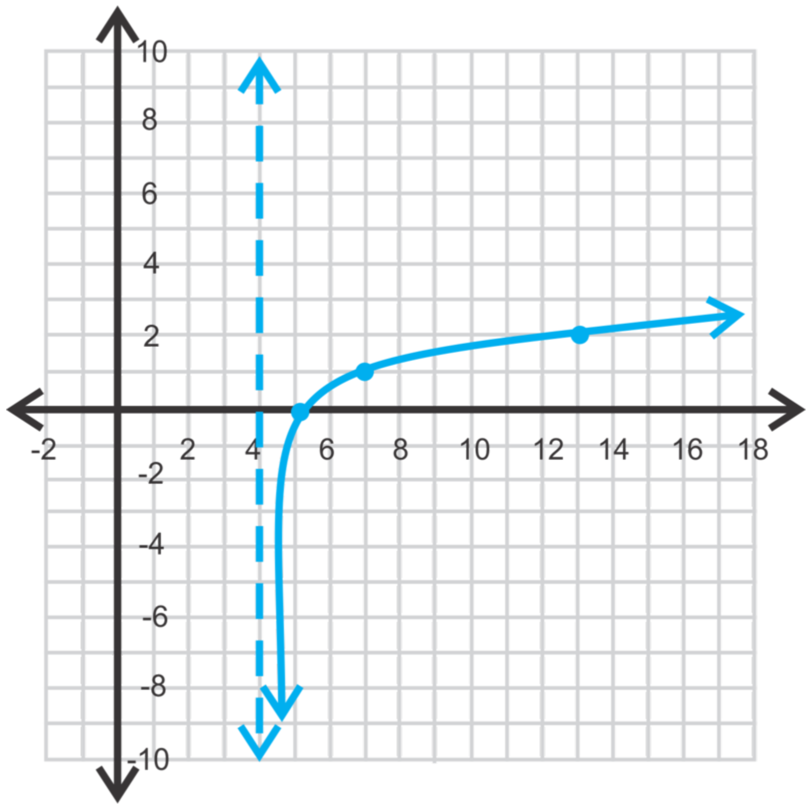


Recall that an exponential function has a horizontal asymptote. Because the logarithm is its inverse, it will have a *vertical* asymptote. The general form of a logarithmic function is $f(x) = \log_b(x-h)+k$ and the vertical asymptote is $x = h$. The domain is $x > h$ and the range is all real numbers. Lastly, if $b > 1$, the graph moves *up* to the right. If $0 < b < 1$, the graph moves *down* to the right.

Example A

Graph $y = \log_3(x-4)$. State the domain and range.

Solution:



To graph a logarithmic function without a calculator, start by drawing the vertical asymptote, at $x = 4$. We know the graph is going to have the general shape of the first function above. Plot a few "easy" points, such as (5, 0), (7, 1), and (13, 2) and connect.

The domain is $x > 4$ and the range is all real numbers.

Example B

Is (16, 1) on $y = \log(x-6)$?

Solution: Plug in the point to the equation to see if it holds true.

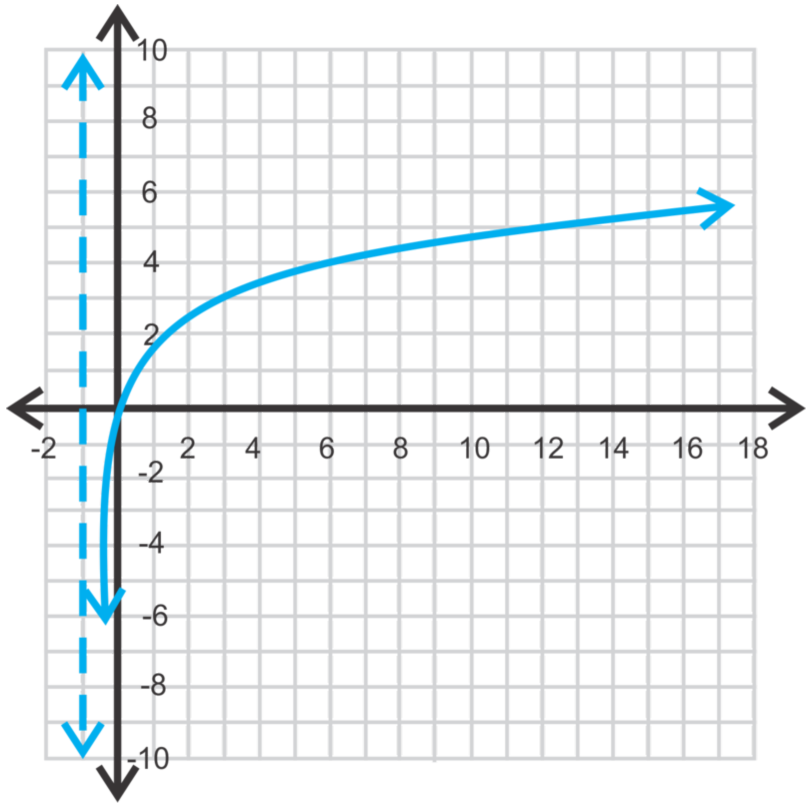
$$\begin{aligned}1 &= \log(16-6)\\1 &= \log 10\\1 &= 1\end{aligned}$$

Yes, this is true, so (16, 1) is on the graph.

Example C

Graph $f(x) = 2\ln(x+1)$.

Solution: To graph a natural log, we need to use a graphing calculator. Press $Y =$ and enter in the function, $Y = 2\ln(x+1)$, GRAPH.



Guided Practice

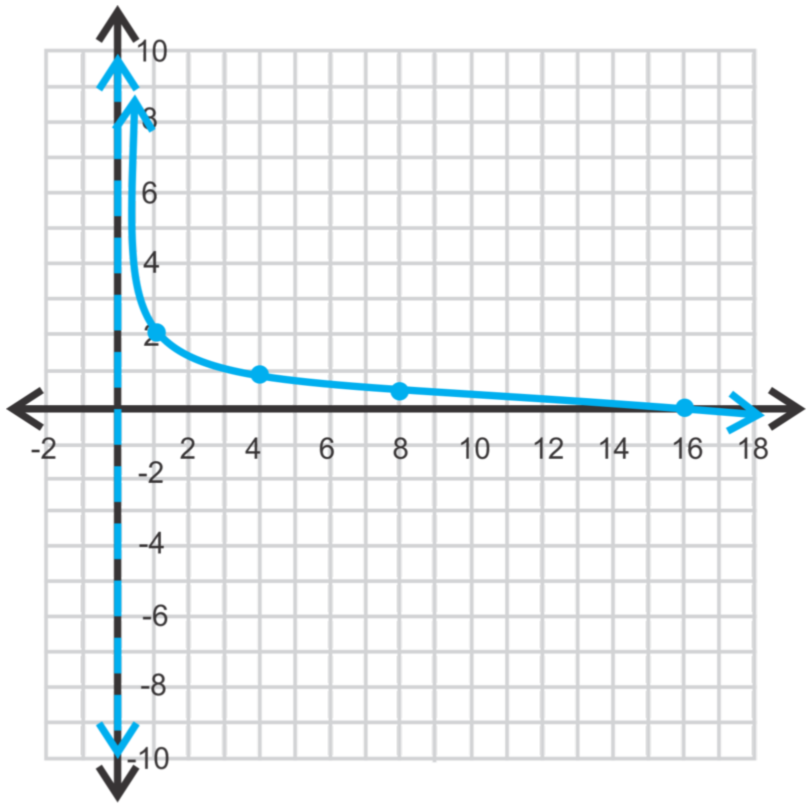
1. Graph $y = \log_{\frac{1}{4}} x + 2$ in an appropriate window.

2. Graph $y = -\log x$ using a graphing calculator. Find the domain and range.

3. Is (-2, 1) on the graph of $f(x) = \log_{\frac{1}{2}}(x+4)$?

Answers

1. First, there is a vertical asymptote at $x = 0$. Now, determine a few easy points, points where the log is easy to find; such as (1, 2), (4, 1), (8, 0.5), and (16, 0).

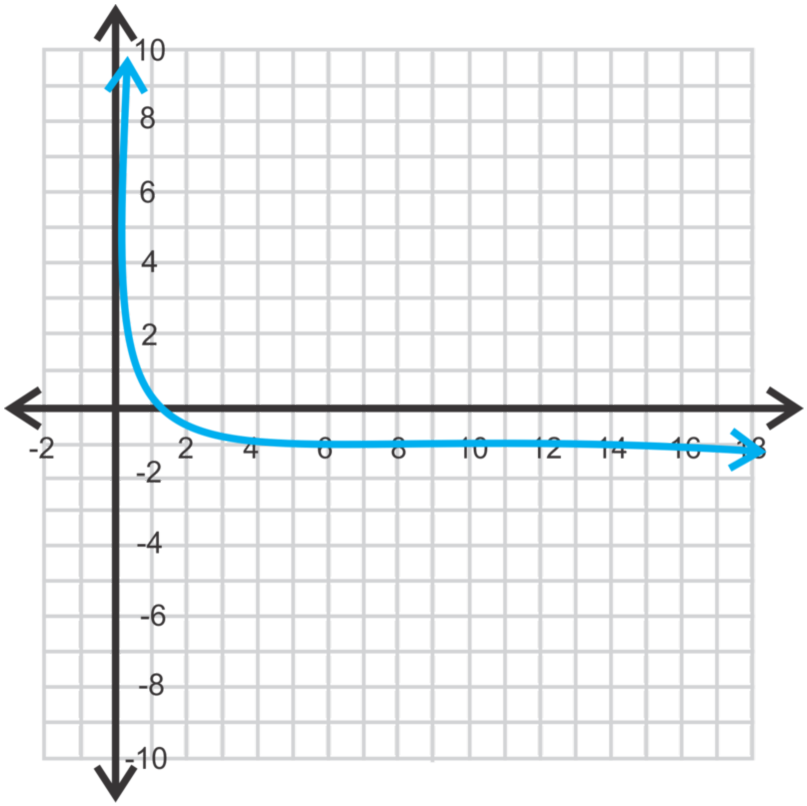


To graph a logarithmic function using a TI-83/84, enter the function into $Y =$ and use the *Change of Base Formula*. The keystrokes would be:

$Y = \frac{\log(x)}{\log\left(\frac{1}{4}\right)} + 2$, GRAPH

To see a table of values, press $2^{nd} \rightarrow$ GRAPH.

2. The keystrokes are $Y = -\log(x)$, GRAPH.



The domain is $x > 0$ and the range is all real numbers.

3. Plug (-2, 1) into $f(x) = \log_{\frac{1}{2}}(x+4)$ to see if the equation holds true.

$$1 = \log_{\frac{1}{2}}(-2+4)$$
$$1 = \log_{\frac{1}{2}} 2 \rightarrow \frac{1}{2}^{x} = 2$$
$$1 \neq -1$$

Therefore, (-2, 1) is not on the graph. However, (-2, -1) is.

Problem Set

Graph the following logarithmic functions without using a calculator. State the equation of the asymptote, the domain and the range of each function.

1. $y = \log_5 x$
2. $y = \log_2(x+1)$
3. $y = \log(x) - 4$
4. $y = \log_{\frac{1}{3}}(x-1) + 3$
5. $y = -\log_{\frac{1}{2}}(x+3) - 5$
6. $y = \log_4(2-x) + 2$

Graph the following logarithmic functions using your graphing calculator.

7. $y = \ln(x+6) - 1$
8. $y = -\ln(x-1) + 2$
9. $y = \ln(1-x) + 3$

10. Is (3, 8) on the graph of $y = \log_3(2x-3)+7$?
11. Is (9, -2) on the graph of $y = \log_{\frac{1}{4}}(x-5)$?
12. Is (4, 5) on the graph of $y = 5\log_2(8-x)$?

16.5 Solving Exponential and Logarithmic Equations

Objective

To solve exponential and logarithmic equations.

Review Queue

Solve the following equations.

1. $2^x = 32$

2. $x^2 - 9x + 20 = 0$

3. $\sqrt{x-5} + 3 = 11$

4. $8^x = 128$

Solving Exponential Equations

Objective

To learn how to solve exponential equations.

Guidance

Until now, we have only solved pretty basic exponential equations, like #1 in the Review Queue above. We know that $x = 5$, because $2^5 = 32$. Ones like #4 are a little more challenging, but if we put everything into a power of 2, we can set the exponents equal to each other and solve.

$$\begin{aligned} 8^x &= 128 \\ 2^{3x} &= 2^7 \\ 3x &= 7 \qquad\qquad \text{So}\ , 8^{\frac{7}{3}} = 128. \\ x &= \frac{7}{3} \end{aligned}$$

But, what happens when the power is not easily found? We must use logarithms, followed by the Power Property to solve for the exponent.

Example A

Solve $6^x = 49$. Round your answer to the nearest three decimal places.

Solution: To solve this exponential equation, let's take the logarithm of both sides. The easiest logs to use are either ln (the natural log), or log (log, base 10). We will use the natural log.

$$\begin{aligned} 6^x &= 49 \\ \ln 6^x &= \ln 49 \\ x \ln 6 &= \ln 49 \\ x &= \frac{\ln 49}{\ln 6} \approx 2.172 \end{aligned}$$

Example B

Solve $10^{x-3} = 100^{3x+11}$.

Solution: Change 100 into a power of 10.

$$\begin{aligned} 10^{x-3} &= 10^{2(3x+11)} \\ x-3 &= 6x+22 \\ -25 &= 5x \\ -5 &= x \end{aligned}$$

Example C

Solve $8^{2x-3} - 4 = 5$.

Solution: Add 4 to both sides and then take the log of both sides.

$$\begin{aligned} 8^{2x-3} - 4 &= 5 \\ 8^{2x-3} &= 9 \\ \log 8^{2x-3} &= \log 9 \\ (2x-3)\log 8 &= \log 9 \\ 2x-3 &= \frac{\log 9}{\log 8} \\ 2x &= 3 + \frac{\log 9}{\log 8} \\ x &= \frac{3}{2} + \frac{\log 9}{2\log 8} \approx 2.56 \end{aligned}$$

Notice that we did not find the numeric value of $\log 9$ or $\log 8$ until the very end. This will ensure that we have the most accurate answer.

Guided Practice

Solve the following exponential equations.

1. $4^{x-8} = 16$

2. $2(7)^{3x+1} = 48$

3. $\frac{2}{3} \cdot 5^{x+2} + 9 = 21$

Answers

1. Change 16 to 4^2 and set the exponents equal to each other.

$$\begin{aligned}4^{x-8} &= 16\\4^{x-8} &= 4^2\\x-8 &= 2\\x &= 10\end{aligned}$$

2. Divide both sides by 2 and then take the log of both sides.

$$\begin{aligned}2(7)^{3x+1} &= 48\\7^{3x+1} &= 24\\\ln 7^{3x+1} &= \ln 24\\(3x+1)\ln 7 &= \ln 24\\3x+1 &= \frac{\ln 24}{\ln 7}\\3x &= -1+\frac{\ln 24}{\ln 7}\\x &= -\frac{1}{3}+\frac{\ln 24}{3\ln 7} \approx 0.211\end{aligned}$$

3. Subtract 9 from both sides and multiply both sides by $\frac{3}{2}$. Then, take the log of both sides.

$$\begin{aligned}\frac{2}{3}\cdot 5^{x+2}+9 &= 21\\\frac{2}{3}\cdot 5^{x+2} &= 12\\5^{x+2} &= 18\\(x+2)\log 5 &= \log 18\\x &= \frac{\log 18}{\log 5}-2 \approx -0.204\end{aligned}$$

Problem Set

Use logarithms and a calculator to solve the following equations for x. Round answers to three decimal places.

1. $5^x = 65$
2. $2^x = 90$
3. $6^{x+1}+3 = 13$
4. $6(11^{3x-2}) = 216$
5. $8+13^{2x-5} = 35$
6. $\frac{1}{2}\cdot 7^{x-3}-5 = 14$

Solve the following exponential equations without a calculator.

7. $4^x = 8$
8. $5^{2x+1} = 125$
9. $9^3 = 3^{4x-6}$
10. $7(2^{x-3}) = 56$
11. $16^x \cdot 4^{x+1} = 32^{x+1}$
12. $3^{3x+5} = 3\cdot 9^{x+3}$

Solving Logarithmic Equations

Objective

To solve a logarithmic equation with any base.

Guidance

A logarithmic equation has the variable within the log. To solve a logarithmic equation, you will need to use the inverse property, $b^{\log_b x} = x$, to cancel out the log.

Example A

Solve $\log_2(x+5) = 9$.

Solution: There are two different ways to solve this equation. The first is to use the definition of a logarithm.

$$\begin{aligned} \log_2(x+5) &= 9 \\ 2^9 &= x+5 \\ 512 &= x+5 \\ 507 &= x \end{aligned}$$

The second way to solve this equation is to put everything into the exponent of a 2, and then use the inverse property.

$$\begin{aligned} 2^{\log_2(x+5)} &= 2^9 \\ x+5 &= 512 \\ x &= 507 \end{aligned}$$

Make sure to check your answers for logarithmic equations. There can be times when you get an extraneous solution. $\log_2(507+5) = 9 \rightarrow \log_2 512 = 9 \boxed{\checkmark}$

Example B

Solve $3\ln(-x) - 5 = 10$.

Solution: First, add 5 to both sides and then divide by 3 to isolate the natural log.

$$\begin{aligned} 3\ln(-x) - 5 &= 10 \\ 3\ln(-x) &= 15 \\ \ln(-x) &= 5 \end{aligned}$$

Recall that the inverse of the natural log is the natural number. Therefore, everything needs to be put into the exponent of e in order to get rid of the log.

$$\begin{aligned} e^{\ln(-x)} &= e^5 \\ -x &= e^5 \\ x &= -e^5 \approx -148.41 \end{aligned}$$

Checking the answer, we have $3\ln(-(-e^5)) - 5 = 10 \rightarrow 3\ln e^5 - 5 = 10 \rightarrow 3 \cdot 5 - 5 = 10$ $\boxed{\checkmark}$

Example C

Solve $\log 5x + \log(x-1) = 2$

Solution: Condense the left-hand side using the Product Property.

$$\begin{aligned}
\log 5x + \log(x-1) &= 2\\
\log[5x(x-1)] &= 2\\
\log(5x^2 - 5x) &= 2
\end{aligned}$$

Now, put everything in the exponent of 10 and solve for x.

$$\begin{aligned}
10^{\log(5x^2-5x)} &= 10^2\\
5x^2 - 5x &= 100\\
x^2 - x - 20 &= 0\\
(x-5)(x+4) &= 0\\
x &= 5, -4
\end{aligned}$$

Now, check both answers.

$$\begin{aligned}
\log 5(5) + \log(5-1) &= 2\\
\log 25 + \log 4 &= 2 \ \boxed{\checkmark}\\
\log 100 &= 2
\end{aligned}
\qquad
\begin{aligned}
\log 5(-4) + \log((-4)-1) &= 2\\
\log(-20) + \log(-5) &= 2 \ \boxed{\times}
\end{aligned}$$

-4 is an extraneous solution. In the step $\log(-20) + \log(-5) = 2$, we cannot take the log of a negative number, therefore -4 is not a solution. 5 is the only solution.

Guided Practice

Solve the following logarithmic equations.

1. $9 + 2\log_3 x = 23$

2. $\ln(x-1) - \ln(x+1) = 8$

3. $\frac{1}{2}\log_5(2x+5) = 5$

Answers

1. Isolate the log and put everything in the exponent of 3.

$$\begin{aligned}
9 + 2\log_3 x &= 23\\
2\log_3 x &= 14\\
\log_3 x &= 7\\
x &= 3^7 = 2187
\end{aligned}
\qquad
\begin{aligned}
9 + 2\log_3 2187 &= 23\\
9 + 2 \cdot 7 &= 23 \ \boxed{\checkmark}\\
9 + 14 &= 23
\end{aligned}$$

2. Condense the left-hand side using the Quotient Rule and put everything in the exponent of e.

$$\begin{aligned}
\ln(x-1)-\ln(x+1) &= 8 \\
\ln\left(\frac{x-1}{x+1}\right) &= 8 \\
\frac{x-1}{x+1} &= \ln 8 \\
x-1 &= (x+1)\ln 8 \\
x-1 &= x\ln 8+\ln 8 \\
x-x\ln 8 &= 1+\ln 8 \\
x(1-\ln 8) &= 1+\ln 8 \\
x &= \frac{1+\ln 8}{1-\ln 8} \approx -2.85
\end{aligned}$$

Checking our answer, we get $\ln(-2.85-1)-\ln(2.85+1)=8$, which does not work because the first natural log is of a negative number. Therefore, there is no solution for this equation.

3. Multiply both sides by 2 and put everything in the exponent of a 5.

$$\begin{aligned}
\frac{1}{2}\log_5(2x+5) &= 2 \\
\log_5(2x+5) &= 4 \\
2x+5 &= 625 \\
2x &= 620 \\
x &= 310
\end{aligned}$$

$$\text{Check}: \quad \begin{aligned}
\frac{1}{2}\log_5(2\cdot 310+5) &= 2 \\
\frac{1}{2}\log_5 625 &= 2 \; \boxed{\checkmark} \\
\frac{1}{2}\cdot 4 &= 2
\end{aligned}$$

Problem Set

Use properties of logarithms and a calculator to solve the following equations for x. Round answers to three decimal places and check for extraneous solutions.

1. $\log_7(2x+3)=3$
2. $8\ln(3-x)=5$
3. $4\log_3 3x-\log_3 x=5$
4. $\log(x+5)+\log x=\log 14$
5. $2\ln x-\ln x=0$
6. $3\log_3(x-5)=3$
7. $\frac{2}{3}\log_3 x=2$
8. $5\log\frac{x}{2}-3\log\frac{1}{x}=\log 8$
9. $2\ln x^{e+2}-\ln x=10$
10. $2\log_6 x+1=\log_6(5x+4)$
11. $2\log_{\frac{1}{2}} x+2=\log_{\frac{1}{2}}(x+10)$
12. $3\log_{\frac{2}{3}} x-\log_{\frac{2}{3}} 27=\log_{\frac{2}{3}} 8$